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THE ELEMENTS

OF

RAILROAD ENGINEERING

Prepared for Students of
The International Correspondence Schools
SCRANTON, PA.

Volume I

ARITHMETIC
ALGEBRA
LOGARITHMS
GEOMETRY AND
TRIGONOMETRY

ELEMENTARY MECHANICS
HYDROMECHANICS
PNEUMATICS
STRENGTH OF
MATERIALS

WITH PRACTICAL QUESTIONS AND EXAMPLES

First Edition

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THE COLLIERY ENGINEER CO.
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PREFACE.

The Instruction and Question Papers which are furnished to the students of The International Correspondence Schools become so badly worn and soiled that, when a student has completed his Course, he has worn out his Instruction Papers, and they are no longer suitable for reference or review. Since the Instruction Papers are very valuable, especially to those who have studied them, there has grown up a demand for Sets of the Instruction and Question Papers, indexed for convenient reference, and durably bound for preservation. Again, many of our students can spare but little time for study, and are, therefore, a long while passing through their Courses. These students also desire the Papers in bound volumes to use for reference. Other students begin Courses, but for various reasons are unable to complete them, and feel that, having paid for their Scholarships, they ought to have the text-books, even though they can not finish their Courses.

For these reasons, we have decided to publish all of the Instruction and Question Papers of our different Technical Courses in volumes bound in Half Leather, to make a small advance in our prices, and to furnish a set to each student as soon as his Scholarship is paid for, whether he has completed his studies or not.

The volumes for the present course, the Railroad Engineering, are five in number:

Volume I contains the Instruction and Question Papers on Arithmetic, Algebra, Logarithms, Geometry and Trigonometry, Mechanics, Hydromechanics, Pneumatics, and Strength of Materials.

Volume II contains the Instruction and Question Papers

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on Surveying, Land Surveying, Mapping, Railroad Location, Railroad Construction, Track Work, and Railroad Structures.

Volume III contains the Plates and the instructions for drawing them. It forms a very complete Course in Mechanical Drawing.

Volume IV contains the Tables and Formulas in common use. The student who has finished his Course will find these Tables of great service. All the principal formulas. with the definitions of the letters used in them, are conveniently arranged for reference, so that the student can save the labor and time of hunting them out in the Instruction Papers.

Volume V contains the answers to the questions in the Question Papers.

These Instruction Papers are written from a practical standpoint, and contain only such information as the student requires in order to obtain a good working knowledge of the subjects which form the groundwork of a course in the elements of Railroad Engineering. There is no padding, and the student gets in a clear and concise form the exact information which he desires.

To keep the student always interested in his work, we do not give him half a dozen ways of doing the same thing; neither do we enter into speculative discussions of different subjects, all of which tend to confuse the student and leave him in doubt as to which one (or any one) is correct; we give him the best or most suitable rule, formula, or method that we know of, without mentioning any of the others. In short, the Papers are written for practical men, and all the difficulties encountered by a student studying by himself, particularly those which are due to a definition or explanation being too technical, abstract, or not clear enough to be readily understood, have been carefully considered and overcome.

THE INTERNATIONAL CORRESPONDENCE SCHOOLS.

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ARITHMETIC.

DEFINITIONS.

1. **Arithmetic** is the art of reckoning, or the study of numbers.

2. A **unit** is *one*, or a single thing, as *one*, *one* boy, *one* horse, *one* dozen.

3. A **number** is a unit or a collection of units, as *one*, *three* apples, *five* boys.

4. The **unit of a number** is one of the collection of units which constitutes the number. Thus, the unit of *twelve* is *one*, of *twenty* dollars is *one* dollar.

5. A **concrete number** is a number applied to some particular kind of object or quantity, as three *horses*, five *dollars*, ten *pounds*.

6. An **abstract number** is a number that is not applied to any object or quantity, as *three*, *five*, *ten*.

7. **Like numbers** are numbers which express units of the *same kind*, as 6 *days* and 10 *days*, 2 *feet* and 5 *feet*.

8. **Unlike numbers** are numbers which express units of *different kinds*, as ten *months* and eight *miles*, seven *dollars* and five *feet*.

NOTATION AND NUMERATION.

9. Numbers are expressed in three ways: (1) by words; (2) by figures; (3) by letters.

10. **Notation** is the art of expressing numbers by figures or letters.

11. **Numeration** is the art of reading the numbers which have been expressed by figures or letters.

12. The **Arabic notation** is the method of expressing numbers by figures. This method employs ten different **figures** to represent numbers, viz.:

Figures	0	1	2	3	4	5	6	7	8	9
Names	*naught, cipher, or zero.*	*one*	*two*	*three*	*four*	*five*	*six*	*seven*	*eight*	*nine*

The first character (0) is called **naught, cipher,** or **zero,** and, when standing alone, has no value.

The other nine figures are called **digits,** and each one has a value of its own.

Any whole number is called an **integer.**

13. As there are only ten *figures* used in expressing numbers, each *figure* must express a different *value* at different times.

14. The value of a figure depends upon its *position* in relation to others.

15. Figures have **simple** values and **local** or **relative** values.

16. The **simple** value of a figure is the value it expresses when standing alone.

17. The **local** or **relative** value is the *increased* value it expresses by having other figures placed on its right.

For instance, if we see the figure 6 standing alone, thus. 6

we consider it as *six units*, or simply **six.**

Place another 6 to the *left* of it; thus. 66

The original figure is still *six units*, but the second one is *ten times* 6, or 6 **tens.**

If a third 6 be now placed still one place further to the *left*, it is increased in value *ten times* more, thus making it 6 **hundreds** 666

A fourth 6 would be 6 **thousands** 6666

A fifth 6 would be 6 **tens of thousands,** or **sixty thousand** 66666

A sixth 6 would be 6 **hundreds of thousands** . 666666

A seventh 6 would be 6 **millions** 6666666

The entire line of seven figures is read *six millions, six hundred sixty-six thousands, six hundred sixty-six.*

18. The **increased value** of each of these figures is its *local* or *relative* value. Each figure is *ten times* greater in value than the one immediately on its *right.*

19. The **cipher** (0) has no value itself, but it is useful in determining the place of other figures. To represent the number *four hundred five*, two digits only are necessary, one to represent *four hundred*, and the other to represent *five units;* but if these two digits are placed together, as 45, the 4 (being in the second place) will mean 4 *tens.* To mean 4 *hundreds*, the 4 should have two figures on its right, and a *cipher* is therefore inserted in the place usually given to *tens*, to show that the number is composed of *hundreds* and *units* only, and that there are no *tens.* *Four hundred five* is therefore expressed as 405. If the number were *four thousand and five*, two ciphers would be inserted; thus, 4005. If it were *four hundred fifty*, it would have the *cipher* at the right-hand side to show that there were no *units*, and only *hundreds and tens;* thus, 450. *Four thousand and fifty* would be expressed 4050, the first cipher indicating that there are no hundreds and the second that there are no units.

NOTE.—When speaking of the figures of a number by referring to them as first figure, second figure, etc., always begin to count at the *left.* Thus, in the number 41,625, 4 is the first figure, 6 the third figure, 5 the fifth or last figure, etc.

20. In *reading* figures, it is usual to point off the number into groups of three figures each, beginning with the right-hand or **units** column, a comma (,) being used to point off these groups.

Billions.			*Millions.*			*Thousands.*			*Units.*		
Hundreds of Billions.	Tens of Billions.	Billions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds of Units.	Tens of Units.	Units.
4	3	2,	1	9	8,	7	6	5,	4	3	2

In *pointing off* these figures, begin at the right-hand figure and count—*units*, *tens*, *hundreds;* the next group of three figures is *thousands*, therefore, we insert a comma (,) before beginning with them. Beginning at the figure 5, we say *thousands*, *tens of thousands*, *hundreds of thousands*, and insert another comma; we next read *millions*, *tens of millions*, *hundreds of millions*, and insert another comma; we then read *billions*, *tens of billions*, *hundreds of billions*.

The entire line of figures would be read: *Four hundred thirty-two billions, one hundred ninety-eight millions, seven hundred sixty-five thousands, four hundred thirty-two.* When we thus *read* a line of figures it is called **numeration,** and if the **numeration** be changed back to *figures*, it is called **notation.**

For instance, the writing of the figures,

72,584,623,

would be the **notation,** and the **numeration** would be *seventy-two millions*, *five hundred eighty-four thousands*, *six hundred twenty-three.*

21. Note.—It is customary to leave the *s* off the words millions, thousands, etc., in cases like the above, both in speaking and writing; hence, the above would usually be expressed, seventy-two million, five hundred eighty-four thousand, six hundred twenty-three.

22. The four fundamental processes of Arithmetic are **addition, subtraction, multiplication,** and **division.** They are called fundamental processes, because all operations in Arithmetic are based upon them.

ADDITION.

23. **Addition** is the *process* of *finding* the *sum* of *two* or *more* numbers. The sign of addition is +. It is read *plus*, and means *more.* Thus, $5 + 6$ is read 5 *plus* 6, and means that 5 and 6 are to be added.

24. The sign of equality is =. It is read *equals* or *is equal to.* Thus, $5 + 6 = 11$ may be read 5 plus 6 *equals* 11.

25. *Like numbers* can be added, but *unlike numbers* cannot. Thus, 6 dollars *can* be added to 7 dollars, and the *sum* will be 13 dollars, but 6 dollars *cannot* be added to 7 *feet.*

26. The following table gives the sum of any two numbers from 1 to 12:

TABLE 1.

1 and 1 are 2	2 and 1 are 3	3 and 1 are 4	4 and 1 are 5
1 and 2 are 3	2 and 2 are 4	3 and 2 are 5	4 and 2 are 6
1 and 3 are 4	2 and 3 are 5	3 and 3 are 6	4 and 3 are 7
1 and 4 are 5	2 and 4 are 6	3 and 4 are 7	4 and 4 are 8
1 and 5 are 6	2 and 5 are 7	3 and 5 are 8	4 and 5 are 9
1 and 6 are 7	2 and 6 are 8	3 and 6 are 9	4 and 6 are 10
1 and 7 are 8	2 and 7 are 9	3 and 7 are 10	4 and 7 are 11
1 and 8 are 9	2 and 8 are 10	3 and 8 are 11	4 and 8 are 12
1 and 9 are 10	2 and 9 are 11	3 and 9 are 12	4 and 9 are 13
1 and 10 are 11	2 and 10 are 12	3 and 10 are 13	4 and 10 are 14
1 and 11 are 12	2 and 11 are 13	3 and 11 are 14	4 and 11 are 15
1 and 12 are 13	2 and 12 are 14	3 and 12 are 15	4 and 12 are 16
5 and 1 are 6	6 and 1 are 7	7 and 1 are 8	8 and 1 are 9
5 and 2 are 7	6 and 2 are 8	7 and 2 are 9	8 and 2 are 10
5 and 3 are 8	6 and 3 are 9	7 and 3 are 10	8 and 3 are 11
5 and 4 are 9	6 and 4 are 10	7 and 4 are 11	8 and 4 are 12
5 and 5 are 10	6 and 5 are 11	7 and 5 are 12	8 and 5 are 13
5 and 6 are 11	6 and 6 are 12	7 and 6 are 13	8 and 6 are 14
5 and 7 are 12	6 and 7 are 13	7 and 7 are 14	8 and 7 are 15
5 and 8 are 13	6 and 8 are 14	7 and 8 are 15	8 and 8 are 16
5 and 9 are 14	6 and 9 are 15	7 and 9 are 16	8 and 9 are 17
5 and 10 are 15	6 and 10 are 16	7 and 10 are 17	8 and 10 are 18
5 and 11 are 16	6 and 11 are 17	7 and 11 are 18	8 and 11 are 19
5 and 12 are 17	6 and 12 are 18	7 and 12 are 19	8 and 12 are 20
9 and 1 are 10	10 and 1 are 11	11 and 1 are 12	12 and 1 are 13
9 and 2 are 11	10 and 2 are 12	11 and 2 are 13	12 and 2 are 14
9 and 3 are 12	10 and 3 are 13	11 and 3 are 14	12 and 3 are 15
9 and 4 are 13	10 and 4 are 14	11 and 4 are 15	12 and 4 are 16
9 and 5 are 14	10 and 5 are 15	11 and 5 are 16	12 and 5 are 17
9 and 6 are 15	10 and 6 are 16	11 and 6 are 17	12 and 6 are 18
9 and 7 are 16	10 and 7 are 17	11 and 7 are 18	12 and 7 are 19
9 and 8 are 17	10 and 8 are 18	11 and 8 are 19	12 and 8 are 20
9 and 9 are 18	10 and 9 are 19	11 and 9 are 20	12 and 9 are 21
9 and 10 are 19	10 and 10 are 20	11 and 10 are 21	12 and 10 are 22
9 and 11 are 20	10 and 11 are 21	11 and 11 are 22	12 and 11 are 23
9 and 12 are 21	10 and 12 are 22	11 and 12 are 23	12 and 12 are 24

This table should be carefully committed to memory. Since 0 has no value, the sum of any number and 0 is the number itself; thus, 17 and 0 are 17.

27. For *addition*, place the numbers to be added directly under each other, taking care to place *units* under *units*, *tens* under *tens*, *hundreds* under *hundreds*, and so on.

When the numbers are thus written, the *right-hand figure* of *one number* is placed *directly under the right-hand figure* of the *number above it*, thus bringing the unit figures of all the numbers to be added in the same vertical line. Proceed as in the following examples:

28. Example.—What is the sum of 131, 222, 21, 2, and 413?

Solution.—

$$\begin{array}{rrl} & 131 & \\ & 222 & \\ & 21 & \\ & 2 & \\ & 413 & \\ \cline{2-2} \textit{sum} & 789 & \text{Ans.} \end{array}$$

Explanation.—After placing the numbers in proper order, begin at the bottom of the right-hand or *units* column, and add, mentally repeating the different sums. Thus, three and two are five and one are six and two are eight and one are nine, the sum of the numbers in *units* column. Place the 9 directly beneath as the first or *units* figure in the sum.

The sum of the numbers in the next or *tens* column equals 8 *tens*, which is the second or *tens* figure in the sum.

The sum of the numbers in the next or *hundreds* column equals 7 *hundreds*, which is the third or *hundreds* figure in the sum.

The sum or answer is 789.

29. Example.—What is the sum of 425, 36, 9,215, 4, and 907?

Solution.—

$$\begin{array}{rrl} & 425 & \\ & 36 & \\ & 9215 & \\ & 4 & \\ & 907 & \\ \cline{2-2} & 27 & \\ & 60 & \\ & 1500 & \\ & 9000 & \\ \cline{2-2} \textit{sum} & 10587 & \text{Ans.} \end{array}$$

Explanation.—The sum of the numbers in the first or units column is seven and four are eleven and five are sixteen and six are twenty-two and five are twenty-seven, or 27 units; i. e., two tens and seven units. Write 27 as shown.

The sum of the numbers in the second or tens column is six tens, or 60. Write 60 underneath 27 as shown. The sum of the numbers in the third or hundreds column is 15 hundreds, or 1,500. Write 1,500 under the two preceding results as shown. There is only one number in the fourth or thousands column, nine, which represents 9,000. Write 9,000 under the three preceding results. Adding these four results, the sum is 10,587, which is the sum of 425, 36, 9,215, 4, and 907.

Note.—It frequently happens, when adding a long column of figures, that the sum of two numbers, one of which does not occur in the addition table, is required. Thus, in the first column above, the sum of 16 and 6 was required. We know from the table that $6 + 6 = 12$; hence, the first figure of the sum is 2. Now, the sum of any number less than 20 and of any number less than 10 must be less than thirty, since $20 + 10 = 30$; therefore, the sum is 22. Consequently, in cases of this kind, add the first figure of the larger number to the smaller number and, if the result is greater than 9, increase the second figure of the larger number by 1. Thus, $44 + 7 = ?$ $4 + 7 = 11$; hence, $44 + 7 = 51$.

30. The addition may also be performed as follows:

$$\begin{array}{rr}
 & 425 \\
 & 36 \\
 & 9215 \\
 & 4 \\
 & 907 \\
\cline{2-2}
\textit{sum} & 10587 \text{ Ans.}
\end{array}$$

Explanation.—The sum of the numbers in *units* column = 27 *units*, or 2 *tens* and 7 *units*. Write the 7 *units* as the first or right-hand figure in the sum. Reserve the two *tens* and add them to the figures in *tens* column. The sum of the figures in the *tens* column, plus the 2 *tens* reserved and carried from the *units* column = 8, which is written down as the second figure in the sum. There is nothing to carry to the next column, because 8 is less than 10. The sum of the numbers in the next column is 15 *hundreds*, or 1 *thousand* and 5 *hundreds*. Write down the 5 as the third or *hundreds* figure in the sum and carry the 1 to the next column. $1 + 9 = 10$, which is written down at the left of the other figures.

The second method saves space and figures, but the first is to be preferred when adding a long column.

31. Example.—Add the numbers in the column below·

Solution.—
890
82
90
393
281
80
770
83
492
80
383
84
191
sum 3899 Ans.

Explanation.—The sum of the digits in the first column equals 19 *units*, or 1 *ten* and 9 *units*. Write down the 9 and carry 1 to the next column. The sum of the digits in the second column + 1 = 109 *tens*, or 10 *hundreds* and 9 *tens*. Write down the 9 and carry the 10 to the next column. The sum of the digits in this column plus the 10 reserved = 38. The entire sum is 3,899.

32. Rule.—I. *Begin at the right, add each column separately, and write the sum, if it be only one figure, under the column added.*

II. *If the sum of any column consists of two or more figures, put the right-hand figure of the sum under that column, and add the remaining figure or figures to the next column.*

33. Proof.—*To prove addition, add each column from top to bottom. If you obtain the same result as by adding from bottom to top, the work is probably correct.*

EXAMPLES FOR PRACTICE.

34. Find the sum of

	Ans.
(*a*) 104 + 203 + 613 + 214.	(*a*) 1,134.
(*b*) 1,875 + 3,143 + 5,826 + 10,832.	(*b*) 21,676.
(*c*) 4,865 + 2,145 + 8,173 + 40,084.	(*c*) 55,267.
(*d*) 14,204 + 8,173 + 1,065 + 10,042.	(*d*) 33,484.

(*e*)	10,832 + 4,145 + 3,133 + 5,872.	Ans.	(*e*) 23,982.
(*f*)	214 + 1,231 + 141 + 5,000.		(*f*) 6,586.
(*g*)	123 + 104 + 425 + 126 + 327.		(*g*) 1,105.
(*h*)	6,354 + 2,145 + 2,042 + 1,111 + 3,333.		(*h*) 14,985.

SUBTRACTION.

35. In Arithmetic, **subtraction** is the process of finding how much greater one number is than another.

The greater of the two numbers is called the **minuend.**

The smaller of the two numbers is called the **subtrahend.**

The number left after subtracting the *subtrahend* from the *minuend* is called the **difference,** or **remainder.**

36. The sign of subtraction is – . It is read **minus,** and means *less*. Thus, 12 – 7 is read 12 *minus* 7, and means that 7 is to be taken from 12.

37. Example.—From 7,568 take 3,425.

Solution.—

minuend 7568
subtrahend 3425
remainder 4143 Ans.

Explanation.—Begin at the right-hand or *units* column and subtract in succession each figure in the subtrahend from the one directly above it in the minuend, and write the remainders below the line. The result is the entire remainder.

38. When there are more figures in the *minuend* than in the *subtrahend*, and when some figures in the *minuend* are *less* than the figures directly under them in the *subtrahend*, proceed as in the following example:

Example.—From 8,453 take 844.

Solution.—

minuend 8453
subtrahend 844
remainder 7609 Ans.

Explanation.—Begin to subtract at the right-hand or *units* column. We cannot take 4 from 3, and must, therefore borrow 1 from 5 in *tens* column and annex it to the 3 in *units* column. The 1 *ten* = 10 *units*, which added to the 3 in *units* column = 13 *units*. 4 from 13 = 9, the first or *units* figure in the remainder.

Since we borrowed 1 from the 5, only 4 remains; 4 from 4 = 0, the second or *tens* figure. We cannot take 8 from 4, and must, therefore, borrow 1 from 8 in *thousands* column. Since 1 *thousand* = 10 *hundreds*, 10 *hundreds* + 4 *hundreds* = 14 *hundreds*, and 8 from 14 = 6, the third or *hundreds* figure in the remainder.

Since we borrowed 1 from 8, only 7 remains, from which there is nothing to subtract; therefore, 7 is the next figure in the remainder or answer.

The operation of borrowing is placing 1 before the figure following the one from which it is borrowed. In the above example the 1 borrowed from 5 is placed before 3, making it 13, from which we subtract 4. The 1 borrowed from 8 is placed before 4, making 14, from which 8 is taken.

39. EXAMPLE.—Find the difference between 10,000 and 8,763.

SOLUTION.—

$$\begin{array}{rr} \textit{minuend} & 10000 \\ \textit{subtrahend} & 8763 \\ \hline \textit{remainder} & 1237 \end{array} \text{ Ans.}$$

EXPLANATION.—In the above example we borrow 1 from the second column and place it before 0, making 10; 3 from 10 = 7. In the same way we borrow 1 and place it before the next cipher, making 10; but as we have borrowed 1 from this column and taken it to the *units* column, only 9 remains, from which to subtract 6; 6 from 9 = 3. For the same reason we subtract 7 from 9 and 8 from 9 for the next two figures, and obtain a total remainder of 1,237.

40. Rule.—*Place the subtrahend or smaller number under the minuend or larger number, in the same manner as for addition, and proceed as in Arts.* ***37, 38,*** *and* ***39.***

41. Proof.—*To prove an example in subtraction, add the remainder to the subtrahend. The sum should equal the minuend. If it does not, a mistake has been made, and the work should be done over.*

Proof of the above example:

subtrahend	8763
remainder	1237
minuend	10000

EXAMPLES FOR PRACTICE.

42. From

	Ans.
(*a*) 94,278 take 62,574.	(*a*) 31,704.
(*b*) 53,714 take 25,824.	(*b*) 27,890.
(*c*) 71,832 take 58,109.	(*c*) 13,723.
(*d*) 20,804 take 10,408.	(*d*) 10,396.
(*e*) 310,465 take 102,141.	(*e*) 208,324.
(*f*) (81,043 + 1,041) take 14,831.	(*f*) 67,253.
(*g*) (20,482 + 18,216) take 21,214.	(*g*) 17,484.
(*h*) (2,040 + 1,213 + 542) take 3,791.	(*h*) 4.

MULTIPLICATION.

43. To **multiply** a number is to *add* it to itself a certain number of times.

44. **Multiplication** is the process of multiplying one number by another.

The *number* thus added to itself, or the number to be multiplied, is called the **multiplicand.**

The *number* which shows how many times the *multiplicand* is to be taken, or the *number* by which we *multiply*, is called the **multiplier.**

The result obtained by multiplying is called the **product.**

45. The sign of multiplication is ×. It is read *times* or *multiplied by*. Thus, 9 × 6 is read 9 *times* 6, or 9 *multiplied by* 6.

46. It matters not in what order the numbers to be multiplied together are placed. Thus, 6 × 9 is the same as 9 × 6.

47. In the following table, the product of any two numbers (neither of which exceeds twelve) may be found:

TABLE 2.

1 times 1 is 1	2 times 1 are 2	3 times 1 are 3
1 times 2 are 2	2 times 2 are 4	3 times 2 are 6
1 times 3 are 3	2 times 3 are 6	3 times 3 are 9
1 times 4 are 4	2 times 4 are 8	3 times 4 are 12
1 times 5 are 5	2 times 5 are 10	3 times 5 are 15
1 times 6 are 6	2 times 6 are 12	3 times 6 are 18
1 times 7 are 7	2 times 7 are 14	3 times 7 are 21
1 times 8 are 8	2 times 8 are 16	3 times 8 are 24
1 times 9 are 9	2 times 9 are 18	3 times 9 are 27
1 times 10 are 10	2 times 10 are 20	3 times 10 are 30
1 times 11 are 11	2 times 11 are 22	3 times 11 are 33
1 times 12 are 12	2 times 12 are 24	3 times 12 are 36
4 times 1 are 4	5 times 1 are 5	6 times 1 are 6
4 times 2 are 8	5 times 2 are 10	6 times 2 are 12
4 times 3 are 12	5 times 3 are 15	6 times 3 are 18
4 times 4 are 16	5 times 4 are 20	6 times 4 are 24
4 times 5 are 20	5 times 5 are 25	6 times 5 are 30
4 times 6 are 24	5 times 6 are 30	6 times 6 are 36
4 times 7 are 28	5 times 7 are 35	6 times 7 are 42
4 times 8 are 32	5 times 8 are 40	6 times 8 are 48
4 times 9 are 36	5 times 9 are 45	6 times 9 are 54
4 times 10 are 40	5 times 10 are 50	6 times 10 are 60
4 times 11 are 44	5 times 11 are 55	6 times 11 are 66
4 times 12 are 48	5 times 12 are 60	6 times 12 are 72
7 times 1 are 7	8 times 1 are 8	9 times 1 are 9
7 times 2 are 14	8 times 2 are 16	9 times 2 are 18
7 times 3 are 21	8 times 3 are 24	9 times 3 are 27
7 times 4 are 28	8 times 4 are 32	9 times 4 are 36
7 times 5 are 35	8 times 5 are 40	9 times 5 are 45
7 times 6 are 42	8 times 6 are 48	9 times 6 are 54
7 times 7 are 49	8 times 7 are 56	9 times 7 are 63
7 times 8 are 56	8 times 8 are 64	9 times 8 are 72
7 times 9 are 63	8 times 9 are 72	9 times 9 are 81
7 times 10 are 70	8 times 10 are 80	9 times 10 are 90
7 times 11 are 77	8 times 11 are 88	9 times 11 are 99
7 times 12 are 84	8 times 12 are 96	9 times 12 are 108
10 times 1 are 10	11 times 1 are 11	12 times 1 are 12
10 times 2 are 20	11 times 2 are 22	12 times 2 are 24
10 times 3 are 30	11 times 3 are 33	12 times 3 are 36
10 times 4 are 40	11 times 4 are 44	12 times 4 are 48
10 times 5 are 50	11 times 5 are 55	12 times 5 are 60
10 times 6 are 60	11 times 6 are 66	12 times 6 are 72
10 times 7 are 70	11 times 7 are 77	12 times 7 are 84
10 times 8 are 80	11 times 8 are 88	12 times 8 are 96
10 times 9 are 90	11 times 9 are 99	12 times 9 are 108
10 times 10 are 100	11 times 10 are 110	12 times 10 are 120
10 times 11 are 110	11 times 11 are 121	12 times 11 are 132
10 times 12 are 120	11 times 12 are 132	12 times 12 are 144

This table should be carefully committed to memory.

Since 0 has no value, the product of 0 and any number is 0.

48. To multiply a number by one figure only:

EXAMPLE.—Multiply 425 by 5.

SOLUTION.— *multiplicand* 425
multiplier 5
product 2125 Ans.

EXPLANATION.—For convenience, the *multiplier* is generally written *under* the *right-hand figure* of the *multiplicand.* On looking in the multiplication table, we see that 5 × 5 are 25. *Multiplying* the *first figure* at the *right* of the *multiplicand*, or 5, by the *multiplier* 5, it is seen that 5 times 5 units are 25 units, or 2 tens and 5 units. Write the *5 units* in *units place* in the *product*, and *reserve* the 2 tens to *add* to the *product* of *tens.* Looking in the multiplication table again, we see that 5 × 2 are 10. *Multiplying* the *second figure* of the *multiplicand* by the *multiplier* 5, we see *that* 5 *times* 2 tens are 10 tens, *plus* the 2 tens *reserved*, are 12 tens, or 1 hundred plus 2 tens. Write the *2 tens* in tens place, and *reserve* the 1 hundred to *add* to the product of *hundreds.* Again, we see by the multiplication table that 5 × 4 are 20. *Multiplying* the *third* or *last figure* of the *multiplicand* by the *multiplier* 5, we see that 5 times 4 hundreds are 20 hundreds, *plus* the 1 hundred *reserved*, are 21 hundreds, or 2 thousands *plus* 1 hundred, which we write in *thousands* and *hundreds places*, respectively.

Hence, the *product* is 2,125.

This *result* is the same as adding 425 five times. Thus,

425
425
425
425
425
sum 2125 Ans.

EXAMPLES FOR PRACTICE.

49. Find the product of

		Ans.
(*a*)	61,483 × 6.	(*a*) 368,898.
(*b*)	12,375 × 5.	(*b*) 61,875.
(*c*)	10,426 × 7.	(*c*) 72,982.
(*d*)	10,835 × 3.	(*d*) 32,505.

(*e*) 98,376 × 4.	Ans.	(*e*) 393,504.
(*f*) 10,873 × 8.		(*f*) 86,984.
(*g*) 71,543 × 9.		(*g*) 643,887.
(*h*) 218,734 × 2.		(*h*) 437,468.

50. To multiply a number by two or more figures:

EXAMPLE.—Multiply 475 by 234.

$$
\begin{array}{rr}
\text{SOLUTION.— } \textit{multiplicand} & 475 \\
\textit{multiplier} & \underline{234} \\
 & 1900 \\
 & 1425 \\
 & \underline{950} \\
\textit{product} & 111150 \text{ Ans.}
\end{array}
$$

EXPLANATION.—For convenience, the *multiplier* is generally written *under* the *multiplicand*, placing units under units, tens under tens, etc.

We *cannot* multiply by 234 at one operation; we must, therefore, *multiply* by the *parts* and then *add* the **partial products.**

The parts by which we are to multiply are 4 units, 3 tens, and 2 hundreds. 4 times 475 = 1,900, the *first partial product;* 3 times 475 = 1,425, the *second partial product*, the *right-hand figure* of which is *written directly under* the *figure multiplied by*, or 3; 2 times 475 = 950, the *third partial product*, the *right-hand figure* of which is *written directly under* the *figure multiplied by*, or 2.

The sum of these *three partial products* is 111,150, which is the *entire product.*

51. Rule.—I. *Write the multiplier under the multiplicand, so that units are under units, tens under tens, etc.*

II. *Begin at the right and multiply each figure of the multiplicand by each successive figure of the multiplier, placing the right-hand figure of each partial product directly under the figure used as a multiplier.*

III. *The sum of the partial products will equal the required product.*

52. Proof.—*Review the work carefully, or multiply the multiplier by the multiplicand; if the results agree, the work is correct.*

53. When there is a *cipher* in the *multiplier*, multiply the entire multiplicand by it; since the result will be zero, place a cipher under the cipher in the multiplier. Thus,

(*a*)	(*b*)	(*c*)	(*d*)
0	2	15	708
× 0	× 0	× 0	× 0
0 Ans.	0 Ans.	0 Ans.	0 Ans.

(*e*)	(*f*)	(*g*)
3114	4008	31264
203	305	1002
9342	20040	62528
62280	120240	3126400
632142 Ans.	1222440 Ans.	31326528 Ans.

In examples (*e*), (*f*), and (*g*), we multiply by 0 as directed above; then multiply by the next figure of the multiplier and place the first figure of the product alongside the 0, as shown.

EXAMPLES FOR PRACTICE.

54. Find the product of

			Ans.	
(*a*)	3,842 × 26.	(*a*)	99,892.	
(*b*)	3,716 × 45.	(*b*)	167,220.	
(*c*)	1,817 × 124.	(*c*)	225,308.	
(*d*)	675 × 38.	(*d*)	25,650.	
(*e*)	1,875 × 33.	(*e*)	61,875.	
(*f*)	4,836 × 47.	(*f*)	227,292.	
(*g*)	5,682 × 543.	(*g*)	3,085,326	
(*h*)	3,257 × 246.	(*h*)	801,222.	
(*i*)	2,875 × 302.	(*i*)	868,250.	
(*j*)	17,819 × 1,004.	(*j*)	17,890,276.	
(*k*)	38,674 × 205.	(*k*)	7,928,170.	
(*l*)	18,304 × 100.	(*l*)	1,830,400.	
(*m*)	7,834 × 10.	(*m*)	78,340.	
(*n*)	87,543 × 1,000.	(*n*)	87,543,000.	
(*o*)	48,763 × 100.	(*o*)	4,876,300.	

DIVISION.

55. Division is the process of finding how many times one number is contained in another of the same kind.

The number to be *divided* is called the **dividend.**

The number by which we *divide* is called the **divisor.**

The number which *shows* how many times the *divisor* is *contained* in the *dividend* is called the **quotient.**

56. The sign of division is ÷. It is read *divided by.* 54 ÷ 9 is read 54 *divided by* 9. Another way to write 54 *divided by* 9 is $\frac{54}{9}$. Thus, 54 ÷ 9 = 6, or $\frac{54}{9} = 6$.

In both of these cases 54 is the *dividend* and 9 is the *divisor.*

Division is the *reverse* of **multiplication.**

57. To divide when the divisor consists of but one figure, proceed as in the following example:

EXAMPLE.—What is the quotient of 875 ÷ 7?

```
                    divisor dividend quotient
SOLUTION.—                7)875(125  Ans.
                            7
                            ——
                            17
                            14
                            ———
                             35
                             35
                             ——
                 remainder    0
```

EXPLANATION.—7 is contained in 8 *hundreds* 1 *hundred* times. Place the one as the first or left-hand figure of the quotient. Multiply the divisor 7 by the 1 *hundred* of the quotient, and place the product 7 *hundreds* under the 8 *hundreds* in the dividend, and subtract. Beside the remainder 1, bring down the next or *tens* figure of the quotient, in this case 7, making 17 *tens;* 7 is contained in 17, 2 times. Write the 2 as the second figure of the quotient. Multiply the divisor 7 by the 2 in the quotient, and subtract the product from 17. Beside the remainder 3, bring down the next or *units* figure of the dividend, in this case 5, making

35 *units.* 7 is contained in 35, 5 times, which is placed in the quotient. Multiplying the divisor by the last figure of the quotient, 5 times 7 = 35, which subtracted from 35, under which it is placed, leaves 0. Therefore, the quotient is 125. This method is called **long division.**

58. In **short division,** only the divisor, dividend, and quotient are written, the operations being performed mentally.

$$\begin{array}{rl} & \quad\textit{dividend} \\ \textit{divisor} & 7\,)\,\underline{8^{1}7^{3}5} \\ \textit{quotient} & \quad 1\;\;2\;\;5 \quad \text{Ans.} \end{array}$$

The mental operation is as follows: 7 is contained in 8, once and one remainder; 1 placed before 7 makes 17; 7 is contained in 17, 2 times and 3 over; the 3 placed before 5 makes 35; 7 is contained in 35, 5 times. These partial quotients placed in order as they are found, make the entire quotient 125.

The small figures are placed in the example given to better illustrate the explanation; they are never written when actually performing division in this way.

59. If the *divisor* consists of *2 or more* figures, proceed as in the following example:

Example.—Divide 2,702,826 by 63.

Solution.—

$$\begin{array}{rccccccclr} \textit{divisor} & \multicolumn{7}{c}{\textit{dividend}} & \textit{quotient} \\ 63\,) & 2&7&0&2&8&2&6 & (\,42902 & \text{Ans.} \\ & 2&5&2 \\ \cline{2-4} & &1&8&2 \\ & &1&2&6 \\ \cline{2-5} & & &5&6&8 \\ & & &5&6&7 \\ \cline{3-7} & & & & &1&2&6 \\ & & & & &1&2&6 \\ \cline{6-8} & & & & & & &0 \end{array}$$

Explanation.—As 63 is not contained in the first two figures, 27, we must use the first three figures, 270. Now, by trial, we must find how many times 63 is contained in 270;

6 is contained in the first two figures of 270, 4 times. Place the 4 as the first or left-hand figure in the quotient. Multiply the divisor 63 by 4, and subtract the product 252 from 270. The remainder is 18, beside which we write the next figure of the dividend, 2, making 182. Now, 6 is contained in the first two figures of 182, 3 times, but on multiplying 63 by 3, we see that the product 189 is too great, so we try 2 as the second figure of the quotient. Multiplying the divisor 63 by 2, and subtracting the product 126 from 182, the remainder is 56, beside which we bring down the next figure of the dividend, making 568; 6 is contained in 56 about 9 times. Multiply the divisor 63 by 9 and subtract the product 567 from 568. The remainder is 1, and bringing down the next figure of the dividend, 2, gives 12. As 12 is smaller than 63, we write 0 in the quotient and bring down the next figure, 6, making 126. 63 is contained in 126, 2 times, without a remainder. Therefore, 42,902 is the quotient.

60. Rule.—I. *Write the divisor at the left of the dividend, with a line between them.*

II. *Find how many times the divisor is contained in the lowest number of the left-hand figures of the dividend that will contain it, and write the result at the right of the dividend, with a line between, for the first figure of the quotient.*

III. *Multiply the divisor by this quotient; write the product under the partial dividend used, and subtract, annexing to the remainder the next figure of the dividend. Divide as before, and thus continue until all the figures of the dividend have been used.*

IV. *If any partial dividend will not contain the divisor, write a cipher in the quotient, annex the next figure of the dividend and proceed as before.*

V. *If there be a remainder at last, write it after the quotient, with the divisor underneath.*

61. Proof.—*Multiply the quotient by the divisor, and add the remainder, if there be any, to the product. The result will be the dividend.*

divisor dividend quotient

Thus, $63)4235(67\frac{14}{63}$ Ans.

$$
\begin{array}{lr}
 & 378 \\
\hline
 & 455 \\
 & 441 \\
\hline
\textit{remainder} & 14 \\
\text{Proof,}\quad \textit{quotient} & 67 \\
\textit{divisor} & 63 \\
\hline
 & 201 \\
 & 402 \\
\hline
 & 4221 \\
\textit{remainder} & 14 \\
\hline
\textit{dividend} & 4235
\end{array}
$$

EXAMPLES FOR PRACTICE.

62. Divide the following:

	Ans.
(*a*) 126,498 by 58.	(*a*) 2,181.
(*b*) 3,207,594 by 767.	(*b*) 4,182.
(*c*) 11,408,202 by 234.	(*c*) 48,753.
(*d*) 2,100,315 by 581.	(*d*) 3,615.
(*e*) 969,936 by 4,008.	(*e*) 242.
(*f*) 7,481,888 by 1,021.	(*f*) 7,328.
(*g*) 1,525,915 by 5,003.	(*g*) 305.
(*h*) 1,646,301 by 381.	(*h*) 4,321.

CANCELATION.

63. Cancelation is the process of shortening operations in division by casting out equal factors from both dividend and divisor.

64. The **factors** of a number are *those numbers* which, when *multiplied* together, will *equal that number*. Thus, 5 and 3 are factors of 15, since $5 \times 3 = 15$. Likewise, 8 and 7 are the factors of 56, since $8 \times 7 = 56$.

65. A **prime number** is one which cannot be divided by any number except itself and 1. Thus, 2, 3, 11, 29, etc., are prime numbers.

66. A **prime factor** is any factor that is a prime number.

Any number that is not a prime is called a **composite** number, and may be produced by multiplying together its prime factors. Thus, 60 is a composite number, and is equal to the product of its prime factors, 2 × 2 × 3 × 5.

Numbers are said to be **prime to each other** when no two of them can be divided by any number except 1; the numbers themselves *may* be either prime or composite. Thus, the numbers 3, 5, and 11 are prime to each other, so also are 22, 25, and 21—all composite numbers.

67. Canceling *equal factors* from *both dividend and divisor* does *not* change the *quotient.*

The *canceling* of a *factor* in *both dividend and divisor* is the *same* as *dividing them both* by the *same number*, which, by the principle of division does not *change the quotient.*

Write the *numbers* which make the *dividend* above the *line*, and those which make the *divisor* below it.

68. EXAMPLE.—Divide 4 × 45 × 60 by 9 × 24.

SOLUTION.—Placing the dividend over the divisor, and canceling

$$\frac{\cancel{4} \times \overset{5}{\cancel{45}} \times \overset{10}{\cancel{60}}}{\cancel{9} \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}} = \frac{50}{1} = 50. \quad \text{Ans.}$$

EXPLANATION.—The 4 in the dividend and 24 in the divisor are both divisible by 4, since 4 divided by 4 equals 1, and 24 divided by 4 equals 6. Cross off the four and write the 1 over it; also, cross off the 24 and write the 6 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times 45 \times 60}{9 \times \underset{6}{\cancel{24}}}.$$

60 in the dividend and 6 in the divisor are divisible by 6, since 60 divided by 6 equals 10, and 6 divided by 6 equals 1. Cross off the 60 and write 10 over it; also, cross off the 6 and write 1 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times 45 \times \overset{10}{\cancel{60}}}{9 \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}}.$$

Again, 45 in the dividend and 9 in the divisor are divisible by 9, since 45 divided by 9 equals 5, and 9 divided by 9 equals 1. Cross off the 45 and write the 5 over it; also, cross off the 9 and write the 1 under it. Thus,

$$\frac{\overset{1}{\cancel{4}} \times \overset{5}{\cancel{45}} \times \overset{10}{\cancel{60}}}{\underset{1}{\cancel{9}} \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}}.$$

Since there are no two remaining numbers (one in the dividend and one in the divisor) divisible by any number except 1, without a remainder, it is impossible to cancel further.

Multiply all the uncanceled numbers in the dividend together, and divide their product by the product of all the uncanceled numbers in the divisor. The result will be the quotient. The product of all the uncanceled numbers in the dividend equals $5 \times 1 \times 10 = 50$; the product of all the uncanceled numbers in the divisor equals $1 \times 1 = 1$.

$$\text{Hence, } \frac{\overset{1}{\cancel{4}} \times \overset{5}{\cancel{45}} \times \overset{10}{\cancel{60}}}{\underset{1}{\cancel{9}} \times \underset{\underset{1}{\cancel{6}}}{\cancel{24}}} = \frac{1 \times 5 \times 10}{1 \times 1} = 50. \quad \text{Ans.}$$

It is usual to omit the 1's when canceling them, instead of writing them as above.

69. Rule.—I. *Cancel the common factors from both the dividend and divisor.*

II. *Then divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor, and the result will be the quotient.*

EXAMPLES FOR PRACTICE.

70. Divide

	Ans.
(*a*) 14 × 18 × 16 × 40 by 7 × 8 × 6 × 5 × 3.	(*a*) 32.
(*b*) 3 × 65 × 50 × 100 × 60 by 30 × 60 × 13 × 10.	(*b*) 250.
(*c*) 8 × 4 × 3 × 9 × 11 by 11 × 9 × 4 × 3 × 8.	(*c*) 1.
(*d*) 164 × 321 × 6 × 7 × 4 by 82 × 321 × 7.	(*d*) 48.
(*e*) 50 × 100 × 200 × 72 by 1,000 × 144 × 100.	(*e*) 5.
(*f*) 48 × 63 × 55 × 49 by 7 × 21 × 11 × 48.	(*f*) 105.
(*g*) 110 × 150 × 84 × 32 by 11 × 15 × 100 × 64.	(*g*) 42.
(*h*) 115 × 120 × 400 × 1,000 by 23 × 1,000 × 60 × 800.	(*h*) 5.

FRACTIONS.

71. A **fraction** is a *part* of a *whole number: One-half, one-third, two-fifths* are fractions.

72. *Two* numbers are required to express a fraction, one called the **numerator,** and the other the **denominator.**

73. The *numerator* is placed above the *denominator*, with a *line* between them; as, $\frac{2}{3}$. 3 is the *denominator*, and shows into how many *equal parts* the *unit* or *one* is divided. The *numerator* 2 shows how many of these *equal parts* are taken or considered. The *denominator* also indicates the *names* of the parts.

$\frac{1}{2}$ is read one-half.
$\frac{3}{4}$ is read three-fourths.
$\frac{3}{8}$ is read three-eighths.
$\frac{5}{16}$ is read five-sixteenths.
$\frac{29}{47}$ is read twenty-nine-forty-sevenths.

74. In the expression "$\frac{3}{4}$ of an apple," the *denominator* 4 shows that the apple is to be (or has been) cut into 4 *equal parts*, and the *numerator* 3 shows that *three of these parts*, or *fourths*, are taken or considered.

If each of the *parts*, or *fourths*, of the apple were cut in *two equal pieces*, there would then be twice as many pieces as before, or $4 \times 2 = 8$ pieces in all; one of these pieces would be called one-eighth, and would be expressed in figures as $\frac{1}{8}$. Three of these pieces would be called three-eighths, and written $\frac{3}{8}$. The words three-fourths, three-eighths, five-sixteenths, etc., are abbreviations of three one-fourths, three one-eighths, five one-sixteenths, etc. It is evident that the larger the *denominator*, the greater is the number of parts into which anything is divided; consequently, the parts themselves are smaller, and the value of the fraction is less for the same number of parts taken. In other words, $\frac{7}{9}$, for example, is smaller than $\frac{7}{8}$, because if an object be divided into 9 parts, the parts are smaller than if the same object had been divided into 8 parts; and, since $\frac{1}{9}$ is smaller than $\frac{1}{8}$, it is clear that 7 one-ninths is a smaller amount than 7 one-eighths. Hence, also, $\frac{3}{8}$ is less than $\frac{3}{4}$.

75. The **value** of a fraction is the *numerator* divided by the *denominator;* as, $\frac{4}{2} = 2$, $\frac{6}{2} = 3$.

76. The line between the *numerator* and *denominator* means *divided by*, or ÷.

$\frac{3}{4}$ is equivalent to $3 \div 4$.
$\frac{5}{8}$ is equivalent to $5 \div 8$.

77. The *numerator* and *denominator* of a fraction, when considered together, are called the **terms** of a fraction.

78. The *value* of a fraction whose *numerator* and *denominator* are equal is 1.

$\frac{4}{4}$, or four-fourths, $= 1$.
$\frac{8}{8}$, or eight-eighths, $= 1$.
$\frac{64}{64}$, or sixty-four-sixty-fourths, $= 1$.

79. A **proper fraction** is a fraction whose *numerator* is *less* than its *denominator.* Its *value* is *less* than 1; as, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{1}{16}$.

80. An **improper fraction** is a fraction whose *numerator equals* or is *greater than* the *denominator.* Its *value* is *one* or *more* than *one;* as, $\frac{4}{4}$, $\frac{9}{8}$, $\frac{42}{32}$.

81. A **mixed number** is a *whole* number and a *fraction* united. $4\frac{2}{3}$ is a mixed number, and is equivalent to $4 + \frac{2}{3}$. It is read *four and two-thirds.*

REDUCTION OF FRACTIONS.

82. **Reduction of fractions** is the process of changing their form without changing their *value.*

83. A *fraction* is reduced to *higher terms* by *multiplying both terms* of the *fraction* by the *same number.* Thus, $\frac{3}{4}$ is reduced to $\frac{6}{8}$ by multiplying both terms by 2.

$$\frac{3 \times 2}{4 \times 2} = \frac{6}{8}.$$

The *value* is not changed, since $\frac{3}{4} = \frac{6}{8}$. For, suppose that an object, say an apple, is divided into 8 equal parts. If these parts be arranged into 4 piles, each containing 2 parts, it is evident that each pile will be composed of the same amount of the entire apple as would have been the case had the apple been originally cut into 4 equal parts. Now, if one of these piles (containing 2 parts) be removed, there will be 3 piles left, each containing 2 equal parts, or 6 equal parts in all, i. e., six-eighths. But, since one pile, or one quarter, was removed, there are three-quarters left. Hence, $\frac{3}{4} = \frac{6}{8}$. The same course of reasoning may be applied to any similar case. Therefore, multiplying both terms of a fraction by the same number does not alter its value.

84. To reduce a fraction to an equal fraction having a given denominator:

Example.—Reduce $\frac{7}{8}$ to an equal fraction having 96 for a denominator.

Solution.—Both the numerator and the denominator must be multiplied by the same number in order not to change the value of the fraction. The denominator must be multiplied by some number which will, in this case, make the product 96; this number is evidently 96 ÷ 8 = 12, since 8 × 12 = 96. Hence, $\frac{7 \times 12}{8 \times 12} = \frac{84}{96}$. Ans.

85. Rule.—*Divide the given denominator by the denominator of the given fraction, and multiply both terms of the fraction by the result.*

Example.—Reduce $\frac{3}{4}$ to 100ths.

Solution.— 100 ÷ 4 = 25; hence, $\frac{3 \times 25}{4 \times 25} = \frac{75}{100}$. Ans.

86. A *fraction* is reduced to *lower terms* by *dividing both terms* by the *same number*. Thus, $\frac{8}{10}$ is reduced to $\frac{4}{5}$ by dividing both terms by 2.

$$\frac{8 \div 2}{10 \div 2} = \frac{4}{5}.$$

That $\frac{8}{10} = \frac{4}{5}$ is readily seen from the explanation given in Art. **83**; for, multiplying both terms of the fraction $\frac{4}{5}$ by 2, $\frac{4 \cdot 2}{5 \cdot 2} = \frac{8}{10}$, and, if $\frac{4}{5} = \frac{8}{10}$, $\frac{8}{10}$ must equal $\frac{4}{5}$. Hence, dividing both terms of a fraction by the same number does not alter its value.

87. A *fraction* is reduced to *lowest terms* when its *numerator and denominator* cannot both be *divided* by the *same* *number* without a remainder; as, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{11}{24}$, $\frac{8}{15}$. In other words, the numerator and denominator are prime to each other.

EXAMPLES FOR PRACTICE.

88. Reduce the following:

(*a*) $\frac{7}{16}$ to 128ths.	Ans.	(*a*) $\frac{56}{128}$.
(*b*) $\frac{24}{132}$ to its lowest terms.		(*b*) $\frac{2}{11}$.
(*c*) $\frac{64}{1000}$ to its lowest terms.		(*c*) $\frac{8}{125}$.
(*d*) $\frac{5}{7}$ to 49ths.		(*d*) $\frac{35}{49}$.
(*e*) $\frac{13}{16}$ to 10,000ths.		(*e*) $\frac{8125}{10000}$.

89. To reduce a whole number or mixed number to an improper fraction:

EXAMPLE.—How many *fourths* in 5?

SOLUTION.—Since there are 4 *fourths* in 1 ($\frac{4}{4} = 1$), in 5 there will be 5×4 fourths, or 20 fourths; i. e., $5 \times \frac{4}{4} = \frac{20}{4}$. Ans.

EXAMPLE.—Reduce $8\frac{3}{4}$ to an improper fraction.

SOLUTION.—$8 \times \frac{4}{4} = \frac{32}{4}$. $\frac{32}{4} + \frac{3}{4} = \frac{35}{4}$. Ans.

90. Rule.—*Multiply the whole number by the denominator of the fraction, add the numerator to the product, and place the denominator under the result. If it is desired to reduce a whole number to a fraction, multiply the whole number by the denominator of the given fraction, and write the result over the denominator.*

EXAMPLES FOR PRACTICE.

91. Reduce to improper fractions:

(*a*) $4\frac{1}{8}$.	Ans.	(*a*) $\frac{33}{8}$.
(*b*) $5\frac{1}{9}$.		(*b*) $\frac{46}{9}$.
(*c*) $10\frac{2}{10}$.		(*c*) $\frac{102}{10}$.
(*d*) $37\frac{3}{4}$.		(*d*) $\frac{151}{4}$.
(*e*) $50\frac{4}{5}$.		(*e*) $\frac{254}{5}$.
(*f*) Reduce 7 to a fraction whose denominator is 16.		(*f*) $\frac{112}{16}$.

92. To reduce an improper fraction to a whole or mixed number:

EXAMPLE.—Reduce $\frac{21}{4}$ to a mixed number.

SOLUTION.—4 is contained in 21, 5 times and 1 remaining (see Art. **75**); as this is also divided by 4, its value is $\frac{1}{4}$. Therefore, $5 + \frac{1}{4}$, or $5\frac{1}{4}$, is the number. Ans.

93. Rule.—*Divide the numerator by the denominator, the quotient will be the whole number; the remainder, if there be any, will be the numerator of the fractional part of which the denominator is the same as the denominator of the improper fraction.*

EXAMPLES FOR PRACTICE.

94. Reduce to whole or mixed numbers:

		Ans.
(*a*)	$\frac{145}{6}$.	(*a*) $24\frac{1}{6}$.
(*b*)	$\frac{185}{3}$.	(*b*) $61\frac{2}{3}$.
(*c*)	$\frac{701}{6}$.	(*c*) $116\frac{5}{6}$.
(*d*)	$\frac{149}{3}$.	(*d*) $49\frac{2}{3}$.
(*e*)	$\frac{76}{19}$.	(*e*) 4.
(*f*)	$\frac{125}{25}$.	(*f*) 5.

95. A **common denominator** of *two* or *more fractions* is a number which will contain (i. e., which may be divided by) all of the *denominators* of the *fractions* without a remainder. The **least common denominator** is the least number that will contain all of the denominators of the fractions without a remainder.

96. To find the least common denominator:

EXAMPLE.—Find the least common denominator of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{9}$, and $\frac{1}{16}$.

SOLUTION.—We first place the denominators in a row, separated by commas.

$$\begin{array}{r|rrrr} 2 & 4, & 3, & 9, & 16 \\ \hline 2 & 2, & 3, & 9, & 8 \\ \hline 3 & 1, & 3, & 9, & 4 \\ \hline & 1, & 1, & 3, & 4 \end{array}$$

$2 \times 2 \times 3 \times 3 \times 4 = 144$, the least common denominator. Ans.

EXPLANATION.—Divide the numbers by some prime number that will divide at least two of them without a remainder (if possible), bringing down to the row below those denominators which will not contain the divisor without a remainder. Dividing each of the numbers by 2, the second row becomes 2, 3, 9, 8, since 2 will not divide 3 and 9 without a remainder. Dividing again by 2, the result is 1, 3, 9, 4.

Dividing the third row by 3, the result is 1, 1, 3, 4. Since the remaining numbers are prime to each other, we cease dividing further. The product of all the divisors and of the numbers prime to each other, is $2 \times 2 \times 3 \times 3 \times 4 = 144$, which is the required least common denominator.

97. Example.—Find the least common denominator of $\frac{4}{9}$, $\frac{5}{12}$, $\frac{7}{18}$.

Solution.—

$$\begin{array}{r|l} 3 & 9,\ 12,\ 18 \\ \hline 3 & 3,\ \ 4,\ \ 6 \\ \hline 2 & 1,\ \ 4,\ \ 2 \\ \hline & 1,\ \ 2,\ \ 1 \end{array}$$

$3 \times 3 \times 2 \times 2 = 36$. Ans.

98. To reduce two or more fractions to fractions having a common denominator:

Example.—Reduce $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{1}{2}$ to fractions having a common denominator.

Solution.—The common denominator is any number which will contain 3, 4, and 2. The *least* common denominator is 12, because it is the smallest number which can be divided by 3, 4, and 2 without a remainder. $\frac{2}{3} = \frac{8}{12}$, $\frac{3}{4} = \frac{9}{12}$, $\frac{1}{2} = \frac{6}{12}$.

Reducing $\frac{2}{3}$ (see Art. **84**), 3 is contained in 12, 4 times. By multiplying both numerator and denominator of $\frac{2}{3}$ by 4, we find

$\frac{2 \times 4}{3 \times 4} = \frac{8}{12}$. In the same way we find $\frac{3}{4} = \frac{9}{12}$ and $\frac{1}{2} = \frac{6}{12}$.

99. Rule.—*Divide the common denominator by the denominator of the given fraction, and multiply both terms of the fraction by the quotient.*

EXAMPLES FOR PRACTICE.

100. Reduce to fractions having a common denominator:

(*a*) $\frac{3}{4}$, $\frac{5}{8}$, $\frac{7}{8}$.	Ans.	(*a*) $\frac{6}{8}$, $\frac{5}{8}$, $\frac{7}{8}$.
(*b*) $\frac{3}{16}$, $\frac{3}{4}$, $\frac{7}{32}$.		(*b*) $\frac{6}{32}$, $\frac{24}{32}$, $\frac{7}{32}$.
(*c*) $\frac{7}{8}$, $\frac{7}{88}$, $\frac{10}{11}$.		(*c*) $\frac{77}{88}$, $\frac{7}{88}$, $\frac{80}{88}$.
(*d*) $\frac{3}{5}$, $\frac{5}{8}$, $\frac{11}{40}$.		(*d*) $\frac{24}{40}$, $\frac{25}{40}$, $\frac{11}{40}$.
(*e*) $\frac{4}{10}$, $\frac{6}{40}$, $\frac{9}{20}$.		(*e*) $\frac{16}{40}$, $\frac{6}{40}$, $\frac{18}{40}$.
(*f*) $\frac{7}{15}$, $\frac{17}{80}$, $\frac{21}{80}$.		(*f*) $\frac{14}{80}$, $\frac{17}{80}$, $\frac{21}{80}$.

ADDITION OF FRACTIONS.

101. *Fractions cannot be added unless they have a common denominator.* We cannot add $\frac{3}{4}$ to $\frac{7}{8}$ as they now stand, since the denominators represent parts of different sizes. Fourths cannot be added to eighths.

Suppose we divide an apple into 4 equal parts, and then divide 2 of these parts into two equal parts. It is evident that we shall have 2 one-fourths and 4 one-eighths. Now, if we add these parts, the result is $2 + 4 = 6$ something. But what is this something? It is not fourths, for six fourths are $1\frac{1}{2}$, and we had only 1 apple to begin with; neither is it eighths, for six eighths are $\frac{3}{4}$, which is less than 1 apple. By reducing the quarters to eighths, we have $\frac{2}{4} = \frac{4}{8}$, and adding the other 4 eighths, $4 + 4 = 8$ eighths. This result is correct, since $\frac{8}{8} = 1$. Or, we can, in this case, reduce the eighths to quarters. Thus, $\frac{4}{8} = \frac{2}{4}$; whence, adding $2 + 2 = 4$ quarters, a correct result since $\frac{4}{4} = 1$.

Before adding, fractions should be reduced to a common denominator, preferably the *least* common denominator.

102. Example.—Find the sum of $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{8}$.

Solution.—The *least common denominator*, or the *least number* which will contain all the *denominators*, is 8.

$$\frac{1}{2} = \frac{4}{8},\ \frac{3}{4} = \frac{6}{8},\ \frac{5}{8} = \frac{5}{8}.$$

Explanation.—As the *denominator* tells or indicates the names of the *parts*, the *numerators* only are added in order to obtain the total number of *parts* indicated by the *denominator*. Thus, 4 one-eighths plus 6 one-eighths plus 5 one-eighths =

$$\frac{4}{8} + \frac{6}{8} + \frac{5}{8} = \frac{4 + 6 + 5}{8} = \frac{15}{8} = 1\frac{7}{8}.\quad \text{Ans.}$$

103. Example.—What is the sum of $12\frac{3}{4}$, $14\frac{5}{8}$, and $7\frac{5}{16}$?

Solution.—The least common denominator in this case is 16.

$$\begin{array}{r} 12\frac{3}{4} = 12\frac{12}{16} \\ 14\frac{5}{8} = 14\frac{10}{16} \\ 7\frac{5}{16} = 7\frac{5}{16} \\ \hline \end{array}$$

$$\textit{sum} = 33 + \frac{27}{16} = 33 + 1\frac{11}{16} = 34\frac{11}{16}.\quad \text{Ans.}$$

The sum of the fractions $= \frac{27}{16}$ or $1\frac{11}{16}$, which added to the sum of the whole numbers $= 34\frac{11}{16}$.

Example.—What is the sum of 17, $13\frac{3}{16}$, $\frac{9}{32}$, and $3\frac{1}{4}$?

Solution.—The least common denominator is 32. $13\frac{3}{16} = 13\frac{6}{32}$, $3\frac{1}{4} = 3\frac{8}{32}$.

$$\begin{array}{rr} & 17 \\ & 13\frac{6}{32} \\ & \frac{9}{32} \\ & 3\frac{8}{32} \\ \hline \textit{sum} & 33\frac{23}{32}. \end{array} \quad \text{Ans.}$$

104. Rule I.—*Reduce the given fractions to fractions having the least common denominator, and write the sum of the numerators over the common denominator.*

II. *When there are mixed numbers and whole numbers, add the fractions first, and if their sum is an improper fraction, reduce it to a mixed number, and add the whole number with the other whole numbers.*

EXAMPLES FOR PRACTICE.

105. Find the sum of

		Ans.	
(*a*)	$\frac{4}{6}$, $\frac{7}{24}$, $\frac{5}{8}$.	(*a*)	$1\frac{7}{12}$.
(*b*)	$\frac{2}{3}$, $\frac{5}{15}$, $\frac{24}{45}$.	(*b*)	$1\frac{8}{15}$.
(*c*)	$\frac{1}{2}$, $\frac{3}{8}$, $\frac{5}{16}$.	(*c*)	$1\frac{3}{16}$.
(*d*)	$\frac{5}{6}$, $\frac{11}{72}$, $\frac{13}{18}$.	(*d*)	$1\frac{51}{72}$.
(*e*)	$\frac{10}{11}$, $\frac{6}{33}$, $\frac{23}{66}$.	(*e*)	$1\frac{29}{66}$.
(*f*)	$\frac{23}{45}$, $\frac{11}{15}$, $\frac{14}{45}$.	(*f*)	$1\frac{5}{9}$.
(*g*)	$\frac{4}{11}$, $\frac{7}{22}$, $\frac{14}{22}$.	(*g*)	$1\frac{7}{22}$.
(*h*)	$\frac{3}{7}$, $\frac{14}{49}$, $\frac{2}{7}$.	(*h*)	1.

SUBTRACTION OF FRACTIONS.

106. Fractions cannot be *subtracted* without first reducing them to a *common denominator*. This can be shown in the same manner as in the case of addition of fractions.

Example.—Subtract $\frac{3}{8}$ from $\frac{13}{16}$.

Solution.—The common denominator is 16.

$$\tfrac{3}{8} = \tfrac{6}{16}. \quad \tfrac{13}{16} - \tfrac{6}{16} = \frac{13 - 6}{16} = \tfrac{7}{16}. \quad \text{Ans.}$$

107. Example.—From 7 take $\frac{5}{8}$.

Solution.— $1 = \frac{8}{8}$; therefore, since $7 = 6 + 1$, $7 = 6 + \frac{8}{8} = 6\frac{8}{8}$, or $6\frac{8}{8} - \frac{5}{8} = 6\frac{3}{8}$. Ans.

108. Example.—What is the difference between $17\frac{9}{16}$ and $9\frac{15}{32}$?

Solution.—The common denominator of the fractions is 32. $17\frac{9}{16} = 17\frac{18}{32}$.

$$\begin{array}{rr} \textit{minuend} & 17\frac{18}{32} \\ \textit{subtrahend} & 9\frac{15}{32} \\ \hline \textit{difference} & 8\frac{3}{32} \end{array} \text{ Ans.}$$

109. Example.—From $9\frac{1}{4}$ take $4\frac{7}{16}$.

Solution.—The common denominator of the fractions is 16. $9\frac{1}{4} = 9\frac{4}{16}$.

$$\begin{array}{rrr} \textit{minuend} & 9\frac{4}{16} & \text{or } 8\frac{20}{16} \\ \textit{subtrahend} & 4\frac{7}{16} & 4\frac{7}{16} \\ \hline \textit{difference} & 4\frac{13}{16} & 4\frac{13}{16} \end{array} \text{ Ans.}$$

Explanation.—As the *fraction* in the *subtrahend* is greater than the fraction in the *minuend*, it *cannot* be subtracted; therefore, *borrow* 1, or $\frac{16}{16}$, from the 9 in the *minuend* and *add* it to the $\frac{4}{16}$; $\frac{4}{16} + \frac{16}{16} = \frac{20}{16}$. $\frac{7}{16}$ from $\frac{20}{16} = \frac{13}{16}$. Since 1 was *borrowed* from 9, 8 *remains*; 4 from $8 = 4$; $4 + \frac{13}{16} = 4\frac{13}{16}$.

110. Example.—From 9 take $8\frac{3}{16}$.

Solution.—

$$\begin{array}{rrr} \textit{minuend} & 9 & \text{or } 8\frac{16}{16} \\ \textit{subtrahend} & 8\frac{3}{16} & 8\frac{3}{16} \\ \hline \textit{difference} & \frac{13}{16} & \frac{13}{16} \end{array} \text{ Ans.}$$

Explanation.—As there is no *fraction* in the *minuend* from which to take the *fraction* in the *subtrahend*, *borrow* 1, or $\frac{16}{16}$, from 9. $\frac{3}{16}$ from $\frac{16}{16} = \frac{13}{16}$. Since 1 was *borrowed* from 9, only 8 is left. 8 from $8 = 0$.

111. Rule I.—*Reduce the fractions to fractions having a common denominator. Subtract one numerator from the other and place the remainder over the common denominator.*

II. *When there are mixed numbers, subtract the fractions and whole numbers separately, and place the remainders side by side.*

III. *When the fraction in the subtrahend is greater than the fraction in the minuend, borrow 1 from the whole number in the minuend and add it to the fraction in the minuend, from which subtract the fraction in the subtrahend.*

IV. *When the minuend is a whole number, borrow 1; reduce it to a fraction whose denominator is the same as the denominator of the fraction in the subtrahend, and place it over that fraction for subtraction.*

EXAMPLES FOR PRACTICE.

112. Subtract

	Ans.
(*a*) $\frac{10}{24}$ from $\frac{11}{12}$.	(*a*) $\frac{1}{2}$.
(*b*) $\frac{7}{14}$ from $\frac{17}{28}$.	(*b*) $\frac{3}{28}$.
(*c*) $\frac{4}{30}$ from $\frac{5}{10}$.	(*c*) $\frac{11}{30}$.
(*d*) $\frac{15}{35}$ from $\frac{45}{70}$.	(*d*) $\frac{3}{14}$.
(*e*) $\frac{15}{16}$ from $\frac{57}{48}$.	(*e*) $\frac{1}{4}$.
(*f*) $13\frac{1}{4}$ from $30\frac{1}{2}$.	(*f*) $17\frac{1}{4}$.
(*g*) $12\frac{1}{8}$ from 27.	(*g*) $14\frac{7}{8}$.
(*h*) $5\frac{1}{4}$ from 30.	(*h*) $24\frac{3}{4}$.

MULTIPLICATION OF FRACTIONS.

113. In *multiplication* of fractions it is not necessary to *reduce* the *fractions* to fractions having a *common denominator.*

114. Multiplying the *numerator* or *dividing* the *denominator multiplies* the fraction.

EXAMPLE.—Multiply $\frac{3}{4}$ by 4.

SOLUTION.— $\frac{3}{4}\times 4=\frac{3\times 4}{4}=\frac{12}{4}=3$. Ans.

Or $\frac{3}{4}\times 4=\frac{3}{4\div 4}=\frac{3}{1}=3$. Ans.

The word "of" in multiplication of fractions means the same as ×, or times. Thus,

$$\frac{3}{4}\text{ of }4=\frac{3}{4}\times 4=3.$$
$$\frac{1}{8}\text{ of }\frac{5}{16}=\frac{1}{8}\times\frac{5}{16}=\frac{5}{128}.$$

EXAMPLE.—Multiply $\frac{3}{8}$ by 2.

SOLUTION.— $2\times\frac{3}{8}=\frac{3\times 2}{8}=\frac{6}{8}=\frac{3}{4}$. Ans.

Or $2\times\frac{3}{8}=\frac{3}{8\div 2}=\frac{3}{4}$. Ans.

115. EXAMPLE.—What is the product of $\frac{4}{16}$ and $\frac{7}{8}$?

SOLUTION.— $\frac{4}{16}\times\frac{7}{8}=\frac{4\times 7}{16\times 8}=\frac{28}{128}=\frac{7}{32}$. Ans.

or, by cancelation, $\frac{\cancel{4}\times 7}{\underset{4}{\cancel{16}}\times 8}=\frac{7}{4\times 8}=\frac{7}{32}$. Ans.

116. EXAMPLE.—What is $\frac{4}{8}$ of $\frac{3}{4}$ of $\frac{16}{32}$?

SOLUTION.— $\frac{\cancel{4}\times 3\times\cancel{16}}{8\times\cancel{4}\times\underset{2}{\cancel{32}}}=\frac{3}{8\times 2}=\frac{3}{16}$. Ans.

117. Example.—What is the product of $9\frac{3}{4}$ and $5\frac{5}{8}$?

Solution.—$9\frac{3}{4} = \frac{39}{4}$; $5\frac{5}{8} = \frac{45}{8}$.

$$\frac{39}{4} \times \frac{45}{8} = \frac{39 \times 45}{4 \times 8} = \frac{1755}{32} = 54\frac{27}{32}. \quad \text{Ans}$$

118. Example.—Multiply $15\frac{7}{8}$ by 3.

Solution.—

$$\begin{array}{r} 15\frac{7}{8} \\ 3 \\ \hline 47\frac{5}{8} \end{array} \quad \text{or} \quad \begin{array}{r} 15\frac{7}{8} \\ 3 \\ \hline 45 + \frac{21}{8} = 45 + 2\frac{5}{8} = 47\frac{5}{8}. \quad \text{Ans.} \end{array}$$

119. Rule.—I. *Divide the product of the numerators by the product of the denominators. All factors common to the numerators and denominators should first be cast out by cancelation.*

II. *To multiply one mixed number by another, reduce them both to improper fractions.*

III. *To multiply a mixed number by a whole number, first multiply the fractional part by the multiplier, and if the product is an improper fraction, reduce it to a mixed number, and add the whole number part to the product of the multiplier and whole number.*

EXAMPLES FOR PRACTICE.

120. Find the product of

	Ans.
(*a*) $7 \times \frac{3}{19}$.	(*a*) $1\frac{2}{19}$.
(*b*) $14 \times \frac{5}{16}$.	(*b*) $4\frac{3}{8}$.
(*c*) $\frac{21}{32} \times \frac{5}{14}$.	(*c*) $\frac{15}{64}$.
(*d*) $\frac{16}{27} \times 4$.	(*d*) $2\frac{10}{27}$.
(*e*) $\frac{19}{18} \times 7$.	(*e*) $7\frac{7}{18}$.
(*f*) $17\frac{18}{21} \times 7$.	(*f*) 125.
(*g*) $\frac{105}{224} \times 32$.	(*g*) 15.
(*h*) $\frac{15}{28} \times 14$.	(*h*) $7\frac{1}{2}$.

DIVISION OF FRACTIONS.

121. In *division* of fractions it is not necessary to *reduce* the *fractions* to fractions having a *common denominator.*

122. Dividing the *numerator* or *multiplying* the *denominator*, *divides* the fraction.

Example.—Divide $\frac{6}{8}$ by 3.

Solution.—When *dividing* the *numerator*, we have

$$\frac{6}{8} \div 3 = \frac{6 \div 3}{8} = \frac{2}{8} = \frac{1}{4}. \quad \text{Ans.}$$

When *multiplying* the *denominator*, we have

$$\tfrac{6}{8} \div 3 = \frac{6}{8 \times 3} = \tfrac{6}{24} = \tfrac{1}{4}.\quad \text{Ans.}$$

EXAMPLE.—Divide $\frac{3}{16}$ by 2.

SOLUTION.— $$\tfrac{3}{16} \div 2 = \frac{3}{16 \times 2} = \tfrac{3}{32}.\quad \text{Ans.}$$

EXAMPLE.—Divide $\frac{14}{32}$ by 7.

SOLUTION.— $$\tfrac{14}{32} \div 7 = \frac{14 \div 7}{32} = \tfrac{2}{32} = \tfrac{1}{16}.\quad \text{Ans.}$$

123. To **invert** a fraction is to *turn* it *upside down*; that is, make the *numerator* and *denominator change places*. Invert $\frac{3}{4}$ and it becomes $\frac{4}{3}$.

124. EXAMPLE.—Divide $\frac{9}{16}$ by $\frac{3}{16}$.

SOLUTION.—1. The fraction $\frac{3}{16}$ is contained in $\frac{9}{16}$, 3 times, for the *denominators* are the same, and one *numerator* is *contained* in the other 3 times. 2. If we now *invert* the *divisor* $\frac{3}{16}$, and *multiply*, the solution is

$$\frac{9}{16} \times \frac{16}{3} = \frac{\overset{3}{\cancel{9}} \times \cancel{16}}{\cancel{16} \times \cancel{3}} = 3.\quad \text{Ans.}$$

This brings the *same quotient* as in the first case.

125. EXAMPLE.—Divide $\frac{3}{8}$ by $\frac{1}{4}$.

SOLUTION.—We cannot divide $\frac{3}{8}$ by $\frac{1}{4}$, as in the first case above, for the *denominators* are *not* the same, therefore, we must solve as in the second case.

$$\tfrac{3}{8} \div \tfrac{1}{4} = \tfrac{3}{8} \times \tfrac{4}{1} = \frac{3 \times \cancel{4}}{\underset{2}{\cancel{8}} \times 1} = \frac{3}{2} \text{ or } 1\tfrac{1}{2}.\quad \text{Ans.}$$

126. EXAMPLE.—Divide 5 by $\frac{10}{16}$.

SOLUTION.— $\frac{10}{16}$ inverted becomes $\frac{16}{10}$.

$$5 \times \frac{16}{10} = \frac{\cancel{5} \times \overset{8}{\cancel{16}}}{\underset{2}{\cancel{10}}} = 8.\quad \text{Ans.}$$

127. EXAMPLE.—How many times is $3\frac{3}{4}$ contained in $7\frac{7}{16}$?

SOLUTION.— $3\frac{3}{4} = \frac{15}{4}$; $7\frac{7}{16} = \frac{119}{16}$.

$\frac{15}{4}$ inverted equals $\frac{4}{15}$.

$$\frac{119}{16} \times \frac{4}{15} = \frac{119 \times \cancel{4}}{\underset{4}{\cancel{16}} \times 15} = \frac{119}{60} = 1\tfrac{59}{60}.\quad \text{Ans.}$$

128. Rule.—*Invert the divisor, and proceed as in multiplication.*

129. We have learned that a line placed between two numbers indicates that the number above the line is to be divided by the number below it. Thus, $\frac{18}{3}$ shows that 18 is to be divided by 3. This is also true if a fraction or a fractional expression be placed above or below a line.

$\cfrac{9}{\frac{3}{8}}$ means that 9 is to be divided by $\frac{3}{8}$; $\cfrac{3 \times 7}{\cfrac{8 + 4}{16}}$ means that 3×7 is to be divided by the value of $\frac{8 + 4}{16}$.

$\cfrac{\frac{1}{4}}{\frac{3}{8}}$ is the same as $\frac{1}{4} \div \frac{3}{8}$.

It will be noticed that there is a heavy line between the 9 and the $\frac{3}{8}$. This is necessary, since otherwise there would be nothing to show as to whether 9 was to be divided by $\frac{3}{8}$, or $\frac{9}{3}$ was to be divided by 8. Whenever a heavy line is used, as shown here, it indicates that *all above the line* is to be divided by *all below it*.

EXAMPLES FOR PRACTICE.

130. Divide

(*a*)	15 by $6\frac{3}{7}$.	Ans.	(*a*)	$2\frac{1}{3}$.
(*b*)	30 by $\frac{6}{8}$.		(*b*)	40.
(*c*)	172 by $\frac{4}{5}$.		(*c*)	215.
(*d*)	$\frac{14}{18}$ by $1\frac{7}{16}$.		(*d*)	$\frac{112}{207}$.
(*e*)	$\frac{103}{6}$ by $14\frac{2}{3}$.		(*e*)	$1\frac{15}{88}$.
(*f*)	$\frac{142}{27}$ by $17\frac{1}{9}$.		(*f*)	$\frac{71}{231}$.
(*g*)	$\frac{14}{18}$ by $\frac{145}{72}$.		(*g*)	$\frac{56}{145}$.
(*h*)	$\frac{128}{18}$ by $72\frac{1}{3}$.		(*h*)	$\frac{64}{651}$.

131. Whenever an expression like one of the three following is obtained, it may always be simplified by transposing the denominator from *above* to *below* the line, or from *below* to *above*, as the case may be, taking care however to indicate that the denominator when so transferred is a multiplier.

1. $\cfrac{\frac{3}{4}}{9} = \frac{3}{9 \times 4} = \frac{3}{36} = \frac{1}{12}$; for, regarding the fraction above the heavy line as the numerator of a fraction whose denominator is 9, $\cfrac{\frac{3}{4} \times 4}{9 \times 4} = \frac{3}{9 \times 4}$, as before.

2. $\frac{9}{\frac{3}{4}} = \frac{9 \times 4}{3} = 12.$ The proof is the same as in the first case.

3. $\frac{\frac{5}{9}}{\frac{3}{4}} = \frac{5 \times 4}{3 \times 9} = \frac{20}{27}$; for, regarding $\frac{5}{9}$ as the numerator of a fraction whose denominator is $\frac{3}{4}$, $\dfrac{\frac{5}{9} \times 9}{\frac{3}{4} \times 9} = \dfrac{5}{\frac{3 \times 9}{4}}$; and $\dfrac{5 \times 4}{\frac{3 \times 9}{4} \times 4} = \frac{5 \times 4}{3 \times 9} = \frac{20}{27}$, as above.

This principle may be used to great advantage in cases like $\dfrac{\frac{1}{4} \times 310 \times \frac{27}{12} \times 72}{40 \times 4\frac{1}{2} \times 5\frac{1}{6}}$. Reducing the mixed numbers to fractions, the expression becomes $\dfrac{\frac{1}{4} \times 310 \times \frac{27}{12} \times 72}{40 \times \frac{9}{2} \times \frac{31}{6}}$. Now transferring the denominators of the fractions and canceling,

$$\frac{1 \times 310 \times 27 \times 72 \times 2 \times 6}{40 \times 9 \times 31 \times 4 \times 12} = \frac{1 \times \overset{\cancel{10}}{\cancel{310}} \times \overset{3}{\cancel{27}} \times \overset{\overset{3}{\cancel{6}}}{\cancel{72}} \times \cancel{2} \times \overset{3}{\cancel{6}}}{\underset{\underset{\cancel{2}}{\cancel{4}}}{\cancel{40}} \times \cancel{9} \times \cancel{31} \times \underset{2}{\cancel{4}} \times \cancel{12}} = \frac{27}{2} = 13\tfrac{1}{2}.$$

Greater exactness in results can usually be obtained by using this principle than can be obtained by reducing the fractions to decimals. The principle, however, should not be employed *if a sign of addition or subtraction occurs either above or below the dividing line.*

DECIMALS.

132. Decimals are *tenth* fractions; that is, the parts of a unit are expressed on the scale of ten, as *tenths*, *hundredths*, *thousandths*, etc.

133. The *denominator*, which is always ten or a multiple of ten, as 10, 100, 1,000, etc., is *not* expressed as it would be in common fractions, by writing it under the *numerator*, with a line between them; as, $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$. The denominator is always understood, the numerator consisting of the figures on the right of the *unit* figure. In order to distinguish the unit figure, a period (.), called the **decimal point**, is placed between the unit figure and the next figure on the right. The decimal point may be regarded in two ways: first, as indicating that the number on the right is the numerator of a fraction whose denominator is 10, 100, 1,000, etc.; and, second, as a part of the Arabic system of notation, each figure on the right being 10 times as large as the next succeeding figure, and 10 times as small as the next preceding figure, serving merely to point out the unit figure.

134. The *reading* of a *decimal number* depends upon the *number* of *decimal places* in it, or the *number* of *figures* to the *right* of the unit figure.

The first figure to the right of the unit figure expresses *tenths*.

The second figure to the right of the unit figure expresses *hundredths*.

The third figure to the right of the unit figure expresses *thousandths*.

The fourth figure to the right of the unit figure expresses *ten-thousandths*.

The fifth figure to the right of the unit figure expresses *hundred-thousandths*.

The sixth figure to the right of the unit figure expresses *millionths*.

Thus:

$.3 = \frac{3}{10}$ = 3 tenths.
$.03 = \frac{3}{100}$ = 3 hundredths.
$.003 = \frac{3}{1000}$ = 3 thousandths.
$.0003 = \frac{3}{10000}$ = 3 ten-thousandths.
$.00003 = \frac{3}{100000}$ = 3 hundred-thousandths.
$.000003 = \frac{3}{1000000}$ = 3 millionths.

The first figure to the right of the unit figure is called the *first* **decimal place**; the second figure, the *second decimal place*, etc. We see in the above that the *number* of *decimal places* in a decimal equals the *number of ciphers* to the *right* of the figure 1 in the *denominator* of its *equivalent fraction*. This fact kept in mind will be of much assistance in reading and writing decimals.

Whatever may be written to the *left* of a *decimal point* is a whole number. The decimal point affects only the figures to its *right*.

When a *whole number* and *decimal* are written together, the expression is a *mixed number*. Thus, 8.12 and 17.25 are mixed numbers.

The relation of decimals and whole numbers to each other is clearly shown by the following table :

hundreds of millions.	tens of millions.	millions.	hundreds of thousands.	tens of thousands.	thousands.	hundreds.	tens.	units.	decimal point.	tenths.	hundredths.	thousandths.	ten-thousandths.	hundred-thousandths.	millionths.	ten-millionths.	hundred-millionths.
9	8	7	6	5	4	3	2	1	.	2	3	4	5	6	7	8	9

The figures to the *left* of the *decimal point* represent *whole numbers;* those to the *right* are *decimals.*

In *both* the decimals and whole numbers, the *units* place is made the *starting point* of notation and numeration. The *decimals decrease* on the scale of *ten* to the *right*, and the *whole* numbers *increase* on the scale of *ten* to the *left*. The *first* figure to the *left* of units is *tens*, and the *first* figure to the *right* of units is *tenths.* The *second* figure to the *left* of units is *hundreds*, and the *second* figure to the *right* is *hundredths.* The *third* figure to the *left* is *thousands*, and the *third* to the *right* is *thousandths*, and so on ; the *whole* numbers on the *left* and the *decimals* on the *right.* The figures equally distant from units place correspond in name. The *decimals* have the ending *ths*, which distinguishes them from *whole* numbers. The following is the numeration of the number in the above table : Nine hundred eighty-seven million, six hundred fifty-four thousand, three hundred twenty-one, and twenty-three million, four hundred fifty-six thousand, seven hundred eighty-nine hundred millionths.

The *decimals* increase to the *left*, on the scale of *ten*, the same as *whole* numbers; for, beginning at, say, 4-*thousandths*, in the table, the next figure to the left is *hundredths*, which is ten times as great, and the next *tenths*, or ten times the *hundredths*, and so on through both decimals and whole numbers.

135. *Annexing* or *taking away* a *cipher* at the *right* of a *decimal* does *not* affect its value.

$.5$ is $\frac{5}{10}$; $.50$ is $\frac{50}{100}$, but $\frac{5}{10} = \frac{50}{100}$; therefore, $.5 = .50$.

136. *Inserting* a *cipher* between a *decimal* and the *decimal point divides* the decimal by 10.

$$.5 = \frac{5}{10};\ \frac{5}{10} \div 10 = \frac{5}{100} = .05.$$

137. *Taking away* a *cipher* from the *left* of a *decimal multiplies* the *decimal* by 10.

$$.05 = \frac{5}{100};\ \frac{5}{100} \times 10 = \frac{5}{10} = .5.$$

138. In some cases it is convenient to express a mixed decimal fraction in the form of a common (improper) fraction. To do so it is only necessary to write the entire number, omitting the decimal point, as the numerator of the fraction, and the denominator of the decimal part as the denominator of the fraction. Thus, $127.483 = \frac{127483}{1000}$; for, $127.483 = 127\frac{483}{1000} = \frac{127000+483}{1000} = \frac{127483}{1000}$.

ADDITION OF DECIMALS.

139. The only respect in which *addition* of *decimals* differs from *addition* of *whole numbers*, is that while the unit figures are placed under each other in both cases, the right-hand figures are not necessarily in line when adding decimals.

Whole numbers begin at units and increase on the scale of 10, to the left. Decimals decrease on the scale of 10, to the right. Whole numbers are to the left of the decimal point and decimals are to the right of it. In whole numbers the *right-hand* side of a column of figures to be added, must be in line, and in decimals, the *left-hand* side must be in line, which brings the decimal points directly under each other.

whole numbers	*decimals*	*mixed numbers*
342	.342	342.032
4234	.4234	4234.5
26	.26	26.6782
3	.03	3.06
sum 4605 Ans.	*sum* 1.0554 Ans.	*sum* 4606.2702 Ans.

140. A decimal, as .342, ought really to be expressed as 0.342, but it is quite customary to omit the cipher on the left of the decimal point, though many authors use it.

EXAMPLE.—What is the sum of 242, .36, 118.725, 1.005, 6, and 100.1?

SOLUTION.—
242.
.36
118.725
1.005
6.
100.1
sum 468.190 Ans.

141. Rule.—*Place the numbers to be added so that the decimal points will be directly under each other. Add as in whole numbers, and place the decimal point in the sum directly under the decimal points above.*

EXAMPLES FOR PRACTICE.

142. Find the sum of

		Ans.
(*a*)	.2143, .105, 2.3042, and 1.1417.	(*a*) 3.7652.
(*b*)	783.5, 21.473, .2101, and .7816.	(*b*) 805.9647.
(*c*)	21.781, 138.72, 41.8738, .72, and 1.413.	(*c*) 204.5078.
(*d*)	.3724, 104.15, 21.417, and 100.042.	(*d*) 225.9814.
(*e*)	200.172, 14.105, 12.1465, .705, and 7.2.	(*e*) 234.3285.
(*f*)	1,427.16, .244, .32, .032, and 10.0041.	(*f*) 1,437.7601.
(*g*)	2,473.1, 41.65, .7243, 104.067, and 21.073.	(*g*) 2,640.6143.
(*h*)	4,107.2, .00375, 21.716, 410.072, and .0345.	(*h*) 4,539.02625.

SUBTRACTION OF DECIMALS.

143. For the same reason as in addition of decimals, the *left-hand* figures of *decimal numbers* are placed in line and the *decimal points* under each other.

EXAMPLE.—Subtract .132 from .3063.

SOLUTION.—
minuend .3063
subtrahend .132
difference .1743 Ans.

144. EXAMPLE.—What is the difference between 7.895 and .725?

SOLUTION.—
minuend 7.895
subtrahend .725
difference 7.170 or 7.17 Ans.

145. EXAMPLE.—Subtract .625 from 11.

SOLUTION.—
minuend 11.000
subtrahend .625
difference 10.375 Ans.

146. Rule.—*Place the subtrahend under the minuend, so that the decimal points will be directly under each other. Subtract, as in whole numbers, and place the decimal point in the remainder, directly under the decimal points above.*

When the figures in the decimal part of the subtrahend extend beyond those in the minuend, place ciphers in the minuend above them, and subtract as before.

EXAMPLES FOR PRACTICE.

147. From

		Ans.
(*a*)	407.385 take 235.0004.	(*a*) 172.3846.
(*b*)	22.718 take 1.7042.	(*b*) 21.0138.
(*c*)	1,368.17 take 13.6817.	(*c*) 1,354.4883.
(*d*)	70.00017 take 7.000017.	(*d*) 63.000153.
(*e*)	630.630 take .6304.	(*e*) 629.9996.
(*f*)	421.73 take 217.162.	(*f*) 204.568.
(*g*)	1.000014 take .00001.	(*g*) 1.000004.
(*h*)	.783652 take .542314.	(*h*) .241338.

MULTIPLICATION OF DECIMALS.

148. In multiplication of decimals, we do not place the decimal points directly under each other as in addition and subtraction. We pay no attention for the time being to the decimal points. Place the multiplier under the multiplicand, so that the *right-hand* figure of the one is under the *right-hand* figure of the other, and proceed exactly as in multiplication of whole numbers. After multiplying, *count the number of decimal places in both multiplicand and multiplier, and point off the same number in the product.*

Example.—Multiply .825 by 13.

Solution.—	*multiplicand*	.825	
	multiplier	13	
		2475	
		825	
	product	10.725	Ans.

In this example there are three decimal places in the multiplicand and none in the multiplier; therefore, 3 decimal places are pointed off in the product.

149. Example.—What is the product of 426 and the decimal .005 ?

Solution.—	*multiplicand*	426	
	multiplier	.005	
	product	2.130	or 2.13 Ans.

In this example there are 3 decimal places in the multiplier and none in the multiplicand; therefore, 3 decimal places are pointed off in the product.

150. It is *not* necessary to multiply by the ciphers on the *left* of a *decimal;* they merely determine the number of decimal places. Ciphers to the *right* of a decimal should be removed, as they only make more figures to deal with, and do not change the value.

151. Example.—Multiply 1.205 by 1.15.

Solution.—	*multiplicand*	1.205	
	multiplier	1.15	
		6025	
		1205	
		1205	
	product	1.38575	Ans.

In this example there are 3 decimal places in the multiplicand, and 2 in the multiplier; therefore, 3 + 2, or 5, decimal places must be pointed off in the product.

152. EXAMPLE.—Multiply .232 by .001.

SOLUTION.— *multiplicand* .232
multiplier .001

product .000232 Ans.

In this example we multiply the multiplicand by the digit in the multiplier, which makes 232 in the product, but since there are 3 decimal places in each, the multiplier and the multiplicand, we must prefix 3 ciphers to the 232, to make 3 + 3, or 6, decimal places in the product.

153. Rule.—*Place the multiplier under the multiplicand, disregarding the position of the decimal points. Multiply as in whole numbers, and in the product point off as many decimal places as there are decimal places in both multiplier and multiplicand, prefixing ciphers if necessary.*

EXAMPLES FOR PRACTICE.

154. Find the product of

		Ans.
(*a*)	.000492 × 4.1418.	(*a*) .0020377656.
(*b*)	4,003.2 × 1.2.	(*b*) 4,803.84.
(*c*)	78.6531 × 1.03.	(*c*) 81.012693.
(*d*)	.3685 × .042.	(*d*) .015477.
(*e*)	178,352 × .01.	(*e*) 1,783.52.
(*f*)	.00045 × .0045.	(*f*) .000002025.
(*g*)	.714 × .00002.	(*g*) .00001428.
(*h*)	.00004 × .008.	(*h*) .00000032.

DIVISION OF DECIMALS.

155. In division of decimals we pay *no* attention to the decimal point until *after* the division is performed. The *number of decimal places* in the *dividend must equal* (*be made to equal by annexing ciphers*) *the number of decimal places in the divisor.* Divide exactly as in whole numbers. *Subtract the number of decimal places in the divisor from the number of decimal places in the dividend, and point off as many decimal* *places in the quotient as there are units in the remainder thus found.*

Example.—Divide .625 by 25.

divisor dividend quotient
Solution.— 2 5) .6 2 5 (.0 2 5 Ans.
5 0
1 2 5
1 2 5
remainder 0

In this example there are no decimal places in the divisor, and 3 decimal places in the dividend; therefore, there are 3 minus 0, or 3, decimal places in the quotient. One cipher has to be prefixed to the 25, to make the 3 decimal places.

156. Example.—Divide 6.035 by .05.

divisor dividend quotient
Solution.— .0 5) 6.0 3 5 (1 2 0.7 Ans.
5
1 0
1 0
3 5
3 5
remainder 0

In this example we divide by 5, as if the cipher were not before it. There is one more decimal place in the dividend than in the divisor; therefore, one decimal place is pointed off in the quotient.

157. Example.—Divide .125 by .005.

divisor dividend quotient
Solution.— .0 0 5) .1 2 5 (2 5 Ans.
1 0
2 5
2 5
remainder 0

In this example there are the same number of decimal places in the dividend as in the divisor; therefore, the quotient has no decimal places, and is a whole number.

158. Example.—Divide 326 by .25.

divisor *dividend* *quotient*

Solution.— .25) 326.00 (1304 Ans.

25

76

75

100

100

remainder 0

In this problem two ciphers were annexed to the dividend, to make the number of decimal places equal to the number in the divisor. The quotient is a whole number.

159. Example.—Divide .0025 by 1.25.

Solution.— 1.25) .00250 (.002 Ans.

250

remainder 0

Explanation.—In this example we are to divide .0025 by 1.25. Consider the dividend as a whole number, or 25 (disregarding the two ciphers at its left, for the present); also, consider the divisor as a whole number, or 125. It is clearly evident that the dividend 25 will not contain the divisor 125; we must, therefore, annex one cipher to the 25, thus making the dividend 250. 125 is contained twice in 250, so we place the figure 2 in the quotient. In pointing off the decimal places in the quotient, it must be remembered that there were only four decimal places in the dividend; but one cipher was annexed, thereby making 4 + 1, or 5, decimal places. Since there are 5 decimal places in the dividend and 2 decimal places in the divisor, we must point off 5 − 2, or 3, decimal places in the quotient. In order to point off 3 decimal places, two ciphers must be prefixed to the figure 2, thereby making .002 the quotient. It is not necessary to consider the ciphers at the left of a decimal when dividing, except when determining the position of the decimal point in the quotient.

160. **Rule.—I.** *Place the divisor to the left of the dividend, and proceed as in division of whole numbers; in the* *quotient, point off as many decimal places as the number of decimal places in the dividend exceed those in the divisor, prefixing ciphers to the quotient, if necessary.*

II. *If in dividing one number by another there be a remainder, the remainder can be placed over the divisor, as a fractional part of the quotient, but it is generally better to annex ciphers to the remainder, and continue dividing until there are 3 or 4 decimal places in the quotient, and then if there still be a remainder, terminate the quotient by the plus sign* (+), *which shows that it can be carried further.*

161. Example.—What is the quotient of 199 divided by 15?

Solution.—

$$\begin{array}{rl}
15)199(13+\frac{4}{15} & \text{Ans.}\\
15 & \\
\hline
49 & \\
45 & \\
\hline
\textit{remainder}\quad 4 &
\end{array}$$

$$\begin{array}{rl}
\text{Or,}\quad 15)199.000(13.266+ & \text{Ans.}\\
15 & \\
\hline
49 & \\
45 & \\
\hline
40 & \\
30 & \\
\hline
100 & \\
90 & \\
\hline
100 & \\
90 & \\
\hline
\textit{remainder}\quad 10 &
\end{array}$$

$$13\tfrac{4}{15} = 13.266+$$

$$\tfrac{4}{15} = .266+$$

162. It frequently happens, as in the above example, that the division will never terminate. In such cases, decide to how many decimal places the division is to be carried, and carry the work one place further. If the last figure of the quotient thus obtained is 5 or a greater number, increase the preceding figure by 1, and write after it the minus sign (−), thus indicating that the quotient is not quite as large as indicated; if the figure thus obtained is less than 5, write the plus sign (+) after the quotient, thus indicating that the number is slightly greater than as indicated. In the last example, had it been desired to obtain the answer correct to four decimal places, the work would have been carried to five places, obtaining 13.26666, and the answer would have been given as 13.2667 −. This remark applies to any other calculation involving decimals, when it is desired to omit some of the figures in the decimal. Thus, if it is desired to retain three decimal places in the number .2471253, it would be expressed as .247 +; if it was desired to retain five decimal places, it would be expressed as .24713 −. Both the + and − signs are frequently omitted; they are seldom used outside of Arithmetic, except in exact calculations, when it is desired to call particular attention to the fact that the result obtained is not *quite* exact.

EXAMPLES FOR PRACTICE.

163. Divide

(*a*)	101.6688 by 2.36.	(*a*)	43.08.
(*b*)	187.12264 by 123.107.	(*b*)	1.52.
(*c*)	.08 by .008.	(*c*)	10.
(*d*)	.0003 by 3.75.	(*d*)	.00008.
(*e*)	.0144 by .024.	(*e*)	.6.
(*f*)	.00375 by 1.25.	(*f*)	.003.
(*g*)	.004 by 400.	(*g*)	.00001.
(*h*)	.4 by .008.	(*h*)	50.

Ans.

TO REDUCE A FRACTION TO A DECIMAL.

164. Example.—$\frac{3}{4}$ equals what decimal?

Solution.— $4\,)\,\underline{3.00}$
$.75$, or $\frac{3}{4} = .75$. Ans.

Example.—What decimal is equivalent to $\frac{7}{8}$?

Solution.—

```
8 ) 7.000 ( .875
    6 4
    ---
      6 0
      5 6     or 7/8 = .875.  Ans.
      ---
        4 0
        4 0
        ---
          0
```

165. Rule.—*Annex ciphers to the numerator and divide by the denominator. Point off as many decimal places in the quotient as there are ciphers annexed.*

EXAMPLES FOR PRACTICE.

166. Reduce the following common fractions to decimals:

(*a*)	$\frac{15}{32}$.	Ans.	(*a*)	.46875.
(*b*)	$\frac{7}{8}$.		(*b*)	.875.
(*c*)	$\frac{21}{32}$.		(*c*)	.65625.
(*d*)	$\frac{51}{64}$.		(*d*)	.796875.
(*e*)	$\frac{4}{25}$.		(*e*)	.16.
(*f*)	$\frac{5}{8}$.		(*f*)	.625.
(*g*)	$\frac{10}{200}$.		(*g*)	.05.
(*h*)	$\frac{4}{1000}$.		(*h*)	.004.

167. To reduce inches to decimal parts of a foot:

EXAMPLE.—What decimal part of a foot is 9 inches ?

SOLUTION.—Since there are 12 inches in one foot, 1 inch is $\frac{1}{12}$ of a foot, and 9 inches is $9 \times \frac{1}{12}$ or $\frac{9}{12}$ of a foot. This, reduced to a decimal by the above rule, shows what decimal part of a foot 9 inches is.

12) 9.00 (.75 of a foot. Ans.
84
60
60
0

168. Rule I.—*To reduce inches to decimal parts of a foot, divide the number of inches by 12.*

II. *Should the resulting decimal be an unending one and it is desired to terminate the division at some point, say, the fourth decimal place, carry the division one place further, and if the fifth figure is 5 or greater, increase the fourth figure by 1. Omit the signs + and —.*

EXAMPLES FOR PRACTICE.

169. Reduce to the decimal part of a foot:

(*a*)	3 in.		(*a*)	.25.
(*b*)	$4\frac{1}{2}$ in.		(*b*)	.375.
(*c*)	5 in.	Ans.	(*c*)	.4167.
(*d*)	$6\frac{5}{8}$ in.		(*d*)	.5521.
(*e*)	11 in.		(*e*)	.9167.

TO REDUCE A DECIMAL TO A FRACTION.

170. Example.—Reduce .125 to a fraction.

Solution.— $.125 = \frac{125}{1000} = \frac{5}{40} = \frac{1}{8}$. Ans.

Example.—Reduce .875 to a fraction.

Solution.— $.875 = \frac{875}{1000} = \frac{35}{40} = \frac{7}{8}$. Ans.

171. Rule.—*Under the figures of the decimal, place 1 with as many ciphers at its right as there are decimal places in the decimal, and reduce the resulting fraction to its lowest terms by dividing both numerator and denominator by the same number.*

EXAMPLES FOR PRACTICE.

172. Reduce the following to common fractions :

(*a*)	.125.	Ans.	(*a*)	$\frac{1}{8}$.
(*b*)	.625.		(*b*)	$\frac{5}{8}$.
(*c*)	.3125.		(*c*)	$\frac{5}{16}$.
(*d*)	.04.		(*d*)	$\frac{1}{25}$.
(*e*)	.06.		(*e*)	$\frac{3}{50}$.
(*f*)	.75.		(*f*)	$\frac{3}{4}$.
(*g*)	.15625.		(*g*)	$\frac{5}{32}$.
(*h*)	.875.		(*h*)	$\frac{7}{8}$.

173. To express a decimal approximately as a fraction having a given denominator :

174. Example.—Express .5827 in 64ths.

Solution.— $.5827 \times \frac{64}{64} = \frac{37.2928}{64}$, say $\frac{37}{64}$.

Hence, $.5827 = \frac{37}{64}$, nearly. Ans.

Example.—Express .3917 in 12ths.

Solution.— $.3917 \times \frac{12}{12} = \frac{4.7004}{12}$, say $\frac{5}{12}$.

Hence, $.3917 = \frac{5}{12}$, nearly. Ans.

175. Rule.—*Reduce 1 to a fraction having the given denominator. Multiply the given decimal by the fraction so obtained, and the result will be the fraction required*

EXAMPLES FOR PRACTICE.

176. Express

(*a*)	.625 in 8ths.	Ans.	(*a*)	$\frac{5}{8}$.
(*b*)	.3125 in 16ths.		(*b*)	$\frac{5}{16}$.
(*c*)	.15625 in 32ds.		(*c*)	$\frac{5}{32}$.
(*d*)	.77 in 64ths.		(*d*)	$\frac{49}{64}$.
(*e*)	.81 in 48ths.		(*e*)	$\frac{39}{48}$.
(*f*)	.923 in 96ths.		(*f*)	$\frac{89}{96}$.

177. The sign for dollars is $. It is read dollars. $25 is read 25 dollars.

Since there are 100 cents in a dollar, one cent is 1-one-hundredth of a dollar; the first two figures of a decimal part of a dollar represent *cents*. Since a mill is $\frac{1}{10}$ of a cent, or $\frac{1}{1000}$ of a dollar, the third figure represents mills.

Thus, $25.16 is read twenty-five dollars and sixteen cents; $25.168 is read twenty-five dollars, sixteen cents and eight mills.

178. The **vinculum** ——, **parenthesis** ( ), **bracket** [ ], and **brace** { } are called **symbols of aggregation,** and are used to include numbers which are to be considered together; thus, $13 \times \overline{8 - 3}$, or $13 \times (8 - 3)$, shows that 3 is to be taken from 8 before multiplying by 13.

$$13 \times (8 - 3) = 13 \times 5 = 65. \quad \text{Ans.}$$
$$13 \times \overline{8 - 3} = 13 \times 5 = 65. \quad \text{Ans.}$$

When the vinculum or parenthesis is not used, we have

$$13 \times 8 - 3 = 104 - 3 = 101. \quad \text{Ans.}$$

179. In any series of numbers connected by the signs +, −, ×, and ÷, the operations indicated by the signs must be performed in order from left to right, *except* that no addition or subtraction may be performed if a sign of multiplication or division *follows* the number on the *right* of a sign of addition or subtraction, until the indicated multiplication or division has been performed. In all cases the sign of multiplication takes the precedence, the reason being that when two or more numbers or expressions are connected by the sign of multiplication, the numbers thus connected are regarded as factors of the product indicated, and not as separate numbers.

EXAMPLE.—What is the value of $4 \times 24 - 8 + 17$?

SOLUTION.—Performing the operations in order from left to right, $4 \times 24 = 96$; $96 - 8 = 88$; $88 + 17 = 105$. Ans.

180. EXAMPLE.—What is the value of the following expression: $1{,}296 \div 12 + 160 - 22 \times 3\frac{1}{2} = ?$

SOLUTION.—$1{,}296 \div 12 = 108$; $108 + 160 = 268$; here we cannot subtract 22 from 268 because the sign of multiplication *follows* 22; hence, multiplying 22 by $3\frac{1}{2}$, we get 77, and $268 - 77 = 191$. Ans.

Had the above expression been written 1,296 ÷ 12 + 160 − 22 × 3½ ÷ 7 + 25, it would have been necessary to have divided 22 × 3½ by 7 before subtracting, and the final result would have been 22 × 3½ = 77; 77 ÷ 7 = 11; 268 − 11 = 257; 257 + 25 = 282. Ans. In other words, it is necessary to perform *all* of the multiplication or division included between the signs + and −, or − and +, before adding or subtracting. Also, had the expression been written 1,296 ÷ 12 + 160 − 24½ ÷ 7 × 3½ + 25, it would have been necessary to have multiplied 3½ by 7 before dividing 24½, since the sign of multiplication takes the precedence, and the final result would have been 3½ × 7 = 24½; 24½ ÷ 24½ = 1; 268 − 1 = 267; 267 + 25 = 292. Ans.

It likewise follows that if a succession of multiplication and division signs occurs, the indicated operations must not be performed in order, from left to right—the multiplication must be performed first. Thus, 24 × 3 ÷ 4 × 2 ÷ 9 × 5 = ⅕. Ans. In order to obtain the same result that would be obtained by performing the indicated operations in order, from left to right, symbols of aggregation must be used. Thus, by using two vinculums, the last expression becomes $24 \times \overline{3 \div 4} \times \overline{2 \div 9} \times 5 = 20$, the same result that would be obtained by performing the indicated operations in order, from left to right.

EXAMPLES FOR PRACTICE.

181. Find the values of the following expressions:

		Ans.
(*a*)	(8 + 5 − 1) ÷ 4.	(*a*) 3.
(*b*)	5 × 24 − 32.	(*b*) 88.
(*c*)	5 × 24 ÷ 15.	(*c*) 8.
(*d*)	144 − 5 × 24.	(*d*) 24.
(*e*)	(1,691 − 540 + 559) ÷ 3 × 57.	(*e*) 10.
(*f*)	2,080 + 120 − 80 × 4 − 1,670.	(*f*) 210.
(*g*)	$(\overline{90 + 60} \div 25) \times 5 - 29$.	(*g*) 1.
(*h*)	$\overline{90 + 60} \div 25 \times 5$.	(*h*) 1.2.

ARITHMETIC.

(CONTINUED.)

PERCENTAGE.

182. Percentage is the process of calculating by *hundredths.*

183. The *term* **per cent.** is an abbreviation of the Latin words *per centum*, which mean *by the hundred.* A certain per cent. of a number is the number of hundredths of that number which is indicated by the number of units in the per cent. Thus, 6 per cent. of 125 is $125 \times \frac{6}{100} = 7.5$; 25 per cent. of 80 is $80 \times \frac{25}{100} = 20$; 43 per cent. of 432 pounds is $432 \times \frac{43}{100} = 185.76$ pounds.

184. The **sign** of per cent. is %, and is read *per cent.* Thus, 6% is read *six per cent.;* $12\frac{1}{2}\%$ is read *twelve and one-half per cent.*, etc.

When expressing the per cent. of a number to use in calculations, it is customary to express it decimally instead of fractionally. Thus, instead of expressing 6%, 25%, and 43% as $\frac{6}{100}$, $\frac{25}{100}$, and $\frac{43}{100}$, it is usual to express them as .06, .25, and .43.

The following table will show how any per cent. can be expressed either as a decimal or as a fraction:

Per Cent.	Decimal.	Fraction.	Per Cent.	Decimal.	Fraction.
1%.....	.01	$\frac{1}{100}$	150 %....	1.50	$\frac{150}{100}$ or $1\frac{1}{2}$
2%.....	.02	$\frac{2}{100}$ or $\frac{1}{50}$	500 %....	5.00	$\frac{500}{100}$ or 5
5%.....	.05	$\frac{5}{100}$ or $\frac{1}{20}$	$\frac{1}{4}\%$....	.0025	$\frac{\frac{1}{4}}{100}$ or $\frac{1}{400}$
10%.....	.10	$\frac{10}{100}$ or $\frac{1}{10}$	$\frac{1}{2}\%$....	.005	$\frac{\frac{1}{2}}{100}$ or $\frac{1}{200}$
25%.....	.25	$\frac{25}{100}$ or $\frac{1}{4}$	$1\frac{1}{2}\%$....	.015	$\frac{1\frac{1}{2}}{100}$ or $\frac{3}{200}$
50%.....	.50	$\frac{50}{100}$ or $\frac{1}{2}$	$8\frac{1}{3}\%$....	$.08\frac{1}{3}$	$\frac{8\frac{1}{3}}{100}$ or $\frac{1}{12}$
75%.....	.75	$\frac{75}{100}$ or $\frac{3}{4}$	$12\frac{1}{2}\%$....	.125	$\frac{12\frac{1}{2}}{100}$ or $\frac{1}{8}$
100%.....	1.00	$\frac{100}{100}$ or 1	$16\frac{2}{3}\%$....	$.16\frac{2}{3}$	$\frac{16\frac{2}{3}}{100}$ or $\frac{1}{6}$
125%.....	1.25	$\frac{125}{100}$ or $1\frac{1}{4}$	$62\frac{1}{2}\%$....	.625	$\frac{62\frac{1}{2}}{100}$ or $\frac{5}{8}$

185. The names of the different elements used in percentage are: the *base*, the *rate per cent*, the *percentage*, the *amount*, and the *difference*.

186. The **base** is the number on which the per cent. is computed.

187. The **rate** is the number of hundredths of the base to be taken.

188. The **percentage** is the part, or number of *hundredths*, of the base indicated by the rate; or, the percentage is the result obtained by multiplying the base by the rate.

Thus, when it is stated that 7% of \$25 is \$1.75, \$25 is the base, 7% is the rate, and \$1.75 is the percentage.

189. The **amount** is the sum of the base and percentage.

190. The **difference** is the remainder obtained by subtracting the percentage from the base.

Thus, if a man has \$180, and he earns 6% more, he will have, altogether, $\$180 + \$180 \times .06$, or $\$180 + \$10.80 = \$190.80$. Here \$180 is the base; 6%, the rate; \$10.80, the percentage, and \$190.80, the *amount*.

Again, if an engine of 125 horsepower uses 16% of it in overcoming friction and other resistances, the amount left for obtaining useful work is $125 - 125 \times .16 = 125 - 20 = 105$ horsepower. Here 125 is the base; 16%, the rate; 20, the percentage, and 105, the *difference*.

191. From the foregoing it is evident that to find the percentage, the base must be multiplied by the rate. Hence, the following

Rule.—*To find the percentage, multiply the base by the rate expressed decimally.*

Example.—Out of a lot of 300 bushels of apples, 76% were sold. How many bushels were sold?

Solution.—76%, the rate, expressed decimally is .76; the base is 300; hence, the number of bushels sold, or the percentage, is by the above rule,

$$300 \times .76 = 228 \text{ bushels. Ans.}$$

Expressing the rule as a

Formula, *percentage* = *base* × *rate*.

192. When the percentage and rate are given, the base may be found by dividing the percentage by the rate. For, suppose that 12 is 6%, or $\frac{6}{100}$, of some number; then, 1%, or $\frac{1}{100}$, of the number, is $12 \div 6$, or 2. Consequently, if $2 = 1\%$, or $\frac{1}{100}$, 100%, or $\frac{100}{100}$, $= 2 \times 100 = 200$. But, since the same result may be arrived at by dividing 12 by .06, for $12 \div .06 = 200$, it follows that

Rule.—*When the percentage and rate are given, to find the base, divide the percentage by the rate, expressed decimally.*

Formula, *base* = *percentage* ÷ *rate.*

Example.—Bought a certain number of bushels of apples and sold 76% of them. If I sold 228 bushels, how many bushels did I buy?

Solution.—Here 228 is the percentage, and 76%, or .76, is the rate; hence, applying the rule,

$$228 \div .76 = 300 \text{ bushels. Ans.}$$

193. When the base and percentage are given, to find the rate, the rate may be found, expressed decimally, by dividing the percentage by the base. For, suppose that it is desired to find what per cent. 12 is of 200. 1% of 200 is $200 \times .01 = 2$. Now, if 1% is 2, 12 is evidently as many per cent. as the number of times that 2 is contained in 12, or $12 \div 2 = 6\%$. But the same result may be obtained by dividing 12, the percentage, by 200, the base, since $12 \div 200 = .06 = 6\%$. Hence,

Rule.—*When the percentage and base are given, to find the rate, divide the percentage by the base, and the result will be the rate, expressed decimally.*

Formula, *rate* = *percentage* ÷ *base.*

Example.—Bought 300 bushels of apples and sold 228 bushels. What per cent. of the total number of bushels was sold?

Solution.—Here 300 is the base and 228 is the percentage; hence, applying rule, $\text{rate} = 228 \div 300 = .76 = 76\%$. Ans.

Example.—What per cent. of 875 is 25?

Solution.—Here 875 is the base and 25 is the percentage; hence, applying rule, $25 \div 875 = .02\frac{6}{7} = 2\frac{6}{7}\%$. Ans.

Proof.—$875 \times .02\frac{6}{7} = 25$.

EXAMPLES FOR PRACTICE.

194. What per cent. of

	Ans.
(*a*) 360 is 90?	(*a*) 25%.
(*b*) 900 is 360?	(*b*) 40%.
(*c*) 125 is 25?	(*c*) 20%.
(*d*) 150 is 750?	(*d*) 500%.
(*e*) 280 is 112?	(*e*) 40%.
(*f*) 400 is 200?	(*f*) 50%.
(*g*) 47 is 94?	(*g*) 200%.
(*h*) 500 is 250?	(*h*) 50%.

195. The amount may be found, when the base and rate are given, by multiplying the base by 1 plus the rate, expressed decimally. For, suppose that it is desired to find the amount when 200 is the base and 6% is the rate. The percentage is 200 × .06 = 12, and, according to definition, Art. **189,** the amount is 200 + 12 = 212. But the same result may be obtained by multiplying 200 by 1 + .06, or 1.06, since 200 × 1.06 = 212. Hence,

Rule.—*When the base and rate are given, to find the amount, multiply the base by 1 plus the rate, expressed decimally.*

Formula, *amount* = *base* × (1 + *rate*).

Example.—If a man earned $725 in a year, and the next year 10% more, how much did he earn the second year?

Solution.—Here 725 is the base and 10% is the rate, and the amount is required. Hence, applying the rule,

725 × 1.10 = $797.50. Ans.

196. When the base and rate are given, the difference may be found by multiplying the base by 1 minus the rate, expressed decimally. For, suppose that it is desired to find the difference when the base is 200 and the rate is 6%. The percentage is 200 × .06 = 12; and, according to definition, Art. **190,** the difference = 200 − 12 = 188. But the same result may be obtained by multipiying 200 by 1 − .06, or .94, since 200 × .94 = 188. Hence,

Rule.—*When the base and rate are given, to find the difference, multiply the base by 1 minus the rate, expressed decimally.*

Formula, *difference* = *base* × (1 − *rate*).

Example.—Bought 300 bushels of apples, and sold all but 24% of them. How many bushels were sold?

Solution.—Here 300 is the base, 24% is the rate, and it is desired to find the difference. Hence, applying the rule,

$$300 \times (1 - .24) = 228 \text{ bushels. Ans.}$$

197. When the amount and rate are given, the base may be found by dividing the amount by 1 plus the rate. For, suppose that it is known that 212 equals some number increased by 6% of itself. Then it is evident that 212 equals 106% of the number (base) that it is desired to find. Consequently, if $212 = 106\%$, $1\% = \frac{212}{106} = 2$, and $100\% = 2 \times 100 = 200 =$ the base. But the same result may be obtained by dividing 212 by $1 + .06$, or 1.06, since $212 \div 1.06 = 200$. Hence,

Rule.—*When the amount and rate are given, to find the base, divide the amount by 1 plus the rate, expressed decimally.*

Formula, $base = amount \div (1 + rate)$.

Example.—The theoretical discharge of a certain pump, when running at a piston speed of 100 feet per minute, is 278,910 gallons per day of 10 hours. Owing to leakage and other defects, this value is 25% greater than the actual discharge. What is the actual discharge?

Solution.—Here 278,910 equals the actual discharge (base) increased by 25% of itself. Consequently, 278,910 is the amount; 25% is the rate, and, applying rule,

$$\text{actual discharge} = 278{,}910 \div 1.25 = 223{,}128 \text{ gallons. Ans.}$$

198. When the difference and rate are given, the base may be found by dividing the difference by 1 minus the rate. For, suppose that 188 equals some number less 6% of itself. Then, 188 evidently equals $100 - 6 = 94\%$ of some number. Consequently, if $188 = 94\%$, $1\% = 188 \div 94 = 2$, and $100\% = 2 \times 100 = 200$. But the same result may be obtained by dividing 188 by $1 - .06$, or .94, since $188 \div .94 = 200$. Hence,

Rule.—*When the difference and rate are given, to find the base, divide the difference by 1 minus the rate, expressed decimally.*

Formula, $base = difference \div (1 - rate)$.

EXAMPLE.—Bought a certain number of bushels of apples and sold 76% of them. If there were 72 bushels left unsold, how many bushels did I buy?

SOLUTION.—Here 72 is the difference and 76% is the rate. Applying rule,

$$72 \div (1 - .76) = 300 \text{ bushels. Ans.}$$

EXAMPLE.—The theoretical number of foot-pounds of work per minute required to operate a boiler feed-pump is 127,344. If 30% of the total number actually required be allowed for friction, leakage, etc., how many foot-pounds are actually required to work the pump?

SOLUTION.—Here the number actually required is the base; hence, 127,344 is the difference, and 30% is the rate. Applying the rule,

$$127{,}344 \div (1 - .30) = 181{,}920 \text{ foot-pounds. Ans.}$$

199. EXAMPLE.—A certain chimney gives a draft of 2.76 inches of water. By increasing the height 20 feet, the draft was increased to 3 inches of water. What was the gain per cent.?

SOLUTION.—Here it is evident that 3 inches is the amount and that 2.76 inches is the base. Consequently, 3 − 2.76 = .24 inch is the percentage, and it is required to find the rate. Hence, applying the rule given in Art. **193**,

$$\text{gain per cent.} = .24 \div 2.76 = .087 = 8.7\%. \text{ Ans.}$$

200. EXAMPLE.—A certain chimney gave a draft of 3 inches of water. After an economizer had been put in, the draft was reduced to 1.2 inches of water. What was the loss per cent.?

SOLUTION.—Here it is evident that 1.2 inches is the difference (since it equals 3 inches diminished by a certain per cent., loss of itself) and 3 inches is the base. Consequently, 3 − 1.2 = 1.8 inches is the percentage. Hence, applying the rule given in Art. **193**,

$$\text{loss per cent.} = 1.8 \div 3 = .60 = 60\%. \text{ Ans.}$$

201. To find the gain or loss per cent.:

Rule.—*Find the difference between the initial and final values; divide this difference by the initial value.*

EXAMPLE.—If a man buys a house for $1,860, and some time afterwards builds a barn for 25% of the cost of the house, does he gain or lose, and how much per cent., if he sells both house and barn for $2,100?

SOLUTION.—The cost of the barn was $1,860 × .25 = $465; consequently, the initial value, or cost, was $1,860 + $465 = $2,325. Since he sold them for $2,100, he lost $2,325 − $2,100 = $225. Hence, applying rule,

$$225 \div 2{,}325 = .0968 = 9.68\% \text{ loss. Ans.}$$

EXAMPLES FOR PRACTICE.

202. Solve the following:

		Ans.	
(*a*)	What is 12½% of $900?	(*a*)	$112.50.
(*b*)	What is ⅘% of 627?	(*b*)	5.016.
(*c*)	What is 33⅓% of 54?	(*c*)	18.
(*d*)	101 is 68¾% of what number?	(*d*)	$146\frac{10}{11}$.
(*e*)	784 is 83⅓% of what number?	(*e*)	940.8.
(*f*)	What % of 960 is 160?	(*f*)	16⅔%.
(*g*)	What % of $3,606 is $450¾?	(*g*)	12½%.
(*h*)	What % of 280 is 112?	(*h*)	40%.

1. A steam plant consumed an average of 3,640 pounds of coal per day. The engineer made certain alterations which resulted in a saving of 250 pounds per day. What was the per cent. of coal saved? Ans. 7%, nearly.

2. If the speed of an engine running at 126 revolutions per minute should be increased 6½%, how many revolutions per minute would it then make? Ans. 134.19 revolutions.

3. The list price of a lot of silk goods is $1,400; of some laces, $1,150, and of some calico, $340. If 25% discount was allowed on silk, 22% on the laces, and 12½% on the calico, what was the actual cost of the purchase? Ans. $2,244.50.

4. If I lend a man $1,100, and this is 18½% of the amount that I have on interest, how much money have I on interest? Ans. $5,945.95.

5. A test showed that an engine developed 190.4 horsepower, 15% of which was consumed in friction. How much power was available for use? Ans. 161.84 H.P.

6. By adding a condenser to a steam engine, the power was increased 14%, and the consumption of coal per horsepower per hour was decreased 20%. If the engine could originally develop 50 horsepower, and required 3½ pounds of coal per horsepower per hour, what would be the total weight of coal used in an hour, with the condenser, assuming the engine to run full power? Ans. 159.6 pounds.

DENOMINATE NUMBERS.

203. A **denominate number** is a concrete number, and may be either simple or compound, as 8 quarts, 5 feet, 10 inches, etc.

204. A **simple denominate number** consists of units of but one denomination, as 16 cents, 10 hours, 5 dollars, etc.

205. A **compound denominate number** consists of units of two or more denominations of a similar kind, as 3 yards 2 feet 1 inch; 34 square feet 57 square inches.

206. In **whole numbers** and in **decimals** the *law* of increase and decrease is on the scale of 10, but in **compound** or **denominate numbers** the scale varies.

MEASURES.

207. A **measure** is a *standard unit*, established by *law* or *custom*, by which *quantity* of any kind is measured. The *standard unit* of **dry measure** is the Winchester bushel; of **weight,** the pound; of **liquid measure,** the gallon, etc.

208. Measures are of six kinds:

1.	Extension.	4.	Time.
2.	Weight.	5.	Angles.
3.	Capacity.	6.	Money or value.

MEASURES OF EXTENSION.

209. **Measures of extension** are used in measuring lengths, distances, surfaces, and solids.

LINEAR MEASURE.

TABLE 3.

			Abbreviation.	in.	ft.	yd.	rd.	fur.	mi.
12	inches (in.)	= 1 foot . .	ft.						
3	feet	= 1 yard . .	yd.	36 =	3 =	1			
5.5	yards	= 1 rod . . .	rd.	198 =	16½ =	5.5 =	1		
40	rods	= 1 furlong	fur.	7,920 =	660 =	220 =	40 =	1	
8	furlongs	= 1 mile . .	mi.	63,360 =	5,280 =	1,760 =	320 =	8 =	1

SURVEYOR'S LINEAR MEASURE.

TABLE 4.

7.92 inches	= 1 link	li.
25 links	= 1 rod	rd.
4 rods 100 links	= 1 chain	ch.
80 chains	= 1 mile	mi.

mi.		ch.		rd.		li.		in.
1	=	80	=	320	=	8,000	=	63,360

210. The linear unit, generally used by surveyors, is **Gunter's chain,** which is equal to 4 rods, or 66 feet.

211. An **engineer's chain,** used by civil engineers, is 100 feet long, and consists of 100 links. In computations, the links are written as so many hundredths of a chain.

SQUARE MEASURE.

TABLE 5.

144 square inches (sq. in.) . . . = 1 square foot . . . . . sq.ft.
9 square feet . . . . = 1 square yard . . . . . sq.yd.
30¼ square yards . . . . = 1 square rod . . . . sq.rd.
160 square rods . . . . = 1 acre . . . . A.
640 acres . . . . = 1 square mile . . . . sq.mi.

sq.mi. A. sq.rd. sq.yd. sq.ft. sq.in.
1 = 640 = 102,400 = 3,097,600 = 27,878,400 = 4,014,489,600

SURVEYOR'S SQUARE MEASURE.

TABLE 6.

625 square links . . . . = 1 square rod . . . . . . sq.rd.
16 square rods . . . . = 1 square chain . . . . sq.ch.
10 square chains . . . . = 1 acre . . . . A.
640 acres . . . . = 1 square mile . . . . . sq.mi.
36 square miles (6 mi. square) . = 1 township . . . . Tp.

sq.mi. A. sq.ch. sq.rd. sq.li.
1 = 640 = 6,400 = 102,400 = 64,000,000

CUBIC MEASURE.

TABLE 7.

1728 cubic inches (cu. in.) . . . = 1 cubic foot . . . . . . cu.ft.
27 cubic feet . . . . = 1 cubic yard . . . . . cu.yd.
128 cubic feet . . . . = 1 cord . . . . cd.
24¾ cubic feet . . . . = 1 perch . . . . P.

cu.yd. cu.ft. cu.in.
1 = 27 = 46,656

MEASURES OF WEIGHT.

AVOIRDUPOIS WEIGHT.

TABLE 8.

16 ounces (oz.) . . . . = 1 pound . . . . lb.
100 pounds . . . . = 1 hundredweight . . . . cwt.
20 cwt., or 2,000 lb. . . . . . . = 1 ton . . . . T.

T. cwt. lb. oz.
1 = 20 = 2,000 = 32,000

212. The ounce is divided into halves, quarters, etc. Avoirdupois weight is used for weighing coarse and heavy articles. One avoirdupois pound contains 7,000 grains.

LONG TON TABLE.

TABLE 9.

16 ounces	=	1 pound	lb.
112 pounds	=	1 hundredweight	cwt.
20 cwt., or 2,240 lb.	=	1 ton	T.

213. In all the calculations throughout this and the succeeding volumes, 2,000 pounds will be considered one ton, unless the long ton (2,240 pounds) is especially mentioned.

TROY WEIGHT.

TABLE 10.

24 grains (gr.)	=	1 pennyweight	pwt.
20 pennyweights	=	1 ounce	oz.
12 ounces	=	1 pound	lb.

lb.		oz.		pwt.		gr.
1	=	12	=	240	=	5,760

214. Troy weight is used in weighing gold and silverware, jewels, etc. It is used by jewelers.

MEASURES OF CAPACITY.

LIQUID MEASURE.

TABLE 11.

4 gills (gi.)	=	1 pint	pt.
2 pints	=	1 quart	qt.
4 quarts	=	1 gallon	gal.
31½ gallons	=	1 barrel	bbl.
2 barrels, or 63 gallons	=	1 hogshead	hhd.

hhd.		bbl.		gal.		qt.		pt.		gi.
1	=	2	=	63	=	252	=	504	=	2,016

DRY MEASURE.

TABLE 12.

2 pints (pt.)	=	1 quart	qt.
8 quarts	=	1 peck	pk.
4 pecks	=	1 bushel	bu.

bu.		pk.		qt.		pt.
1	=	4	=	32	=	64

MEASURE OF TIME.

TABLE 13.

60 seconds (sec.) . . . . = 1 minute . . . . . . min.
60 minutes . . . . = 1 hour . . . . hr.
24 hours . . . . = 1 day . . . . da.
7 days . . . . = 1 week . . . . wk.
365 days } 12 months } . . . . = 1 common year . . . yr.
366 days . . . . = 1 leap year.
100 years . . . . = 1 century.

NOTE.—It is customary to consider one month as 30 days.

MEASURE OF ANGLES OR ARCS.

TABLE 14.

60 seconds (″) . . . . = 1 minute . . . . ′.
60 minutes . . . . = 1 degree . . . . °.
90 degrees . . . . = 1 right angle or quadrant ∟.
360 degrees . . . . = 1 circle . . . . cir.

cir.
1 = 360° = 21,600′ = 1,296,000″

MEASURE OF MONEY.

UNITED STATES MONEY.

TABLE 15.

10 mills (m.) . . . . = 1 cent . . . . ct.
10 cents . . . . = 1 dime . . . . d.
10 dimes . . . . = 1 dollar . . . . $.
10 dollars . . . . = 1 eagle . . . . E.

E. $ d. ct. m.
1 = 10 = 100 = 1,000 = 10,000

MISCELLANEOUS TABLE.

TABLE 16.

12 things are 1 dozen.
12 dozen are 1 gross.
12 gross are 1 great gross.
2 things are 1 pair.
20 things are 1 score.
1 league is 3 miles.
1 fathom is 6 feet.
1 meter is nearly 39.37 inches.
1 hand is 4 inches.
1 palm is 3 inches.
1 span is 9 inches.
24 sheets are 1 quire.
20 quires, or 480 sheets, are 1 ream.
1 bushel contains 2,150.4 cubic in.
1 U. S. standard gallon (also called a wine gallon) contains 231 cubic in.
1 U. S. standard gallon of water weighs 8.355 pounds, nearly.
1 cubic foot of water contains 7.481 U. S. standard gallons, nearly.
1 British imperial gallon weighs 10 pounds.

It will be of great advantage to the student to carefully memorize all of the above tables.

REDUCTION OF DENOMINATE NUMBERS.

215. Reduction of denominate numbers is the process of changing their denomination without changing their value. They may be changed from a higher to a lower denomination or from a lower to a higher—either is reduction. As,

2 hours = 120 minutes.
32 ounces = 2 pounds.

216. Principle.—Denominate numbers are changed to *lower* denominations by *multiplying*, and to *higher* denominations by *dividing*.

To reduce denominate numbers to lower denominations:

217. EXAMPLE.—Reduce 5 yd. 2 ft. 7 in. to inches.

SOLUTION.—

yd.	ft.	in.
5	2	7

$$\begin{array}{r l}
5 & \\
3 & \\
\hline
15 & \text{ft.} \\
2 & \text{ft.} \\
\hline
17 & \text{ft.} \\
12 & \\
\hline
34 & \\
17 & \\
\hline
204 & \text{in.} \\
7 & \text{in.} \\
\hline
211 & \text{inches. Ans.}
\end{array}$$

EXPLANATION.—Since there are 3 feet in 1 yard, in 5 yards there are 5 × 3, or 15 feet, and 15 feet plus 2 feet = 17 feet. There are 12 inches in a foot; therefore, 12 × 17 = 204 inches, and 204 inches plus 7 inches = 211 inches = number of inches in 5 yards 2 feet and 7 inches. Ans.

218. EXAMPLE.—Reduce 6 hours to seconds.

SOLUTION.—

$$\begin{array}{r l}
6 & \text{hours.} \\
60 & \\
\hline
360 & \text{minutes.} \\
60 & \\
\hline
21600 & \text{seconds. Ans.}
\end{array}$$

Explanation.—As there are 60 minutes in one hour, in six hours there are 6 × 60, or 360 minutes ; as there are no minutes to add, we multiply 360 minutes by 60, to get the number of seconds.

219. In order to avoid mistakes, if any denomination be omitted, represent it by a cipher. Thus, before reducing 3 rods 6 inches to inches, insert a cipher for yards and a cipher for feet; as,

rd.	yd.	ft.	in.
3	0	0	6

220. Rule.—*Multiply the number representing the highest denomination by the number of units in the next lower required to make one of the higher denomination, and to the product add the number of given units of that lower denomination. Proceed in this manner until the number is reduced to the required denomination.*

EXAMPLES FOR PRACTICE.

221. Reduce

	Ans.
(*a*) 4 rd. 2 yd. 2 ft. to ft.	(*a*) 74 ft.
(*b*) 4 bu. 3 pk. 2 qt. to qt.	(*b*) 154 qt.
(*c*) 13 rd. 5 yd. 2 ft. to ft.	(*c*) 231.5 ft.
(*d*) 5 mi. 100 rd. 10 ft. to ft.	(*d*) 28,060 ft.
(*e*) 8 lb. 4 oz. 6 pwt. to gr.	(*e*) 48,144 gr.
(*f*) 52 hhd. 24 gal. 1 pt. to pt.	(*f*) 26,401 pt.
(*g*) 5 cir. 16° 20′ to minutes.	(*g*) 108,980′.
(*h*) 14 bu. to qt.	(*h*) 448 qt.

To reduce lower to higher denominations:

222. Example.—Reduce 211 in. to higher denominations.

Solution.— 12) 211 in.
3) 17 ft. + 7 in.
5 yd. + 2 ft. Ans.

Explanation.—There are 12 inches in 1 foot ; therefore, 211 divided by 12 = 17 feet and 7 inches over. There are 3 feet in 1 yard ; therefore, 17 feet divided by 3 = 5 yards and 2 feet over. The last quotient and the two remainders constitute the answer, 5 yards 2 feet 7 inches.

223. Example.—Reduce 15,735 grains Troy weight to higher denominations.

Solution.—

```
24)15735 gr. (655 pwt.
   144
   ---
    133
    120
    ---
     135
     120
     ---
      15 gr.

20)655 pwt. (32 oz.
   60
   --
    55
    40
    --
    15 pwt.

12)32 oz. (2 lb.
   24
   --
    8 oz.
```

Explanation.—There are 24 grains in 1 pennyweight, and in 15,735 grains there are as many pennyweights as 24 is contained in 15,735, or 655 pennyweights and 15 grains remaining. There are 20 pennyweights in 1 ounce, and in 655 pennyweights there are 32 ounces and 15 pennyweights remaining. There are 12 ounces in 1 pound, and in 32 ounces there are 2 pounds and 8 ounces remaining. The last quotient and the three remainders constitute the answer, 2 pounds 8 ounces 15 pennyweights 15 grains.

The above problem is worked out by long division, because the numbers are too large to solve easily by short division. The student may use either method.

224. Rule.—*Divide the number representing the denomination given by the number of units of this denomination required to make one unit of the next higher denomination. The remainder will be of the same denomination, but the quotient will be of the next higher. Divide this quotient by* *the number of units of its denomination required to make one unit of the next higher. Continue until the highest denomination is reached, or until there is not enough of a denomination left to make one of the next higher. The last quotient and the remainders constitute the required result.*

EXAMPLES FOR PRACTICE.

225. Reduce to units of higher denominations:

(*a*) 7,460 sq. in.; (*b*) 7,580 sq. yd.; (*c*) 148,760 cu. in.; (*d*) 7,896 cu. ft. to cd.; (*e*) 17,651″; (*f*) 1,120 cu. ft. to cd.; (*g*) 8,000 gi.; (*h*) 36,450 lb.

Ans.
(*a*) 5 sq. yd. 6 sq. ft. 116 sq. in.
(*b*) 1 A. 90 sq. rd. 17 sq. yd. 4 sq. ft. 72 sq. in.
(*c*) 3 cu. yd. 5 cu. ft. 152 cu. in.
(*d*) 61 cd. 88 cu. ft.
(*e*) 4° 54′ 11″.
(*f*) 8 cd. 96 cu. ft.
(*g*) 3 hhd. 61 gal.
(*h*) 18 T. 4 cwt. 50 lb.

ADDITION OF DENOMINATE NUMBERS.

226. Example.—Find the sum of 3 cwt. 46 lb. 12 oz.; 8 cwt. 12 lb 13 oz.; 12 cwt. 50 lb. 13 oz.; 27 lb. 4 oz.

Solution.—

T.	cwt.	lb.	oz.	
0	3	46	12	
0	8	12	13	
0	12	50	13	
0	0	27	4	
1	4	37	10	Ans.

Explanation.—Begin to add at the right-hand column: $4 + 13 + 13 + 12 = 42$ ounces; as 16 ounces make 1 pound, 42 ounces $\div 16 = 2$ and a remainder of 10 ounces, or 2 pounds and 10 ounces. Place 10 ounces under ounce column, and add 2 pounds to the next or pound column. Then, $2 + 27 + 50 + 12 + 46 = 137$ pounds; as 100 pounds make a hundredweight, $137 \div 100 = 1$ hundredweight and a remainder of 37 pounds. Place the 37 under the pounds column, and add 1 hundredweight to the next or hundredweight column. Next, $1 + 12 + 8 + 3 = 24$ hundredweight.

20 hundredweight make a ton; therefore 24 ÷ 20 = 1 ton and 4 hundredweight remaining. Hence, the sum is 1 ton 4 hundredweight 37 pounds 10 ounces. Ans.

227. Example.—What is the sum of 2 rd. 3 yd. 2 ft. 5 in.; 6 rd. 1 ft. 10 in.; 17 rd. 11 in.; 4 yd. 1 ft.?

Solution.—

	rd.	yd.	ft.	in.	
	2	3	2	5	
	6	0	1	10	
	17	0	0	11	
	0	4	1	0	
	26	3½	0	2	
or	26	3	1	8	Ans.

Explanation.—The sum of the numbers in the first column = 26 inches, or 2 feet and 2 inches remaining. The sum of the numbers in the next column plus 2 feet = 6 feet, or 2 yards and 0 feet remaining. The sum of the next column plus 2 yards = 9 yards, or 9 ÷ 5½ = 1 rod and 3½ yards remaining. The sum of the next column plus 1 rod = 26 rods. To avoid fractions in the sum, the ½ yard is reduced to 1 foot and 6 inches, which added to 26 rods 3 yards 0 feet and 2 inches = 26 rods 3 yards 1 foot 8 inches. Ans.

228. Example.—What is the sum of 47 ft. and 3 rd. 2 yd. 2 ft. 10 in.?

Solution.—When 47 ft. is reduced it equals 2 rd. 4 yd. 2 ft., which can be added to 3 rd. 2 yd. 2 ft. 10 in. Thus,

	rd.	yd.	ft.	in.	
	3	2	2	10	
	2	4	2	0	
	6	1½	1	10	
or	6	2	0	4	Ans.

229. **Rule.**—*Place the numbers so that like denominations are under each other. Begin at the right-hand column, and add. Divide the sum by the number of units of this denomination required to make one unit of the next higher. Place the remainder under the column added, and carry the quotient to the next column. Continue in this manner until the highest denomination given is reached.*

EXAMPLES FOR PRACTICE.

230. What is the sum of

(*a*) 25 lb. 7 oz. 15 pwt. 23 gr.; 17 lb. 16 pwt.; 15 lb. 4 oz. 12 pwt.; 18 lb. 16 gr.; 10 lb. 2 oz. 11 pwt. 16 gr.?

(*b*) 9 mi. 13 rd. 4 yd. 2 ft.; 16 rd. 5 yd. 1 ft. 5 in.; 16 mi. 2 rd. 3 in.; 14 rd. 1 yd. 9 in.?

(*c*) 3 cwt. 46 lb. 12 oz.; 12 cwt. 9½ lb.; 2¼ cwt. 21⅝ lb.?

(*d*) 10 yr. 8 mo. 5 wk. 3 da.; 42 yr. 6 mo. 7 da.; 7 yr. 5 mo. 18 wk. 4 da.; 17 yr. 17 da.?

(*e*) 17 tons 11 cwt. 49 lb. 14 oz.; 16 tons 47 lb. 13 oz.; 20 tons 13 cwt. 14 lb. 6 oz.; 11 tons 4 cwt. 16 lb. 12 oz.?

(*f*) 14 sq. yd. 8 sq. ft. 19 sq. in.; 105 sq. yd. 16 sq. ft. 240 sq. in.; 42 sq. yd. 28 sq. ft. 165 sq. in.?

Ans.
(*a*) 86 lb. 3 oz. 16 pwt. 7 gr.
(*b*) 25 mi. 47 rd. 1 ft. 5 in.
(*c*) 18 cwt. 2 lb. 14 oz.
(*d*) 78 yr. 1 mo. 3 wk. 3 da.
(*e*) 65 tons 9 cwt. 28 lb. 13 oz.
(*f*) 167 sq. yd. 136 sq. in.

SUBTRACTION OF DENOMINATE NUMBERS.

231. Example.—From 21 rd. 2 yd. 2 ft. 6½ in., take 9 rd 4 yd. 10¼ in.

Solution.—

rd.	yd.	ft.	in.	
21	2	2	6½	
9	4	0	10¼	
11	3½	1	8¼	Ans.

Explanation.—Since 10¼ inches cannot be taken from 6½ inches, we must borrow 1 foot, or 12 inches, from the 2 feet in the next column and add it to the 6½. $6\frac{1}{2} + 12 = 18\frac{1}{2}$. $18\frac{1}{2}$ inches $-$ $10\frac{1}{4}$ inches $=$ $8\frac{1}{4}$ inches. Then, 0 foot from the 1 remaining foot = 1 foot. 4 yards cannot be taken from 2 yards; therefore, we borrow 1 rod, or 5½ yards, from 21 rods and add it to 2. $2 + 5\frac{1}{2} = 7\frac{1}{2}$; $7\frac{1}{2} - 4 = 3\frac{1}{2}$ yards. 9 rods from 20 rods = 11 rods. Hence, the remainder is 11 rods 3½ yards 1 foot 8¼ inches. Ans.

To avoid fractions as much as possible, we reduce the ½ yard to inches, obtaining 18 inches; this added to 8¼ inches, gives 26¼ inches, which equals 2 feet 2¼ inches. Then, 2 feet + 1 foot = 3 feet = 1 yard, and 3 yards + 1 yard = 4 yards. Hence, the above answer becomes 11 rods 4 yards 0 feet 2¼ inches.

232. Example.—What is the difference between 3 rd. 2 yd. 2 ft. 10 in. and 47 ft.?

Solution.—47 ft. = 2 rd. 4 yd. 2 ft.

	rd.	yd.	ft.	in.	
	3	2	2	10	
	2	4	2	0	
	0	$3\frac{1}{2}$	0	10	
or		3	2	4	Ans.

To find (approximately) the interval of time between two dates:

233. Example.—How many years, months, days, and hours between 4 o'clock P.M. of June 15, 1868, and 10 o'clock A.M., September 28, 1891?

Solution.—	yr.	mo.	da.	hr.	
	1891	8	28	10	
	1868	5	15	16	
	23	3	12	18	Ans.

Explanation.—Counting 24 hours in 1 day, 4 o'clock P.M. is the 16th hour from the beginning of the day, or midnight. On September 28, 8 months and 28 days have elapsed, and on June 15, 5 months and 15 days. After placing the earlier date under the later date, subtract as in the previous problems. Count 30 days as 1 month.

234. **Rule.**—*Place the smaller quantity under the larger quantity, with like denominations under each other. Beginning at the right, subtract successively the number in the subtrahend in each denomination from the one above, and place the differences underneath. If the number in the minuend of any denomination is less than the number under it in the subtrahend, one must be borrowed from the minuend of the next higher denomination, reduced and added to it.*

EXAMPLES FOR PRACTICE.

235. From

(*a*) 125 lb. 8 oz. 14 pwt. 18 gr. take 96 lb. 9 oz. 10 pwt. 4 gr.

(*b*) 126 hhd. 27 gal. take 104 hhd. 14 gal. 1 qt. 1 pt.

(*c*) 65 T. 14 cwt. 64 lb. 10 oz. take 16 T. 11 cwt. 14 oz.

(*d*) 148 sq. yd. 16 sq. ft. 142 sq. in. take 132 sq. yd. 136 sq. in.

(*e*) 100 bu. take 28 bu. 2 pk. 5 qt. 1 pt.
(*f*) 14 mi. 34 rd. 16 yd. 13 ft. 11 in. take 3 mi. 27 rd. 11 yd. 4 ft. 10 in.

Ans.
(*a*) 28 lb. 11 oz. 4 pwt. 14 gr.
(*b*) 22 hhd. 12 gal. 2 qt. 1 pt.
(*c*) 49 T. 3 cwt. 63 lb. 12 oz.
(*d*) 16 sq. yd. 16 sq. ft. 6 sq. in.
(*e*) 71 bu. 1 pk. 2 qt. 1 pt.
(*f*) 11 mi. 7 rd. 5 yd. 9 ft. 1 in.

MULTIPLICATION OF DENOMINATE NUMBERS.

236. EXAMPLE.—Multiply 7 lb. 5 oz. 13 pwt. 15 gr. by 12.

SOLUTION.—

lb.	oz.	pwt.	gr.	
7	5	13	15	
			12	
89	8	3	12	Ans.

EXPLANATION.—15 grains × 12 = 180 grains. 180 ÷ 24 = 7 pennyweights and 12 grains remaining. Place the 12 in the grain column and carry the 7 pennyweights to the next. Now, 13 × 12 + 7 = 163 pennyweights; 163 ÷ 20 = 8 ounces and 3 pennyweights remaining. Then, 5 × 12 + 8 = 68 ounces; 68 ÷ 12 = 5 pounds and 8 ounces remaining. Then, 7 × 12 + 5 = 89 pounds. The entire product is 89 pounds 8 ounces 3 pennyweights 12 grains. Ans.

237. **Rule.**—*Multiply the number representing each denomination by the multiplier, and reduce each product to the next higher denomination, writing the remainders under each denomination, and carrying the quotient to the next, as in Addition of Denominate Numbers.*

238. NOTE.—In multiplication and division of denominate numbers, it is sometimes easier to reduce the number to the lowest denomination given before multiplying or dividing, especially if the multiplier or divisor is a decimal. Thus, in the above example, had the multiplier been 1.2, the easiest way to multiply would have been to reduce the number to grains; then, multiply by 1.2, and reduce the product to higher denominations. For example, 7 lb. 5 oz. 13 pwt. 15 gr. = 43,047 gr. 43,047 × 1.2 = 51,656.4 gr. = 8 lb. 11 oz. 12 pwt. 8.4 gr. Also, 43,047 × 12 = 516,564 gr. = 89 lb. 8 oz. 3 pwt. 12 gr., as above. The student may use either method.

EXAMPLES FOR PRACTICE.

239. Multiply

(*a*) 15 cwt. 90 lb. by 5; (*b*) 12 yr. 10 mo. 4 wk. 3 da. by 14; (*c*) 11 mi. 145 rd. by 20; (*d*) 12 gal. 4 pt. by 9; (*e*) 8 cd. 76 cu. ft. by 15; (*f*) 4 hhd. 3 gal. 1 qt. 1 pt. by 12.

Ans.
(*a*) 79 cwt. 50 lb.
(*b*) 180 yr. 11 mo. 2 wk.
(*c*) 229 mi. 20 rd.
(*d*) 112 gal. 2 qt.
(*e*) 128 cd. 116 cu. ft.
(*f*) 48 hhd. 40 gal. 2 qt.

DIVISION OF DENOMINATE NUMBERS.

240. Example.—Divide 48 lb. 11 oz. 6 pwt. by 8.

Solution.—

	lb.	oz.	pwt.	gr.	
8)	48	11	6	0	
	6 lb.	1 oz.	8 pwt.	6 gr.	Ans.

Explanation.—After placing the quantities as above, proceed as follows: 8 is contained in 48 six times without a remainder. 8 is contained in 11 ounces once with 3 ounces remaining. 3 × 20 = 60; 60 + 6 = 66 pennyweights; 66 pennyweights ÷ 8 = 8 pennyweights and 2 remaining; 2 × 24 grains = 48 grains; 48 grains ÷ 8 = 6 grains. Therefore, the entire quotient is 6 pounds 1 ounce 8 pennyweights 6 grains. Ans.

Example.—A silversmith melted up 2 lb. 8 oz. 10 pwt. of silver, which he made into 6 spoons; what was the weight of each spoon?

Solution.—

	lb.	oz.	pwt.		
6)	2	8	10		
		5 oz.	8 pwt.	8 gr.	Ans.

Explanation.—Since we cannot divide 2 pounds by 6, we reduce it to ounces. 2 pounds = 24 ounces, and 24 ounces + 8 ounces = 32 ounces; 32 ounces ÷ 6 = 5 ounces and 2 ounces over. 2 ounces = 40 pennyweights. 40 pennyweights + 10 pennyweights = 50 pennyweights, and 50 pennyweights ÷ 6 = 8 pennyweights and 2 pennyweights over. 2 pennyweights = 48 grains, and 48 grains ÷ 6 = 8 grains. Hence, each spoon contains 5 ounces 8 pennyweights 8 grains. Ans.

241. Example.—Divide 820 rd. 4 yd. 2 ft. by 112.

Solution.—

```
                 rd.    yd. ft.  rd. yd. ft.  in.
            112)820      4   2 ( 7   1   2  5.143  Ans.
                784
                ---
                 36 rd. rem.
                 5.5
                ---
                180
               180
               -----
               198.0 yd.
                 4
               ---
          112) 202 yd. (1 yd.
               112
               ---
                90 yd. rem.
                 3
               ---
               270 ft.
                 2 ft.
               ---
          112) 272 ft. (2 ft.
               224
               ---
                48 ft. rem.
                12
                --
                96
               48
               ---
          112) 576 in. (5.1428 + in., or 5.143 in.
               560
               ---
                160
                112
                ---
                 480
                 448
                 ---
                  320
                  224
                  ---
                   960
                   896
                   ---
                    64
```

Explanation.—The first quotient is 7 rods with 36 rods remaining. 5.5 × 36 = 198 yards; 198 yards + 4 yards = 202 yards; 202 yards ÷ 112 = 1 yard and 90 yards remaining. 90 × 3 = 270 feet; 270 feet + 2 feet = 272 feet; 272 feet ÷ 112 = 2 feet and 48 feet remaining; 48 × 12 = 576 inches; 576 inches ÷ 112 = 5.143 inches, nearly. Ans.

The preceding example is solved by long division, because the numbers are too large to deal with mentally. Instead of expressing the last result as a decimal, it might have been expressed as a common fraction. Thus, $576 \div 112 = 5\frac{16}{112} = 5\frac{1}{7}$ inches. The chief advantage of using a common fraction is that if the quotient be multiplied by the divisor, the result will always be the same as the original dividend.

242. Rule.—*Find how many times the divisor is contained in the first or highest denomination of the dividend. Reduce the remainder (if any) to the next lower denomination, and add to it the number in the given dividend expressing that denomination. Divide this new dividend by the divisor. The quotient will be the next denomination in the quotient required. Continue in this manner until the lowest denomination is reached. The successive quotients will constitute the entire quotient.*

EXAMPLES FOR PRACTICE.

243. Divide

(*a*) 376 mi. 276 rd. by 22; (*b*) 1,137 bu. 3 pk. 4 qt. 1 pt. by 10; (*c*) 84 cwt. 48 lb. 49 oz. by 16; (*d*) 78 sq. yd. 18 sq. ft. 41 sq. in. by 18; (*e*) 148 mi. 64 rd. 24 yd. by 12; (*f*) 100 tons 16 cwt. 18 lb. 11 oz. by 15; (*g*) 36 lb. 18 oz. 18 pwt. 14 gr. by 8; (*h*) 112 mi. 48 rd. by 100.

Ans. (*a*) 17 mi. $41\frac{7}{11}$ rd.
(*b*) 113 bu. 3 pk. 1 qt. $\frac{1}{2}$ pt.
(*c*) 5 cwt. 28 lb. $3\frac{1}{16}$ oz.
(*d*) 4 sq.yd. 4 sq.ft. $2\frac{5}{18}$ sq.in.
(*e*) 12 mi. 112 rd. 2 yd.
(*f*) 6 tons 14 cwt. 41 lb. $3\frac{14}{15}$ oz.
(*g*) 4 lb. 8 oz. 7 pwt. $7\frac{3}{4}$ gr.
(*h*) 1 mi. $38\frac{22}{25}$ rd.

INVOLUTION.

244. Involution is the process of multiplying a number by itself one or more times. The product obtained by multiplying a number by itself is called a **power** of that number.

Thus, the *second* power of 3 is 9, since 3 × 3 are 9.

The *third* power of 3 is 27, since 3 × 3 × 3 are 27
The *fifth* power of 2 is 32, since 2 × 2 × 2 × 2 × 2 are 32.

245. An **exponent** is a *small figure* placed to the *right* and a little *above* a number to show to what *power* it is to be raised, or how many times the number is to be used as a factor, as the small figures ^{2}, ^{3}, and ^{5} below:

$$3^2 = 3 \times 3 = 9.$$
$$3^3 = 3 \times 3 \times 3 = 27.$$
$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32.$$

246. The **root** of a number is that number which, used the required number of times as a factor, produces the number. In the above cases, 3 is a root of 9, since 3 × 3 are 9. It is also a root of 27, since 3 × 3 × 3 are 27. Also, 2 is a root of 32, since 2 × 2 × 2 × 2 × 2 are 32.

247. The *second* power of a number is called its **square.**

Thus, 5^2 is called the **square** of 5, or 5 *squared*, and its value is 5 × 5 = 25.

248. The *third* power of a number is called its **cube.**

Thus, 5^3 is called the *cube* of 5, or 5 *cubed*, and its value is 5 × 5 × 5 = 125.

To find any power of a number:

249. EXAMPLE.—What is the third power, or cube. of 35.

SOLUTION.— 35 × 35 × 35,

$$\text{or}\quad \begin{array}{r} 35\\ 35\\ \hline 175\\ 105\\ \hline 1225\\ 35\\ \hline 6125\\ 3675\\ \hline \end{array}$$

$$\textit{cube} = 42875 \quad \text{Ans.}$$

Example.—What is the fourth power of 15?

Solution.— $15 \times 15 \times 15 \times 15$,

$$\text{or}\quad \begin{array}{r} 15 \\ 15 \\ \hline 75 \\ 15 \\ \hline 225 \\ 15 \\ \hline 1125 \\ 225 \\ \hline 3375 \\ 15 \\ \hline 16875 \\ 3375 \\ \hline \textit{fourth power} = 50625 \end{array} \quad \text{Ans.}$$

250. Example.— 1.2^3 = what?

Solution.— $1.2 \times 1.2 \times 1.2$,

$$\text{or}\quad \begin{array}{r} 1.2 \\ 1.2 \\ \hline 1.44 \\ 1.2 \\ \hline 288 \\ 144 \\ \hline 1.728 \end{array} \quad \text{Ans.}$$

251. Example.—What is the third power, or cube, of $\frac{3}{8}$?

Solution.— $(\frac{3}{8})^3 = \frac{3}{8} \times \frac{3}{8} \times \frac{3}{8} = \frac{3 \times 3 \times 3}{8 \times 8 \times 8} = \frac{27}{512}$. Ans.

252. **Rule I.**—*To raise a whole number, or a decimal, to any power, use it as a factor as many times as there are units in the exponent.*

II. *To raise a fraction to any power, raise both the numerator and denominator to the power indicated by the exponent.*

EXAMPLES FOR PRACTICE.

253. Raise the following to the powers indicated:

(*a*)	85^2.	Ans.	(*a*)	7,225.
(*b*)	$(\frac{12}{13})^2$.		(*b*)	$\frac{144}{169}$.
(*c*)	6.5^2.		(*c*)	42.25.
(*d*)	14^4.		(*d*)	38,416.

(*e*)	$(\frac{3}{4})^3$.	Ans.	(*e*)	$\frac{27}{64}$.
(*f*)	$(\frac{5}{6})^3$.		(*f*)	$\frac{125}{216}$.
(*g*)	$(\frac{7}{2})^3$.		(*g*)	$\frac{343}{8}$.
(*h*)	1.4^5.		(*h*)	5.37824.

EVOLUTION.

254. **Evolution** is the reverse of involution. It is the process of finding the root of a number which is considered as a power.

255. The **square root** of a number is that number which, when used twice as a factor, produces the number.

Thus, 2 is the square root of 4, since 2×2, or 2^2, $= 4$.

256. The **cube root** of a number is that number which, when used three times as a factor, produces the number.

Thus, 3 is the cube root of 27, since $3 \times 3 \times 3$, or 3^3, $=27$.

257. The **radical sign** $\sqrt{}$, when placed before a number, indicates that some root of that number is to be found.

258. The **index** of the root is a *small figure* placed *over* and to the *left* of the *radical sign*, to show what root is to be found.

Thus, $\sqrt[2]{100}$ denotes the *square root* of 100.
$\sqrt[3]{125}$ denotes the *cube root* of 125.
$\sqrt[4]{256}$ denotes the *fourth root* of 256, and so on.

259. When the square root is to be extracted, the index is generally omitted. Thus, $\sqrt{100}$ indicates the square root of 100. Also, $\sqrt{225}$ indicates the square root of 225.

SQUARE ROOT.

260. The *largest* number that can be written with *one* figure is 9, and $9^2 = 81$; the *largest* number that can be written with *two* figures is 99, and $99^2 = 9,801$; with *three* figures 999, and $999^2 = 998,001$; with *four* figures 9,999, and $9,999^2 = 99,980,001$, etc.

In *each* of the above it will be noticed that the square of the number contains just *twice* as many figures as the number.

In order to find the square root of a number, the first step is to find how many figures there will be in the root. This is done by pointing off the number into *periods* of *two* figures each, *beginning at the right.* The number of periods will indicate the number of figures in the root.

Thus, the square root of 83,740,801 must contain 4 figures, since, pointing off the periods, we get 83'74'08'01, or 4 periods; consequently, there must be 4 figures in the root. In like manner, the square root of 50,625 must contain 3 figures, since there are (5'06'25) 3 periods.

261. Example.—Find the square root of 31,505,769.

					root.
Solution.—	(*a*)	5		31505769(	5613 Ans.
		5	(*b*)	25	
	(*d*)	100	(*c*)	650	
		6		636	
		106	(*e*)	1457	
		6		1121	
		1120		33669	
		1		33669	
		1121		0	
		1			
		11220			
		3			
		11223			

Explanation.—Pointing off into periods of two figures each, it is seen that there are four figures in the root. Now, find the largest single number whose square is less than or equal to 31, the first period. This is evidently 5, since $6^2 = 36$, which is greater than 31. Write it to the right, as in long division, and also to the left, as shown at (*a*). This is the first figure of the root. Now, multiply the 5 at (*a*) by the 5 in the root, and write the result under the first period, as shown at (*b*). Subtract, and obtain 6 as a remainder.

Bring down the next period 50, and annex it to the remainder 6, as shown at (*c*), which we call the **dividend.** Add the root already found to the 5 at (*a*), getting 10, and annex a cipher to this 10, thus making it 100, which we call the **trial divisor.** Divide the dividend (*c*) by the trial divisor (*d*), and obtain 6, which is *probably* the next figure of the root. Write 6 in the root, as shown, and also add it to 100, the trial divisor, making it 106. This is called the **complete divisor.**

Multiply this by 6, the second figure in the root, and subtract the result from the dividend (*c*). The remainder is 14, to which annex the next period, making it 1,457, as shown at (*c*), which we call the **new dividend.** Add the second figure of the root to the trial divisor 106, and annex a cipher, thus getting 1,120. Dividing 1,457 by 1,120, we get 1 as the next figure of the root. Adding this last figure of the root to 1,120, multiplying the result by it, and subtracting from 1,457, the remainder is 336.

Annexing the next and last period, 69, the result is 33,669. Now, adding the last figure of the root to 1,121, and annexing a cipher as before, the result is 11,220. Dividing 33,669 by 11,220, the result is 3, the fourth figure in the root. Adding it to 11,220, and multiplying the sum by it, the result is 33,669. Subtracting, there is no remainder; hence, $\sqrt{31,505,769} =$ 5,613. Ans.

262. The square of any number wholly decimal always contains twice as many figures as the number squared. For example, $.1^2 = .01$; $.13^2 = .0169$; $.751^2 = .564001$, etc.

263. It will also be noticed that the number squared is always less than the decimal. Hence, if it be required to find the square root of a decimal, and the decimal has not an even number of figures in it, annex a cipher. The best way to determine the number of figures in the root of a decimal is to begin at the decimal point, and, going towards the *right*, point off the decimal into periods of two figures each. Then, if the last period contains but one figure, annex a cipher.

264. EXAMPLE.—What is the square root of .000576 ?

		root	
SOLUTION.—	2	.00'05'76(.024	Ans.
	2	4	
	40	176	
	4	176	
	44	0	

Explanation.—Beginning at the decimal point, and pointing off the number into periods of two figures each, it is seen that the first period is composed of ciphers; hence, the first figure of the root must be a cipher. The remaining portion of the solution should be perfectly clear from what has preceded.

265. If the number is not a perfect power, the root will consist of an interminable number of decimal places. The result may be carried to any required number of decimal places by annexing periods of two ciphers each to the number.

266. Example.—What is the square root of 3? Find the result to five decimal places

Solution.—

$$\begin{array}{rl}
 & \hspace{6em}\textit{root} \\
1 & 3.00'00'00'00'00(1.73205+ \quad \text{Ans.} \\
\underline{1} & \underline{1} \\
20 & 200 \\
\underline{7} & \underline{189} \\
27 & 1100 \\
\underline{7} & \underline{1029} \\
340 & 7100 \\
\underline{3} & \underline{6924} \\
343 & 1760000 \\
\underline{3} & \underline{1732025} \\
3460 & 27975 \\
\underline{2} & \\
3462 & \\
\underline{2} & \\
346400 & \\
\underline{5} & \\
346405 &
\end{array}$$

Explanation.—Annexing five periods of two ciphers each to the right of the decimal point, the first figure of the root is 1. To get the second figure, we find that, in dividing 200 by 20, it is 10. This is evidently too large.

Trying 9, we add 9 to 20, and multiply 29 by 9, the result is 261, a result which is considerably larger than 200; hence, 9 is too large. In the same way it is found that 8 is also too large. Trying 7, 7 times 27 is 189, a result smaller than 200; therefore, 7 is the second figure of the root. The next two figures, 3 and 2, are easily found. The fifth figure in the root is a cipher, since the trial divisor 34,640 is greater than the new dividend 17,600. In a case of this kind, we annex another cipher to 34,640, thereby making it 346,400, and bring down the next period, making the 17,600, 1,760,000. The next figure of the root is 5, and as we now have five decimal places, we will stop.

The square root of 3 is, then, 1.73205 +. Ans.

267. EXAMPLE.—What is the square root of .3 to five decimal places?

```
                                                  root
SOLUTION.—   5            .30'00'00'00'00( .54772+  Ans.
             5             25
           ---             --
           100              500
             4              416
           ---              ---
           104               8400
             4               7609
           ----              ----
           1080               79100
              7               76629
           ----               -----
           1087                247100
              7                219084
           -----               ------
           10940                28016
               7
           -----
           10947
               7
           ------
           109540
                2
           ------
           109542
```

EXPLANATION.—In the above example, we annex a cipher to .3, making the first period .30, since every period of a decimal, as was mentioned before, must have two figures in it. The remainder of the work should be perfectly clear.

268. If it is required to find the square root of a mixed number, begin at the decimal point, and point off the periods both ways. The manner of finding the root will then be exactly the same as in the previous cases.

269. Example.—What is the square root of 258.2449?

```
SOLUTION.—        1       2'58.24'49(16.07  Ans.
                  1       1
                  -       -
                  20      158
                   6      156
                  --      ---
                  26        22449
                   6        22449
                  --        -----
                  3200          0
                     7
                  ----
                  3207
```

Explanation.—In the above example, since 320 is greater than 224, we place a cipher for the third figure of the root, and annex a cipher to 320, making it 3,200. Then, bringing down the next period 49, 7 is found to be the fourth figure of the root. Since there is no remainder, the square root of 258.2449 is 16.07. Ans.

270. Proof.—*To prove square root, square the result obtained. If the number is an exact power, the square of the root will equal it; if it is not an exact power, the square of the root will very nearly equal it.*

271. Rule I.—*Begin at units place, and separate the number into periods of two figures each, proceeding from left to right with the decimal part, if there is any.*

II. *Find the greatest number whose square is contained in the first or left-hand period. Write this number as the first figure in the root; also, write it at the left of the given number.*

Multiply this number at the left by the first figure of the root, and subtract the result from the first period; then annex the second period to the remainder.

III. *Add the first figure of the root to the number in the first column on the left, and annex a cipher to the result; this is the trial divisor. Divide the dividend by the trial divisor* *for the second figure in the root, and add this figure to the trial divisor to form the complete divisor. Multiply the complete divisor by the second figure in the root, and subtract this result from the dividend. (If this result is larger than the dividend, a smaller number must be tried for the second figure of the root.) Now bring down the third period, and annex it to the last remainder for a new dividend. Add the second figure of the root to the complete divisor, and annex a cipher for a new trial divisor.*

IV. *Continue in this manner to the last period, after which, if any additional places in the root are required, bring down cipher periods, and continue the operation.*

V. *If at any time the trial divisor is not contained in the dividend, place a cipher in the root, annex a cipher to the trial divisor, and bring down another period.*

VI. *If the root contains an interminable decimal, and it is desired to terminate the operation at some point, say, the fourth decimal place, carry the operation one place further, and if the fifth figure is 5 or greater, increase the fourth figure by 1 and omit the sign +.*

272. Short Method.—If the number whose root is to be extracted is not an exact square, the root will be an interminable decimal. It is then usual to extract the root to a certain number of decimal places. In such cases, the work may be greatly shortened as follows: Determine to how many decimal places the work is to be carried, say 5, for example; add to this the number of places in the integral part of the root, say 2, for example, thus determining the number of figures in the root, in this case $5 + 2 = 7$. Divide this number by 2 and take the next higher number. In the above case, we have $7 \div 2 = 3\frac{1}{2}$; hence, we take 4, the next higher number. Now extract the root in the usual manner until the same number of figures have been obtained as was expressed by the number obtained above, in this case 4. Then form the trial divisor in the usual manner, but omitting to annex the cipher; divide the last remainder by the trial divisor, as in long division, obtaining as many figures of the quotient as there are remaining figures of the root, in this case 7 − 4 = 3. The remainder so obtained is the remaining figures of the root.

Consider the example in Art. **267.** Here there are 5 figures in the root. We therefore extract the root to 3 places in the usual manner, obtaining .547 for the first three root figures. The next trial divisor is 1,094 (with the cipher omitted), and the last remainder is 791. Then, 791 ÷ 1,094 = .723, and the next two figures of the root are 72, the whole root being .54772 +. Always carry the division one place further than desired, and if the last figure is 5 or greater, increase the preceding figure by 1. This method should not be used unless the root contains five or more figures.

Note.—If the last figure of the root found in the regular manner is a cipher, carry the process one place further before dividing as described above.

EXAMPLES FOR PRACTICE.

273. Find the square root of

		Ans.
(*a*)	186,624.	(*a*) 432.
(*b*)	2,050,624.	(*b*) 1,432.
(*c*)	29,855,296.	(*c*) 5,464.
(*d*)	.0116964.	(*d*) .1081 +.
(*e*)	198.1369.	(*e*) 14.0761.
(*f*)	994,009.	(*f*) 997.
(*g*)	2.375 to four decimal places.	(*g*) 1.5411.
(*h*)	1.625 to three decimal places.	(*h*) 1.275.
(*i*)	.3025.	(*i*) .55.
(*j*)	.571428.	(*j*) .7559 +.
(*k*)	.78125.	(*k*) .8839.

CUBE ROOT.

274. In the same manner as in the case of square root, it can be shown that the periods into which a number is divided, whose cube root is to be extracted, must contain three figures, except that the first or left-hand period of a whole or mixed number may contain one, two, or three figures.

275. Example.—What is the cube root of 375,741,853,696?

Solution.—

(1)	(2)	(3) *root*
7	49	375'741'853'696(7216 Ans.
7	98	343
14	14700	32741
7	424	30248
210	15124	2493853
2	428	1557361
212	1555200	936492696
2	2161	936492696
214	1557361	0
2	2162	
2160	155952300	
1	129816	
2161	156082116	
1		
2162		
1		
21630		
6		
21636		

Explanation.—Write the work in three columns as follows: On the right place the number whose cube root is to be extracted, and point it off into periods of three figures each. Call this column (3). Find the largest number whose cube is less than or equal to the first period, in this case 7. Write the 7 on the right, as shown, for the first figure of the root, and also on the extreme left at the head of column (1). Multiply the 7 in column (1) by the first figure of the root 7, and write the product 49 at the head of column (2). Multiply the number in column (2) by the first figure of the root 7, and write the product 343 under the figures in the first period. Subtract and bring down the next period, obtaining 32,741 for the dividend. Add the first figure of the root to the number in column (1), obtaining 14, which call the *first correction*. Multiply the first correction by the first figure of the root, add the product to the number in column (2), and obtain 147. Add the first figure of the root to the first correction, and obtain 21, which call the *second correction*. Annex *two* ciphers to the number in column (2), and obtain 14,700 for the trial divisor; also, annex *one* cipher to the second correction, and obtain 210. Dividing the dividend by the trial divisor, we obtain $\frac{32741}{14700} = 2 +$, and write the 2 as the second figure of the root. Add the 2 to the second correction, and obtain 212, which, multiplied by the second figure of the root, and added to the trial divisor, gives 15,124, the complete divisor. This last result, multiplied by the second figure of the root and subtracted from the dividend, gives a remainder of 2,493. Annexing the third period, we obtain 2,493,853 for the new dividend. Adding the second figure of the root to the number in column (1), we get 214 as the new first correction; this, multiplied by the second figure of the root and added to the trial divisor, gives 15,552. Adding the second figure of the root to the first new correction gives 216 as the second new correction. Annexing two ciphers to the number in column (2) gives 1,555,200, the new trial divisor. Annexing one cipher to the second new correction gives 2,160. Dividing the new dividend by the new trial divisor, we obtain $\frac{2493853}{1555200} = 1 +$, and write 1 as the third figure of the root. The remainder of the work should be perfectly clear from what has preceded.

276. In extracting the cube root of a decimal, proceed as above, taking care that each period contains *three* figures. Begin the pointing off at the decimal point, going towards the right. If the last period does not contain three figures, annex ciphers until it does.

277. EXAMPLE.—What is the cube root of .009129329?

SOLUTION.—

		root
2	4	.009'129'329(.209
2	8	8
4	120000	1129329
2	5481	1129329
600	125481	0
9		
609		

EXPLANATION.—Beginning at the decimal point, and pointing off as shown, the largest number whose cube is less than 9 is seen to be 2; hence, 2 is the first figure of the root. When finding the second figure, it is seen that the trial divisor 1,200 is greater than the dividend; hence, write a cipher for the second figure of the root; bring down the next period to form the new dividend; annex two ciphers to the trial divisor to form a new trial divisor; also, annex one cipher to the 60 in column (1). Dividing the new dividend by the new trial divisor, we get $\frac{1129329}{120000} = 9 +$, and write 9 as the third figure of the root. Complete the work as before.

278. EXAMPLE.—What is the cube root of 78,292.892952?

SOLUTION.—

		root
4	16	78'292.892'952 (42.78
4	32	64
8	4800	14292
4	244	10088
120	5044	4204892
2	248	3766483
122	529200	438409952
2	8869	438409952
124	538069	0
2	8918	
1260	54698700	
7	102544	
1267	54801244	
7		
1274		
7		
12810		
8		
12818		

Explanation.—Since we have a mixed number, begin at the decimal point and point off periods of three figures each, in both directions. The first period contains but two figures, and the largest number whose cube is less than 78 is 4; consequently, 4 is the first figure of the root. The remainder of the work should be perfectly clear. When dividing the dividend by the trial divisor for the third figure of the root, the quotient was 8 +; but, on trying it, it was found that 8 was too large, the complete divisor being considerably larger than the trial divisor. Therefore, 7 was used instead of 8.

279. Example.—What is the cube root of 5 to five decimal places?

Solution.—		*root*
1	1	5.000'000'000'000'000(1.70997+
1	2	1
2	300	4000
1	259	3913
30	559	87000000
7	308	78443829
37	8670000	8556171000
7	45981	7889992299
44	8715981	666178701000
7	46062	614014317973
5100	876204300	52164383027
9	461511	
5109	876665811	
9	461592	
5118	87712740300	
9	3590839	
51270	87716331139	
9		
51279		
9		
51288		
9		
512970		
7		
512977		

Explanation.—In the preceding example we annex five periods of ciphers, of three ciphers each, to the 5 for the decimal part of the root, placing the decimal point between the 5 and the first cipher. Since it is easy to see that the next figure of the root will be 5, we increase the last figure by 1, obtaining 1.70998 for the correct root to 5 decimal places. Ans.

280. Example.—What is the cube root of .5 to four decimal places?

Solution.—

```
                                                   root
    7         49          .500'000'000'000(.7937+
    7         98           343
   --         --           ---
   14       14700          157000
    7        1971          150039
   --       -----          ------
  210       16671             6961000
    9        2052             5638257
  ---       -----             -------
  219       1872300           1322743000
    9          7119           1321748953
  ---       -------           ----------
  228       1879419               994047
    9          7128
  ---       -------
 2370       188654700
    3          166579
 ----       ---------
 2373       188821279
    3
 ----
 2376
    3
 ----
23790
    7
-----
23797
```

Explanation.—In the above example we annex two ciphers to the .5 to complete the first period, and three periods of three ciphers each. The cube root of 500 is 7; this we write as the first figure of the root. The remainder of the work should be perfectly plain from the explanations of the preceding examples.

281. Example.—What is the cube root of .05 to four decimal places?

Solution.—

		root
3	9	.050'000'000'000(.3684+
3	18	27
6	2700	23000
3	576	19656
90	3276	3344000
6	612	3180032
96	388800	163968000
6	8704	162685504
102	397504	1282496
6	8768	
1080	40627200	
8	44176	
1088	40671376	
8		
1096		
8		
11040		
4		
11044		

282. Proof.—*To prove cube root, cube the result obtained. If the given number is an exact power, the cube of the root will equal it; if not an exact power, the cube of the root will very nearly equal it.*

283. Rule I.—*Arrange the work in three columns, placing the number whose cube root is to be extracted in the third or right-hand column. Begin at units place, and separate the number into periods of three figures each, proceeding from the decimal point towards the right with the decimal part, if there is any.*

II. *Find the greatest number whose cube is not greater than the number in the first period. Write this number as the first figure of the root; also, write it at the head of the first column. Multiply the number in the first column by the first figure in the root, and write the result in the second* *column. Multiply the number in the second column by the first figure of the root; subtract the product from the first period, and annex the second period to the remainder for a new dividend; add the first figure of the root to the number in the first column for the first correction. Multiply the first correction by the first figure of the root, and add the product to the number in the second column. Add the first figure of the root to the first correction to form the second correction. Annex one cipher to the second correction, and two ciphers to the last number in the second column; the last number in the second column is the trial divisor.*

III. *Divide the dividend by the trial divisor to find the second figure of the root. Add the second figure of the root to the number in the first column, multiply the sum by the second figure of the root, and add the result to the trial divisor to form the complete divisor. Multiply the complete divisor by the second figure of the root, subtract the result from the dividend in the third column, and annex the third period to the remainder for a new dividend. Add the second figure of the root to the number in the first column to form the first correction; multiply the first correction by the second figure of the root, and add the product to the complete divisor. Add the second figure of the root to the first correction to form the second correction. Annex one cipher to the second correction, and two ciphers to the last number in the second column to form the new trial divisor.*

IV. *If there are more periods to be brought down, proceed as before. If there is a remainder after the root of the last period has been found, annex cipher periods, and proceed as before. The figures of the root thus obtained will be decimals.*

V. *If the root contains an interminable decimal, and it is desired to terminate the operation at some point, say the fourth decimal place, carry the operation one place further, and if the fifth figure is 5 or greater, increase the fourth figure by 1 and omit the sign +.*

284. Art. **272** can be applied to cube root (or any other root) as well as to square root. Thus, in the example, Art. **279**, there are to be $5 + 1 = 6$ figures in the root. Extracting the root in the usual manner to $6 \div 2 = 3$, say 4 figures, we get for the first four figures 1,709. The last remainder is 8,556,171, and the next trial divisor, with the ciphers omitted, is 8,762,043. Hence, the next two figures of the root are $8,556,171 \div 8,762,043 = .976$, say .98. Therefore, the root is 1.70998.

ROOTS OF FRACTIONS.

285. If the given number is in the form of a fraction, and it is required to find some root of it, the simplest and most exact method is to reduce the fraction to a decimal and extract the required root of the decimal. If, however, the numerator and denominator of the fraction are perfect powers, extract the required root of each separately, and write the root of the numerator for a new numerator, and the root of the denominator for a new denominator.

286. Example.—What is the square root of $\frac{9}{64}$?

Solution.— $\sqrt{\frac{9}{64}} = \frac{\sqrt{9}}{\sqrt{64}} = \frac{3}{8}$. Ans.

287. Example.—What is the square root of $\frac{5}{8}$?

Solution.—Since $\frac{5}{8} = .625$, $\sqrt{\frac{5}{8}} = \sqrt{.625} = .7906$. Ans.

288. Example.—What is the cube root of $\frac{27}{64}$?

Solution.— $\sqrt[3]{\frac{27}{64}} = \frac{\sqrt[3]{27}}{\sqrt[3]{64}} = \frac{3}{4}$. Ans.

289. Example.—What is the cube root of $\frac{1}{4}$?

Solution.—Since $\frac{1}{4} = .25$, $\sqrt[3]{\frac{1}{4}} = \sqrt[3]{.25} = .62996 +$. Ans.

290. **Rule.**—*Extract the required root of the numerator and denominator separately; or, reduce the fraction to a decimal, and extract the root of the decimal.*

EXAMPLES FOR PRACTICE.

291. Find the cube root of

(*a*) $\frac{27}{512}$.
(*b*) 2 to five decimal places.
(*c*) 4,180,769,192.462 to five decimal places.
(*d*) $\frac{87}{125}$.
(*e*) $\frac{3}{8}$.
(*f*) 513,229.783302144 to three decimal places.

Ans.
(*a*) $\frac{3}{8}$.
(*b*) 1.25992 +.
(*c*) 1,610.96238.
(*d*) .8862+.
(*e*) .7211 +.
(*f*) 80.064.

TO EXTRACT OTHER ROOTS THAN THE SQUARE AND CUBE ROOTS.

292. Example.—What is the fourth root of 256?

Solution.— $\sqrt{256} = 16.$

$\sqrt{16} = 4.$

Therefore, $\sqrt[4]{256} = 4.$ Ans.

In this example, $\sqrt[4]{256}$, the index is 4, which equals 2×2. The root indicated by 2 is the square root; therefore, the square root is extracted twice.

293. Example.—What is the sixth root of 64?

Solution.— $\sqrt{64} = 8.$

$\sqrt[3]{8} = 2.$

Therefore, $\sqrt[6]{64} = 2.$ Ans.

In this example, $\sqrt[6]{64}$, the index is 6, which equals 2×3. The root indicated by 3 is the cube root; therefore, the square and cube roots are extracted in succession.

294. Rule.—*Separate the index of the required root into its factors (2's and 3's), and extract successively the roots indicated by the several factors obtained. The final result will be the required root.*

295. Example.—What is the sixth root of 92,873,580 to two decimal places?

Solution.— $6 = 3 \times 2$. Hence, extract the cube root, and then extract the square root of the result. $\sqrt[3]{92{,}873{,}580} = 452.8601$, and $\sqrt{452.8601} = 21.28 +$. Ans.

296. It matters not which root is extracted first, but it is probably easier and more exact to extract the cube root first.

EXAMPLES FOR PRACTICE.

297. Extract the

(*a*) Fourth root of 100.	Ans.	(*a*) 3.16227+.
(*b*) Fourth root of 3,049,800,625.		(*b*) 235.
(*c*) Sixth root of 9,474,296,896.		(*c*) 46.

RATIO.

298. Suppose that it is desired to compare two numbers, say 20 and 4. If we wish to know how many times larger 20 is than 4, we divide 20 by 4 and obtain 5 for the quotient; thus, $20 \div 4 = 5$. Hence, we say that 20 is 5 times as large as 4, i. e., 20 contains 5 times as many units as 4. Again, suppose we desire to know what part of 20 is 4. We then divide 4 by 20 and obtain $\frac{1}{5}$; thus, $4 \div 20 = \frac{1}{5}$, or .2. Hence, 4 is $\frac{1}{5}$ or .2 of 20. This operation of comparing two numbers is termed *finding the ratio* of the two numbers. Ratio, then, is a comparison. It is evident that the two numbers to be compared must be expressed in the same unit; in other words, the two numbers must both be abstract numbers or concrete numbers of the same kind. For example, it would be absurd to compare 20 horses with 4 birds, or 20 horses with 4. Hence, **ratio** may be defined as a comparison between two numbers of the same kind.

299. A ratio may be *expressed* in three ways; thus, if it is desired to compare 20 and 4, and express this comparison as a ratio, it may be done as follows: $20 \div 4$; $20 : 4$, or $\frac{20}{4}$. All three are read *the ratio of 20 to 4*. The ratio of 4 to 20 would be expressed thus: $4 \div 20$; $4 : 20$, or $\frac{4}{20}$. The first method of expressing a ratio, although correct, is seldom or never used; the second form is the one oftenest met with, while the third is rapidly growing in favor, and is likely to supersede the second. The third form, called the fractional form, is preferred by modern mathematicians, and possesses great advantages to students of Algebra and of higher mathematical subjects. The second form seems to be better adapted to arithmetical subjects, and is one we shall ordinarily adopt. There is still another way of expressing a ratio, though seldom or never used in the case of a simple ratio like that given above. Instead of the colon, a straight vertical line is used; thus, 20 | 4.

300. The **terms** of a ratio are the two numbers to be compared; thus, in the above ratio, 20 and 4 are the terms. When both terms are considered together, they are called a **couplet;** when considered separately, the first term is called the **antecedent,** and the second term the **consequent.** Thus, in the ratio 20 : 4, 20 and 4 form a couplet, and 20 is the antecedent, and 4 the consequent.

301. A ratio may be **direct** or **inverse.** The *direct ratio* of 20 to 4 is 20 : 4, while the *inverse ratio* of 20 to 4 is 4 : 20. The direct ratio of 4 to 20 is 4 : 20, and the inverse ratio is 20 : 4. An inverse ratio is sometimes called a **reciprocal** ratio. The **reciprocal** of a number is 1 divided by the number. Thus, the reciprocal of 17 is $\frac{1}{17}$; of $\frac{3}{8}$ is $1 \div \frac{3}{8} = \frac{8}{3}$; i.e., the reciprocal of a fraction is the fraction inverted. Hence, the inverse ratio of 20 to 4 may be expressed as 4 : 20, or as $\frac{1}{20} : \frac{1}{4}$. Both have equal values; for, $4 \div 20 = \frac{1}{5}$, and $\frac{1}{20} \div \frac{1}{4} = \frac{1}{20} \times \frac{4}{1} = \frac{1}{5}$.

302. The term **vary** implies a ratio. When we say that two numbers vary as some other two numbers, we mean that the ratio between the first two numbers is the same as the ratio between the other two numbers.

303. The **value** of a ratio is the result obtained by performing the division indicated. Thus, the value of the ratio 20:4 is 5, it is the quotient obtained by dividing the antecedent by the consequent.

304. By expressing the ratio in the fractional form, for example, the ratio of 20 to 4 as $\frac{20}{4}$, it is easy to see, from the laws of fractions, that if both terms be multiplied, or both divided by the same number, it will not alter the value of the ratio. Thus,

$$\frac{20}{4} = \frac{20 \times 5}{4 \times 5} = \frac{100}{20}; \text{ and } \frac{20}{4} = \frac{20 \div 4}{4 \div 4} = \frac{5}{1}.$$

305. It is also evident, from the laws of fractions, that multiplying the antecedent or dividing the consequent multiplies the ratio; and dividing the antecedent or multiplying the consequent divides the ratio.

306. When a ratio is expressed in words, as the ratio of 20 to 4, the first number named is always regarded as the antecedent and the second as the consequent, without regard to whether the ratio itself is direct or inverse. *When not otherwise specified, all ratios are understood to be direct.* To express an inverse ratio, the simplest way of doing it is to express it as if it were a direct ratio, with the first number named as the antecedent, and then transpose the antecedent to the place occupied by the consequent and the consequent to the place occupied by the antecedent; or if expressed in the fractional form, invert the fraction. Thus, to express the inverse ratio of 20 to 4, first write it 20 : 4, and then, transposing the terms, as 4 : 20; or as $\frac{20}{4}$, and then inverting as $\frac{4}{20}$. Or, the reciprocals of the numbers may be taken, as explained above. To **invert** a ratio is to transpose its terms.

EXAMPLES FOR PRACTICE.

307. What is the value of the ratio of

	Ans.
(*a*) 98 to 49?	(*a*) 2.
(*b*) \$45 to \$9?	(*b*) 5.
(*c*) $6\frac{1}{4}$ to $\frac{2}{4}$?	(*c*) $12\frac{1}{2}$.
(*d*) 3.5 to 4.5?	(*d*) $.77\frac{7}{9}$.
(*e*) The inverse ratio of 76 to 19?	(*e*) $\frac{1}{4}$.
(*f*) The inverse ratio of 49 to 98?	(*f*) 2.
(*g*) The inverse ratio of 18 to 24?	(*g*) $1\frac{1}{3}$.
(*h*) The inverse ratio of 9 to 15?	(*h*) $1\frac{2}{3}$.
(*i*) The ratio of 10 to 3, multiplied by 3?	(*i*) 10.
(*j*) The ratio of 35 to 49, multiplied by 7?	(*j*) 5.
(*k*) The ratio of 18 to 64, divided by 9?	(*k*) $\frac{1}{32}$.
(*l*) The ratio of 14 to 28, divided by 5?	(*l*) $\frac{1}{10}$.

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308. Instead of expressing the value of a ratio by a single number, as above, it is customary to express it by means of another ratio in which the consequent is 1. Thus, suppose that it is desired to find the ratio of the weights of two pieces of iron, one weighing 45 pounds and the other weighing 30 pounds. The ratio of the heavier to the lighter is then 45 : 30, an inconvenient expression. Using the fractional form, we have $\frac{45}{30}$. Dividing both terms by 30, the consequent, we obtain $\frac{1\frac{1}{2}}{1}$ or $1\frac{1}{2} : 1$. This is the same result as obtained above, for $1\frac{1}{2} \div 1 = 1\frac{1}{2}$, and $45 \div 30 = 1\frac{1}{2}$.

309. A ratio may be squared, cubed, or raised to any power, or any root of it may be taken. Thus, if the ratio of two numbers is 105 : 63, and it is desired to cube this ratio, the cube may be expressed as $105^3 : 63^3$. That this is correct is readily seen; for, expressing the ratio in the fractional form, it becomes $\frac{105}{63}$, and the cube is $\left(\frac{105}{63}\right)^3 = \frac{105^3}{63^3} = 105^3 : 63^3$. Also, if it is desired to extract the cube root of the ratio $105^3 : 63^3$, it may be done by simply dividing the exponents by 3, obtaining 105 : 63. This may be proved in the same way as in the case of cubing the ratio. Thus, $105^3 : 63^3 = \left(\frac{105}{63}\right)^3$, and $\sqrt[3]{\left(\frac{105}{63}\right)^3} = \frac{105}{63} = 105 : 63$.

310. Since $\left(\frac{105}{63}\right)^3 = \left(\frac{5}{3}\right)^3$, it follows that $105^3 : 63^3 = 5^3 : 3^3$ (this expression is read: the ratio of 105 cubed to 63 cubed equals the ratio of 5 cubed to 3 cubed), it follows that the antecedent and consequent may always be multiplied or divided by the same number, irrespective of any indicated powers or roots, without altering the value of the ratio. Thus, $24^2 : 18^2 = 4^2 : 3^2$. For, performing the operations indicated by the exponents, $24^2 = 576$ and $18^2 = 324$. Hence, $576 : 324 = 1\frac{7}{9}$ or $1\frac{7}{9} : 1$. Also, $4^2 = 16$ and $3^2 = 9$; hence, $16 : 9 = 1\frac{7}{9}$ or $1\frac{7}{9} : 1$, the same result as before. Also, $24^2 : 18^2 = \frac{24^2}{18^2} = \left(\frac{24}{18}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2} = 4^2 : 3^2$.

The statement may be proved for roots in the same manner. Thus, $\sqrt[3]{24^3} : \sqrt[3]{18^3} = \sqrt[3]{4^3} : \sqrt[3]{3^3}$. For the $\sqrt[3]{24^3} = 24$ and $\sqrt[3]{18^3} = 18$; and, $24 : 18 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$. Also, $\sqrt[3]{4^3} = 4$ and $\sqrt[3]{3^3} = 3$; $4 : 3 = 1\frac{1}{3}$ or $1\frac{1}{3} : 1$.

NOTE.—If the numbers composing the antecedent and consequent have different exponents, or if different roots of those numbers are indicated, the operations described in Art. **310** cannot be performed. This is evident; for, consider the ratio $4^2 : 8^3$. When expressed in the fractional form, it becomes $\frac{4^2}{8^3}$, which cannot be expressed either as $\left(\frac{4}{8}\right)^2$ or as $\left(\frac{4}{8}\right)^3$, and, hence, cannot be reduced as described above.

PROPORTION.

311. **Proportion** is an equality of ratios, the equality being indicated by the double colon (: :) or by the sign of equality (=). Thus, to write in the form of a proportion the two equal ratios, 8 : 4 and 6 : 3, which both have the same value 2, we may employ one of the three following forms:

$$8 : 4 :: 6 : 3 \qquad (1)$$

$$8 : 4 = 6 : 3 \qquad (2)$$

$$\frac{8}{4} = \frac{6}{3} \qquad (3)$$

312. The first form is the one most extensively used, by reason of its having been exclusively employed in all the older works on mathematics. The second and third forms are being adopted by all modern writers on mathematical subjects, and, in time, will probably entirely supersede the first form. In this paper we shall adopt the second form, unless some statement can be made clearer by using the third form.

313. A proportion may be *read* in two ways. The old way to read the above proportion was—*8 is to 4 as 6 is to 3*; the new way is—*the ratio of 8 to 4 equals the ratio of 6 to 3.* The student may read it either way, but we recommend the latter.

314. Each ratio of a proportion is termed a **couplet.** In the above proportion, 8 : 4 is a couplet, and so is 6 : 3.

315. The numbers forming the proportion are called **terms;** and they are numbered consecutively from left to right, thus:

first second third fourth

$$8 : 4 = 6 : 3$$

Hence, in any proportion, the ratio of the first term to the second term equals the ratio of the third term to the fourth term.

316. The first and fourth terms of a proportion are called the **extremes,** and the second and third terms, the **means.** Thus, in the foregoing proportion, 8 and 3 are the extremes and 4 and 6 are the means.

317. A **direct proportion** is one in which both couplets are direct ratios.

318. An **inverse proportion** is one which requires one of the couplets to be expressed as an inverse ratio. Thus, 8 is to 4 inversely as 3 is to 6 must be written $8 : 4 = 6 : 3$; i. e., the second ratio (couplet) must be inverted.

319. Proportion forms one of the most useful sections of Arithmetic. In our grandfathers' Arithmetics, it was called "The rule of three."

320. Rule I.—*In any proportion, the product of the extremes equals the product of the means.*

Thus, in the proportion,

$$17 : 51 = 14 : 42.$$

$17 \times 42 = 51 \times 14$, since both products equal 714.

321. Rule II.—*The product of the extremes divided by either mean gives the other mean.*

Example.—What is the third term of the proportion $17 : 51 = \quad : 42$?

Solution.—Applying rule II, $17 \times 42 = 714$, and $714 \div 51 = 14$. Ans.

322. Rule III.—*The product of the means divided by either extreme gives the other extreme.*

Example.—What is the first term of the proportion $\quad : 51 = 14 : 42$?

Solution.—Applying rule III, $51 \times 14 = 714$, and $714 \div 42 = 17$. Ans.

323. When stating a proportion in which one of the terms is unknown, represent the missing term by a letter, as x. Thus, the last example would be written,

$$x : 51 = 14 : 42$$

and for the value of x we have $x = \frac{51 \times 14}{42} = 17$.

324. If the same (addition and subtraction excepted) operations be performed upon *all* of the terms of a proportion, the proportion is not thereby destroyed. In other words, if all of the terms of a proportion be (1) multiplied or (2) divided by the same number; (3) if all the terms be raised to the same power; if (4) the same root of all the terms be taken, or (5) if both couplets be inverted, the proportion still holds. We will prove these statements by a numerical example, and the student can satisfy himself by other similar ones. The fractional form will be used, as it is better suited to the purpose. Consider the proportion $8 : 4 = 6 : 3$. Expressing it in the third form, it becomes $\frac{8}{4} = \frac{6}{3}$. What we are to prove is that, if any of the five operations enumerated above be performed upon all of the terms of this proportion, the first fraction will still equal the second fraction.

1. Multiplying all the terms by any number, say 7, $\frac{8 \times 7}{4 \times 7} = \frac{6 \times 7}{3 \times 7}$; or $\frac{56}{28} = \frac{42}{21}$. Now $\frac{56}{28}$ evidently equals $\frac{42}{21}$, since the value of either ratio is 2, and the same is true of the original proportion.

2. Dividing all the terms by any number, say 7, $\frac{8 \div 7}{4 \div 7} = \frac{6 \div 7}{3 \div 7}$; or $\frac{\frac{8}{7}}{\frac{4}{7}} = \frac{\frac{6}{7}}{\frac{3}{7}}$. But $\frac{8}{7} \div \frac{4}{7} = 2$, and $\frac{6}{7} \div \frac{3}{7} = 2$ also, the same as in the original proportion.

3. Raising all the terms to the same power, say the cube, $\frac{8^3}{4^3} = \frac{6^3}{3^3}$. This is evidently true, since $\frac{8^3}{4^3} = \left(\frac{8}{4}\right)^3 = 2^3 = 8$, and $\frac{6^3}{3^3} = \left(\frac{6}{3}\right)^3 = 2^3 = 8$ also.

4. Extracting the same root of all the terms, say the cube root, $\frac{\sqrt[3]{8}}{\sqrt[3]{4}} = \frac{\sqrt[3]{6}}{\sqrt[3]{3}}$. It is evident that this is likewise true, since $\frac{\sqrt[3]{8}}{\sqrt[3]{4}} = \sqrt[3]{\frac{8}{4}} = \sqrt[3]{2}$, and $\frac{\sqrt[3]{6}}{\sqrt[3]{3}} = \sqrt[3]{\frac{6}{3}} = \sqrt[3]{2}$ also.

5. Inverting both couplets, $\frac{4}{8} = \frac{3}{6}$, which is true, since both equal $\frac{1}{2}$.

325. If both terms of either couplet be multiplied or both divided by the same number, the proportion is not destroyed. This should be evident from the preceding article, and also from Art. **304.** Hence, in any proportion, equal factors may be canceled from the terms of a couplet, before applying rules II or III. Thus, the proportion $45 : 9 = x : 7.1$, we may divide both terms of the first couplet by 9 (that is, cancel 9 from both terms), obtaining $5 : 1 = x : 7.1$, whence $x = 7.1 \times 5 \div 1 = 35.5$. (See note in Art. **310.**)

326. The principle of all calculations in proportion is this: *Three of the terms are always given, and the remaining one is to be found.*

327. EXAMPLE.—If 4 men can earn $25 in one week, how much can 12 men earn in the same time?

SOLUTION.—The required term must bear the same relation to the given term of the same kind as one of the remaining terms bears to the other remaining term. We can then form a proportion by which the required term may be found.

The first question the student must ask himself in every calculation by proportion is :

"What is it I want to find?"

In this case it is dollars. We have two sets of men, one set earning $25, and we want to know how many dollars the other set earns. It is evident that the *amount* 12 men earn bears the same relation to the *amount* that 4 men earn as 12 men bears to 4 men. Hence, we have the proportion, the amount 12 men earn is to $25 as 12 men is to 4 men; or, since either extreme equals the product of the means divided by the other extreme, we have

The amount 12 men earn : \$25 = 12 men : 4 men,

or the amount 12 men earn $= \frac{\$25 \times 12}{4} = \$75$. Ans.

Since it matters not which place x or the required term occupies, the problem could be stated as any of the following forms, the value of x being the same in each:

(*a*) $25 : the amount 12 men earn = 4 men : 12 men; or the amount 12 men earn $= \frac{\$25 \times 12}{4}$, or $75, since either mean equals the product of the extremes divided by the other mean.

(*b*) 4 men : 12 men = $25 : the amount 12 men earn; or the amount that 12 men earn $= \frac{\$25 \times 12}{4}$, or $75, since either extreme equals the product of the means divided by the other extreme.

(*c*) 12 men : 4 men = the amount 12 men earn : $25; or the amount that 12 men earn $= \frac{\$25 \times 12}{4}$, or $75, since either mean equals the product of the extremes divided by the other mean.

328. If the proportion is an inverse one, first form it as though it were a direct proportion, and then invert one of the couplets.

EXAMPLES FOR PRACTICE.

329. Find the value of x in each of the following:

(*a*)	$\$16 : \$64 :: x : \$4$.	Ans.	(*a*)	$x = \$1$.
(*b*)	$x : 85 :: 10 : 17$.		(*b*)	$x = 50$.
(*c*)	$24 : x :: 15 : 40$.		(*c*)	$x = 64$.
(*d*)	$18 : 94 :: 2 : x$.		(*d*)	$x = 10\frac{4}{9}$.
(*e*)	$\$75 : \$100 = x : 100$.		(*e*)	$x = 75$.
(*f*)	15 pwt. : $x = 21 : 10$.		(*f*)	$x = 7\frac{1}{7}$ pwt.
(*g*)	x : 75 yd. = $15 : $5.		(*g*)	$x = 225$ yd.

1. If 75 pounds of lead cost $2.10, what would 125 pounds cost at the same rate? Ans. $3.50.

2. If A does a piece of work in 4 days and B does it in 7 days, how long will it take A to do what B does in 63 days? Ans. 36 days.

3. The circumferences of any two circles are to each other as their diameters. If the circumference of a circle 7 inches in diameter is 22 inches, what will be the circumference of a circle 31 inches in diameter? Ans. $97\frac{3}{7}$ inches.

INVERSE PROPORTION.

330. In Art. **318**, an inverse proportion was defined as one which required one of the couplets to be expressed as an inverse ratio. Sometimes the word *inverse* occurs in the statement of the example ; in such cases the proportion can be written directly, merely inverting one of the couplets. But it frequently happens that only by carefully studying the conditions of the example can it be ascertained whether the proportion is direct or inverse. When in doubt, the student can always satisfy himself as to whether the proportion is direct or inverse by first ascertaining what is required, and stating the proportion as a direct proportion. Then, in order that the proportion may be true, if the first term is smaller than the second term, the third term must be smaller than the fourth ; or if the first term is larger than the second term, the third term must be larger than the fourth term. Keeping this in mind, the student can always tell whether the required term will be larger or smaller than the other term of the couplet to which the required term belongs. Having determined this, the student then refers to the example, and ascertains from its conditions whether the required term is to be larger or smaller than the other term of the same kind. If the two determinations agree, the proportion is direct; otherwise, it is inverse, and one of the couplets must be inverted.

331. EXAMPLE.—If A's *rate* of doing work is to B's as 5 : 7, and A does a piece of work in 42 days, in what time will B do it ?

SOLUTION.—The required term is the number of days it will take B to do the work. Hence, stating as a direct proportion,

$$5 : 7 = 42 : x.$$

Now, since 7 is greater than 5, x will be greater than 42. But, referring to the statement of the example, it is easy to see that B works faster than A; hence it will take B a less number of days to do the work than A. Therefore, the proportion is an inverse one, and should be stated

$$5 : 7 = x : 42,$$

from which $x = \frac{5 \times 42}{7} = 30$ days. Ans.

Had the example been stated thus: The time that A requires to do a piece of work is to the time that B requires, as 5 : 7; A can do it in 42 days, in what time can B do it ? it is evident that it would take B a longer time to do the work than it would A; hence, x would be greater than 42, and the proportion would be direct, the value of x being $\frac{7 \times 42}{5} = 58.8$ days.

EXAMPLES FOR PRACTICE.

332. Solve the following:

1. If a pump which discharges 4 gal. of water per min. can fill a tank in 20 hr., how long will it take a pump discharging 12 gal. per min. to fill it? Ans. $6\frac{2}{3}$ hr.

2. If a pump discharges 90 gal. of water in 20 hr., in what time will it discharge 144 gal.? Ans. 32 hr.

3. The weight of any gas (the volume and pressure remaining the same) varies inversely as the absolute temperature. If a certain quantity of some gas weighs 2.927 lb. when the absolute temperature is 525°, what will the same volume of gas weigh when the absolute temperature is 600°, the pressure remaining the same? Ans. 2.561 + lb.

4. If 50 cu. ft. of air weigh 4.2 pounds when the absolute temperature is 562°, what will be the absolute temperature when the same volume weighs 5.8 pounds, the pressure being the same in both cases?

Ans. 407°, very nearly.

POWERS AND ROOTS IN PROPORTION.

333. It was stated in Art. **309** that a ratio could be raised to any power or any root of it might be taken. A proportion is frequently stated in such a manner that one of the couplets must be raised to some power or some root of it must be taken. In all such cases, both terms of the couplet so affected *must be raised to the same power or the same root of both terms must be taken.*

334. Example.—Knowing that the weight of a sphere varies as the cube of its diameter, what is the weight of a sphere 6 inches in diameter if a sphere 8 inches in diameter of the same material weighs 180 pounds?

Solution.—This is evidently a direct proportion. Hence, we write

$$6^3 : 8^3 = x : 180.$$

Dividing both terms of the first couplet by 2^3 (see Art. **310**),

$$3^3 : 4^3 = x : 180, \text{ or } 27 : 64 = x : 180;$$

whence, $x = \dfrac{27 \times 180}{64} = 75\frac{15}{16}$ pounds. Ans.

Example.—A sphere 8 inches in diameter weighs 180 pounds; what is the diameter of another sphere of the same material which weighs $75\frac{15}{16}$ pounds?

Solution.—Since the weights of any two spheres are to each other as the cubes of their diameters, we have the proportion

$$180 : 75\tfrac{15}{16} = 8^3 : x^3;$$

x, the required term, must be cubed, because the other term of the couplet is cubed (see Art. **333**). But, $8^3 = 512$; hence,

$$180 : 75\tfrac{15}{16} = 512 : x^3, \text{ or } x^3 = \frac{75\frac{15}{16} \times 512}{180} = 216;$$

whence, $x = \sqrt[3]{216} = 6$ inches. Ans.

335. Since taking the same root of all of the terms of a proportion does not change its value (Art. **324**), the above example might have been solved by extracting the cube root of all of the numbers, thus obtaining $\sqrt[3]{180} : \sqrt[3]{75\frac{15}{16}} = 8 : x$; whence,

$$x = \frac{8 \times \sqrt[3]{75\frac{15}{16}}}{\sqrt[3]{180}} = 8 \times \sqrt[3]{\frac{75\frac{15}{16}}{180}} = 8\sqrt[3]{\frac{1,215}{2,880}} = 8\sqrt[3]{\frac{27}{64}} =$$

$8 \times \frac{3}{4} = 6$ inches. The process, however, is longer and is not so direct, and the first method is to be preferred.

336. If two cylinders have *equal* volumes, but different diameters, the diameters are to each other inversely as the square roots of their lengths. Hence, if it is desired to find the diameter of a cylinder that is to be 15 inches long, and which shall have the same volume as one that is 9 inches in diameter and 12 inches long, we write the proportion

$$9 : x = \sqrt{15} : \sqrt{12}.$$

Since neither 12 nor 15 are perfect squares, we square all of the terms (Arts. **335** and **324**) and obtain

$$81 : x^2 = 15 : 12; \text{ whence } x^2 = \frac{81 \times 12}{15} = 64.8,$$

and $x = \sqrt{64.8} = 8.05$ inches = diameter of 15-inch cylinder.

EXAMPLES FOR PRACTICE.

337. Solve the following examples:

1. The intensity of light varies inversely as the square of the distance from the source of light. If a gas jet illuminates an object 30 feet away with a certain distinctness, how much brighter will the object be at a distance of 20 feet? Ans. $2\frac{1}{4}$ times as bright.

2. In the last example, suppose that the object had been 40 feet from the gas jet; how bright would it have been compared with its brightness at 30 feet from the gas jet? Ans. $\frac{9}{16}$ as bright.

3. When comparing one light with another, the intensities of their illuminating powers vary as the squares of their distances from the

source. If a man can just distinguish the time indicated by his watch, 50 feet from a certain light, at what distance could he distinguish the time from a light 3 times as powerful? Ans. 86.6+ feet.

4. The quantity of air flowing through a mine varies directly as the square root of the pressure. If 60,000 cubic feet of air flow per minute when the pressure is 2.8 pounds per square foot, how much will flow when the pressure is 3.6 pounds per square foot?

Ans. 68,034 cu. ft. per min., nearly.

5. In the last example, suppose that 70,000 cubic feet per minute had been required; what would be the pressure necessary for this quantity? Ans. 3.81+ lb. per sq. ft.

CAUSES AND EFFECTS.

338. Many examples in proportion may be more easily solved by using the principle of *cause and effect.* That which may be regarded as producing a change or alteration in something, or as accomplishing something, may be called a **cause,** and the change or alteration, or thing accomplished, is the **effect.**

339. *Like causes produce like effects.* Hence, when two causes of the same kind produce two effects of the same kind, the ratio of the causes equals the ratio of the effects; in other words, the first cause is to the second cause as the first effect is to the second effect. Thus, in the question, if 3 men can lift 1,400 pounds, how many pounds can 7 men lift? we call 3 men and 7 men the *causes* (since they accomplish something, viz., the lifting of the weight), the number of pounds lifted, viz., 1,400 pounds and x pounds, are the effects. If we call 3 men the first cause, 1,400 pounds is the first effect; 7 men is the second cause, and x pounds is the second effect. Hence, we may write

$$\begin{array}{ccccccc} \textit{1st cause} & & \textit{2d cause} & & \textit{1st effect} & & \textit{2d effect} \\ 3 & : & 7 & = & 1{,}400 & : & x, \end{array}$$

whence $x = \dfrac{7 \times 1{,}400}{3} = 3{,}266\tfrac{2}{3}$ pounds.

340. The principle of cause and effect is extremely useful in the solution of examples in compound proportion, as we shall now show.

COMPOUND PROPORTION.

341. All the cases of proportion so far considered have been cases of **simple proportion;** i. e., each term has been composed of but one number. There are many cases, however, in which two or all of the terms have more than one number in them; all such cases belong to **compound proportion.** In all examples in compound proportion, both causes or both effects or all four consist of more than two numbers. We will illustrate this by an

Example.—If 40 men earn $1,280 in 16 days, how much will 36 men earn in 31 days?

Solution.—Since 40 men earn something, 40 men is a cause, and since they take 16 days in which to earn something, 16 days is also a cause. For the same reason, 36 men and 31 days are also causes. The effects, that which is earned, are 1,280 dollars and x dollars. Then, 40 men and 16 days make up the first cause, and 36 men and 31 days make up the second cause. $1,280 is the first effect and $\$x$ is the second effect. Hence, we write

$$\begin{array}{ccccccc} \textit{1st cause} & & \textit{2d cause} & & \textit{1st effect} & & \textit{2d effect} \\ \begin{matrix} 40 \\ 16 \end{matrix} & : & \begin{matrix} 36 \\ 31 \end{matrix} & = & 1,280 & : & x \end{array}$$

Now, instead of using the colon to express the ratio, we shall use the vertical line (see Art. **299**), and the above becomes

$$\begin{array}{c|c} 40 & 36 \\ 16 & 31 \end{array} = \begin{array}{c|c} 1,280 & x. \end{array}$$

In the last expression, the product of all of the numbers included between the vertical lines must equal the product of all the numbers without them; i. e., $36 \times 31 \times 1,280 = 40 \times 16 \times x$.

$$\text{Or } x = \frac{36 \times 31 \times \overset{\overset{2}{\cancel{80}}}{\cancel{1280}}}{\cancel{40} \times \cancel{16}} = \$2,232. \quad \text{Ans.}$$

342. The above might have been solved by canceling factors of the numbers in the original proportion. For if any number within the lines has a factor common to any number without the lines, that factor may be canceled from both numbers. Thus, 16 is contained in

$$\begin{array}{c|c} \cancel{40} & 36 \\ \cancel{16} & 31 \end{array} = \begin{array}{c|c} \overset{\overset{2}{\cancel{80}}}{\cancel{1280}} & x, \end{array}$$

1,280, 80 times. Cancel 16 and 1,280, and write 80 above 1,280. 40 is contained in 80, 2 times. Cancel 40 and 80, and write 2 above 80. Now, since there are no more numbers that can be canceled, $x = 36 \times 31 \times 2 = \$2,232$, the same result as was obtained in the last article.

343. Rule.—*Write all the numbers forming the first cause in a vertical column, and draw a vertical line; on the other side of this line write in a vertical column all of the numbers forming the second cause. Write the sign of equality to the right of the second column, and on the right of this form a third column of the numbers composing the first effect, drawing a vertical line to the right; on the other side of this line, write, for a fourth column, the numbers composing the second effect. There must be as many numbers in the second cause as in the first cause, and in the second effect as in the first effect; hence, if any term is wanting, write* x *in its place. Multiply together all of the numbers within the vertical lines, and also all those without the lines (canceling previously, if possible), and divide the product of those numbers which do not contain* x *by the product of the others in which* x *occurs, and the result will be the value of* x.

344. Example.—If 40 men can dig a ditch 720 feet long, 5 feet wide and 4 feet deep in a certain time, how long a ditch 6 feet deep and 3 feet wide could 24 men dig in the same time?

Solution.—Here 40 men and 24 men are the causes and the two ditches are the effects. Hence,

$$\begin{array}{c|ccc|l} & & & 3 & \\ & & & \cancel{18} & \\ & & & \cancel{720} & x \\ \cancel{40} & 24 & = & 5 & 3 \text{ whence, } x = 24 \times 5 \times 4 = 480 \text{ feet. Ans.} \\ & & & 4 & \cancel{6} \end{array}$$

345. Example.—The volume of a cylinder varies directly as its length and directly as the square of its diameter. If the volume of a cylinder 10 inches in diameter and 20 inches long is 1,570.8 cubic inches, what is the volume of another cylinder 16 inches in diameter and 24 inches long?

Solution.—In this example, either the dimensions or the volumes may be considered the causes; say we take the dimensions for the causes. Then, squaring the diameters,

$$\begin{array}{c|ccc|c} 10^2 & 16^2 & = & 1{,}570.8 & x, \\ 20 & 24 & & & \\ & & & & \end{array} \text{ or } \begin{array}{c|ccc|c} 100 & 256 & = & 1{,}570.8 & x; \\ \cancel{20} & \cancel{24} & & & \\ 5 & 6 & & & \end{array}$$

whence, $x = \dfrac{256 \times 6 \times 1{,}570.8}{5 \times 100} = 4{,}825.4976$ cubic inches. Ans.

346. Example.—If a block of granite 8 ft. long, 5 ft. wide and 3 ft. thick weighs 7,200 lb., what will be the weight of a block of granite 12 ft. long, 8 ft. wide and 5 ft. thick?

Solution.—Taking the weights as the effects, we have

$$\begin{array}{c|ll|l} & 4 & & \\ \cancel{8} & \cancel{12} & & \\ \cancel{5} & \cancel{8} & = 7{,}200 & x, \text{ or } x = 4 \times 7{,}200 = 28{,}800 \text{ pounds.} \quad \text{Ans.} \\ \cancel{3} & \cancel{5} & & \end{array}$$

347. Example.—If 12 compositors in 30 days of 10 hours each set up 25 sheets of 16 pages each, 32 lines to the page, in how many days 8 hours long can 18 compositors set up, in the same type, 64 sheets of 12 pages each, 40 lines to the page?

Solution.—Here compositors, days, and hours compose the causes, and sheets, pages, and lines the effects. Hence,

$$\begin{array}{c|cccc|l} 3 & \cancel{3} & & \cancel{5} & & 2 \\ \cancel{12} & \cancel{18} & & \cancel{25} & & \cancel{64} \\ & & & \cancel{4} & & \\ \cancel{30} & x & = & \cancel{16} & & \cancel{12}, \text{ or } x = 3 \times 10 \times 2 = 60 \text{ days.} \quad \text{Ans.} \\ \cancel{6} & & & & & \cancel{4} \\ 10 & \cancel{8} & & \cancel{32} & & \cancel{40} \\ & & & & & \cancel{5} \end{array}$$

348. In examples stated like that in Art. **345**, should an inverse proportion occur, write the various numbers as in the preceding examples, and then transpose those numbers which are said to vary inversely from one side of the vertical line to the other side.

Example.—The centrifugal force of a revolving body varies directly as its weight, as the square of its velocity and inversely as the radius of the circle described by the center of the body. If the centrifugal force of a body weighing 15 pounds is 187 pounds when the body revolves in a circle having a radius of 12 inches, with a velocity of 20 feet per second, what will be the centrifugal force of the same body when the radius is increased to 18 inches and the speed is increased to 24 feet per second?

Solution.—Calling the centrifugal force the effect, we have,

$$\begin{array}{c|ll|l} 15 & 15 & & \\ 20^2 & 24^2 & = \ 187 & x. \\ 12 & 18 & & \end{array}$$

Transposing 12 and 18 (since the radii are to vary inversely) and squaring 20 and 24,

$$\begin{array}{c|ll|l} \cancel{15} & \cancel{15} & & \\ & 2 & & \\ 25 & \cancel{36} & = 187 & x, \text{ or } x = \dfrac{12 \times 2 \times 187}{25} = 179.52 \text{ pounds.} \quad \text{Ans.} \\ \cancel{400} & \cancel{576} & & \\ \cancel{18} & 12 & & \end{array}$$

EXAMPLES FOR PRACTICE.

349. Solve the following by compound proportion:

1. If 12 men dig a trench 40 rods long in 24 days of 10 hours each, how many rods can 16 men dig in 18 days of 9 hours each?
Ans. 36 rods.

2. If a piece of iron 7 ft. long, 4 in. wide, and 6 in. thick weighs 600 lb., how much will a piece of iron weigh that is 16 ft. long, 8 in. wide and 4 in. thick? Ans. 1,828$\frac{4}{7}$ lb.

3. If 24 men can build a wall 72 rods long, 6 feet wide, and 5 feet high in 60 days of 10 hours each, how many days will it take 32 men to build a wall 96 rods long, 4 feet wide and 8 feet high, working 8 hours a day? Ans. 80 days.

4. The horsepower of an engine varies as the mean effective pressure, as the piston speed and as the square of the diameter of the cylinder. If an engine having a cylinder 14 inches in diameter develops 112 horsepower when the mean effective pressure is 48 pounds per square inch and the piston speed is 500 feet per minute, what horsepower will another engine develop if the cylinder is 16 inches in diameter, piston speed is 600 feet per minute, and mean effective pressure is 56 pounds per square inch? Ans. 204.8 horsepower.

5. Referring to the example in Art. **345**, what will be the volume of a cylinder 20 inches in diameter and 24 inches long?
Ans. 7,539.84 cubic inches.

6. Knowing that the product of $3 \times 5 \times 7 \times 9$ is 945, what is the product of $6 \times 15 \times 14 \times 36$? Ans. 45 360.

ALGEBRA.

350. In arithmetic, numbers are represented by the figures 1, 2, 3, 4, etc. There is no reason, however, why numbers may not be represented by other symbols, as letters, if rules are provided for their use.

351. In algebra, numbers are represented both by figures and letters. It will be seen later that the use of letters often simplifies the solution of examples, and shortens calculations.

352. The principal advantage of letters is that they are general in their meaning. Thus, unlike figures, the letter a does not stand for the number one, the letter b for two, c for three, etc., but *any* letter may be taken to represent *any* number, it being only necessary that a letter shall always stand for the same number *in the same example.*

353. To illustrate this difference between letters and figures, consider the following example: If a person exchanges 10 books worth \$3 per volume for cloth at \$2 per yard, how many yards will he obtain? A rule for solving, not only this example, but all others of the same kind, would be to multiply the number of books by the price per volume, and to divide the result by the price of the cloth. This rule is *general*, because it tells what to do with the number of books, and the prices of the books and cloth, *whatever they may be.*

Another and more concise way of stating the rule is to use letters. Thus:

Let $a =$ the number of books,
$b =$ the price per volume,
$c =$ the price of the cloth,
and $d =$ the number of yards of cloth.

Then, according to the rule,

$\frac{\text{number of books} \times \text{price per volume}}{\text{price of cloth}} =$ number of yards of cloth, or $\frac{a \times b}{c} = d.$

In the example in question, $a = 10$, $b = 3$, and $c = 2$. Hence, writing for a, b, and c their values, 10, 3, and 2, d, the number of yards, $= \frac{10 \times 3}{2} = 15$. Here the expression $\frac{10 \times 3}{2}$ corresponds to $\frac{a \times b}{c}$, but this difference is to be noticed: $\frac{10 \times 3}{2}$ applies only to *this* example, and by performing the operations indicated only one answer can be obtained, while $\frac{a \times b}{c}$ is *general* in its application, in the same way that the rule previously given is general. That is, while a, b, and c stand for the numbers 10, 3, and 2 in *this* example, they may stand for other numbers in *another* example; hence, by writing their values in place of the letters, and performing the operations indicated, the answer to any example of the same kind may be obtained. Consequently, while figures or combinations of figures always represent the same numbers, letters are more general, and may represent any numbers, according to the conditions of the example.

354. An **equation** is a statement of equality between two expressions. Thus, $x + y = 8$ is an equation, and means that the sum of the numbers represented by x and y equals 8. Examples are solved in algebra by the aid of equations in which numbers are represented both by letters and figures. An idea of the method of solution may be had from the following simple example: If an iron rail 30 feet long is cut in two so that one part is four times as long as the other, how long is the shorter part?

SOLUTION.—Since any letter may represent any number,
Let $x =$ the length of the shorter part.
Then, $4 \times x$ (written $4x$) $=$ the length of the longer part.

But the sum of the two parts must equal the total length, 30 feet.

Hence, $x + 4x = 30$

Adding x and $4x$, $5x = 30$

Whence, dividing by 5, $x = 6$ ft. Ans.

355. Another application of equations is made in the use of formulas. A **formula** is a statement of a general rule, abridged by means of symbols. In Art. **353**, the expression $\frac{a \times b}{c} = d$ is a formula. All formulas are equations, and the same rules apply to both. An equation is not called a formula, however, unless it is a statement of a general rule. In modern technical works, the rules for solving examples are generally given by formúlas, and it is important to understand how to apply them.

356. **Algebra** treats of the equation and its use. Since the use of equations involves the use of letters, it will be necessary to take up addition, subtraction, multiplication, involution, evolution, etc., where letters are used, before considering equations.

NOTATION.

357. The term **quantity** is used to designate any number that is to be subjected to mathematical processes. A quantity is strictly a concrete number, as 6 books, 5 pounds, 10 yards. *Symbols* used to *represent* numbers, or expressions containing two or more such symbols, as a, x, bd, 10, $(c + 12)$, etc., are often called quantities, the term being a convenient one to use.

358. The **signs** +, −, ×, ÷ are the same in algebra as in arithmetic. The sign of multiplication × is usually omitted, however, multiplication being indicated by simply writing the quantities together. Thus, abc means $a \times b \times c$; $2xy$ means $2 \times x \times y$. Evidently the sign cannot be omitted between two *figures*, as addition instead of multiplication would be indicated. Thus, 24 means $20 + 4$, instead of 2×4.

Another sign of multiplication that is sometimes used between numbers is a dot placed between them. Thus, instead of $2 \times 4 \times 12$, we may indicate the multiplication as $2 \cdot 4 \cdot 12$. This dot always occupies a higher position than the decimal point, so as to avoid confusing the two.

359. A **coefficient** is a figure or letter prefixed to a quantity; it shows how many times the latter is to be taken. Thus, in the expression $4a$, 4 is the coefficient of a, and indicates that a is to be taken four times, or $a + a + a + a$. When several quantities are multiplied together, any of them may be regarded as the coefficient of the others. Thus, in $6axy$, 6 is the coefficient of axy, $6a$ of xy, $6ax$ of y, etc. In general, however, when a coefficient is spoken of, the numerical coefficient only is meant, as the 6 above. When no numerical coefficient is written it is understood to be 1. Thus, cd is the same as $1cd$.

360. The **factors** of a quantity are the quantities which, when multiplied together, will produce it. Thus, 2, 3 and 3 are the factors of 18, since $2 \times 3 \times 3 = 18$; 2, a and b are the factors of $2ab$, since $2 \times a \times b = 2ab$. (Art. **358.**)

361. An **exponent** is a small figure placed at the right and a little above a quantity; it shows how many times the latter is to be taken *as a factor*. Thus, $4^3 = 4 \times 4 \times 4 = 64$; $a^5 = aaaaa$. Any quantity written without an exponent is understood to have an exponent of 1.

The difference between the meaning of a coefficient and an exponent should be clearly understood. A coefficient *multiplies* the quantity which it precedes; it shows that the quantity is to be *added to itself*. Thus, $3a = 3 \times a$, or $a + a + a$. An exponent indicates that a quantity is to be *multiplied by itself*. Thus, $a^3 = a \times a \times a$. A more complete definition of an exponent will be given later.

362. A **power** is the result obtained by taking a quantity two or more times *as a factor*. For example, 16 is the fourth power of 2, because 2 multiplied by itself, until it has been taken four times as a factor, produces 16; a^3 is the third power of a, because $a \times a \times a = a^3$.

363. A **root** of a quantity is one of its equal factors. Thus, 2 is the root of 4, 8 and 16, since $2 \times 2 = 4$, $2 \times 2 \times 2 = 8$ and $2 \times 2 \times 2 \times 2 = 16$, 2 being one of the equal factors in each case. In like manner, a is a root of a^2, a^3, a^5, etc. The symbol which denotes that the second or square root is to be extracted is $\sqrt{\ }$, called the **radical sign.** For other roots the same symbol is used, but with a figure called the *index* of the root, written above it to indicate the root. Thus, $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, etc., signify the square root, cube root, fourth root, etc., of a.

364. The **parenthesis** ( ), **brackets** [ ], and **braces** { } have the same meaning, and signify that the quantities within them are to be subjected to the same operations. Thus, $(a + 2)4$, $[a + 2]4$ and $\{a + 2\}4$ all indicate that $a + 2$ is to be multiplied by 4, the sign of multiplication being omitted. When two or more expressions are enclosed and written together, their product is indicated. Thus, $(a + b)(c - d^2)(x + y)$ indicates that $a + b$, $c - d^2$ and $x + y$ are to be multiplied together.

365. The above symbols are called **symbols of aggregation,** meaning that the quantities enclosed within them are aggregated, or collected into one quantity. A fourth symbol of aggregation is the **vinculum** ———, which is simply a horizontal line drawn above the quantities affected. Thus, $\overline{a + 2} \times 4$ shows that $a + 2$ is to be multiplied by 4. The principal use of the vinculum is in connection with the radical sign (Art. **363**), where it is extended over the whole expression whose root is to be extracted. For example, in $\sqrt{x^2 + 2xy + y^2}$ the vinculum shows that the square root of $x^2 + 2xy + y^2$ is to be extracted. Without the vinculum, as in $\sqrt{\ }x^2 + 2xy + y^2$, the square root of x^2 only is indicated.

366. The **terms** of an algebraic expression are the parts which are connected by the signs + and —. Thus, x^2, $-2xy$ and y^2 are terms of the expression $x^2 - 2xy + y^2$.

367. **Like terms** are those which differ only in their numerical coefficients; all others are **unlike terms.** Thus, $2ab^2$ and $5ab^2$ are like terms; $5ab$ and $5ab^2$ are unlike terms, because one contains b and the other b^2. Differing signs before two terms do not make them unlike.

368. A **monomial** is an expression consisting of only one term, as $4abc$, $3x^2$, $2ax^3$, etc.

369. A **binomial** is an expression consisting of two terms, as $a + b$, $2a + 5b$, etc.

370. A **trinomial** is an expression consisting of three terms, as $a^2 + 2ab + b^2$, $(a + x)^2 - 2(a + x)y + y^2$, etc. The expression $(a + x)$ being treated as one quantity. (Art. **365.**)

371. A **polynomial** is any expression consisting of more than two terms. The term is usually applied only to an expression consisting of four or more terms.

372. The polynomial $a + a^2b + 2a^3 - 3a^4b - a^5$ is said to be arranged according to the *increasing powers* of a, because the exponents of a increase in each term from left to right, the exponent of the first a being 1 understood. (Art. **361.**) The polynomial $a^3b^3 + ab^2 + 4a^4b + 1$ is arranged according to the *decreasing powers* of b, the exponents of b decreasing in order, from left to right.

373. The arrangement of the terms of a polynomial does not affect its value. Thus, $x^2 + 2xy + y^2$ has the same value as $2xy + y^2 + x^2$, just as $2 + 6 + 4$ has the same value as $6 + 4 + 2$.

READING ALGEBRAIC EXPRESSIONS.

374. Quantities like a, x, b^2, etc., are read "a," "x," "b square," etc. In reading monomials in which multiplication is indicated, the word "times" is not used. Thus, abc is read "abc"; $7ad^2b^3$ is read "$7ad$ square b cube."

375. The polynomial $a + a^2b + 2a^3 - 3a^4b - a^5$ is read "a, plus a square b, plus $2a$ cube, minus $3a$ fourth b, minus *a* fifth." Considerable care is required when reading expressions containing polynomials. Thus, if $4\ (a-b)$ were read "$4a$ minus b," the binomial $4a-b$ would be understood. It *should* be read "4 into a minus b," or "4 times a minus b," when it will be understood that 4 multiplies the whole quantity $a-b$, since the word "times" or "into" is not used with monomials. (Art. **374.**) Again $m(m^2+2mn+n^2)$ and $m(m^2+2mn)+n^2$ should each be so read that there can be no doubt as to whether the n^2 is to be multiplied by m.

Let the distinction to be made in reading the following be observed:

$$\sqrt{\frac{m+n}{x-y^2}} \text{ and } \sqrt{m+\frac{n}{x-y^2}}.$$

In the first case, the whole quantity $m+n$ is divided by $x-y^2$, and it would be clear to say, "the square root of the quotient of $m+n$ divided by $x-y^2$." In the second case, where the n only is divided by $x-y^2$, it may be read, "the square root of the quantity m, plus n divided by $x-y^2$." The word "quantity" shows that the square root of the whole expression is taken, and the pause after the m that only the n is divided by $x-y^2$.

376. When a polynomial is affected by an exponent, it should be indicated clearly. Thus,

$$(3a-d^2)\ (3a-d)^2\ (3a-d^2)^2$$

may be read, "the product of $3a-d$ square, $3a-d$ squar*ed* (meaning that the whole quantity is squared), and the square of $3a-d$ square." Otherwise it may be read, "$3a-d$ square, times the square of $3a-d$, times the square of $3a-d$ square."

377. Sometimes expressions like A', B'', c', d'', C_1, a_2, etc., appear in formulas or elsewhere in algebraic problems where it is desirable to have the same letter represent different quantities that are similar, or correspond to one another. The marks $'$, $''$, $'''$, $_1$, $_2$, etc., serve to distinguish the letters. The expressions are also used to designate similar or corresponding lines in geometrical figures, as will appear in mechanical drawing. A', B'', C''', etc., are read, "*a major prime, b major second, c major third,*" etc.; a', b'', c''', etc., are read, "*a minor prime, b minor second, c minor third,*" etc., a_1, B_2, C_3, d_4, etc., are read, "*a minor sub-one, b major sub-two, c major sub-three, d minor sub-four,*" etc.

POSITIVE AND NEGATIVE QUANTITIES.

378. In algebra, the terms **positive** and **negative** are applied to quantities which are opposed, or directly opposite in character. When a quantity is positive it is denoted by writing the plus sign before it, and when it is negative, by writing the minus sign before it. Thus, $+2a$ and $+x^2y$ are positive, and $-2a$ and $-x^2y$ negative quantities. Besides indicating addition and subtraction, the signs $+$ and $-$ denote the *character* of the quantities which they precede.

379. To illustrate the meaning of positive and negative quantities, suppose a person to earn \$1,000 in a year, and that during that time he has incurred a debt of \$500. If he had no other property, he would be worth \$500. Supposing, however, that he has incurred a debt of \$1,500 instead of \$500, he not only would have no property, but he would be \$500 *in debt.* Now, since money earned *adds* to one's property, and debts incurred *subtract* from it, the two are opposed in character, each tending to destroy the other. Hence, if money earned is taken as *positive*, money owed may be called *negative.*

In the preceding illustration, writing the signs $+$ or $-$ before each sum of money, according as it is positive or negative, we have the following statement for each case:

$$+\$1{,}000 - \$500 = +\$500 \text{ and}$$
$$+\$1{,}000 - \$1{,}500 = -\$500.$$

In the first instance, the person had \$500 remaining. In the second, speaking arithmetically, he *owed* \$500; expressed algebraically, he *had minus* \$500.

There are many other common illustrations of positive and negative quantities. The pull exerted by a locomotive may be said to be positive, while the resistance of the train, which is opposed to it, is negative. On the thermometer we speak of a temperature above 0° as +, and one below 0° as −. Suppose a man to row a boat up a river. If he is able to propel the boat at 6 miles an hour in quiet water, and the current is sufficient to carry it down stream at the rate of 2 miles an hour, the 6 and 2 are exactly opposite, relative to the movement of the boat, and if the 6 be taken as positive, the 2 would be negative.

Again, suppose that two men, A and B, start to travel from the same point, A going east and B west. Considering the starting point as zero, if A's direction be taken as +, B's direction will be −; if B's direction be taken as +, A's direction will be −.

380. A quantity is always considered to increase in a *positive* direction, and to decrease in a *negative* direction; hence, *any positive quantity, however small, is always considered to be greater than any negative quantity.* For example, a person who *owns* \$2 is better off than one who *owes* \$10. Also, since a person who owes only \$1 is better off than one who owes \$10; then of two negative quantities, the one with the smaller numerical value is to be regarded as the greater. From this it follows that *zero*, which is *smaller* than any *positive* quantity, is *greater* than any *negative* quantity. For example, we know that a day when the thermometer stands at 0° is warmer than one with the thermometer at − 15°, and the former is a *higher* temperature than the latter; also when the thermometer stands at − 15°, it is warmer than when it stands at − 25°.

381. Two quantities may be compared *arithmetically*, or *geometrically*. To the question, how much greater is a than b? the answer would be $a - b$, and the result would be the **arithmetic ratio.** Arithmetic ratio, then, corresponds to the *difference between two quantities.* To the question, what part of a is b? the answer would be $b \div a$, $b : a$ or $\frac{b}{a}$, and the result would be the **geometric ratio.** Geometric ratio, then, corresponds to the *quotient obtained by dividing* *one quantity by another.* When the word ratio is used alone, geometric ratio is *always* meant.

EXAMPLE.—What is (*a*) the ratio of 16 to 4? (*b*) the arithmetic ratio of 16 to 4?

SOLUTION.—(*a*) The ratio (geometric, understood) of 16 to 4 is 16 : 4, or $\frac{16}{4} = 4$. Ans. (*b*) The arithmetic ratio of 16 to 4 is $16 - 4 = 12$. Ans.

382. The mere fact that a quantity is negative does not *necessarily* imply that it is less than a positive quantity of the same kind. It always becomes less, however, when considered in relation to an increase in the positive direction. Thus, if two men, A and B, start from the same point and travel 100 miles. If A travels east 100 miles and B travels west 100 miles, neither can be said to have traveled any greater distance than the other. But if A's direction be considered + and B's −, then A will have traveled 200 miles further *east* than B.

EXAMPLES FOR PRACTICE.

383. When writing algebraic expressions, if a positive term stands alone, or if the first term of an expression is positive, the plus sign is omitted, it being understood that the term is positive. Thus, $3a$ means the same as $+3a$, and $a - b$ the same as $+a - b$. The minus sign must never be omitted. Polynomials are usually written with a positive term first, and monomials with the letters arranged alphabetically.

Express the following algebraically:

1. Three *x* square *y* square, minus two *cd* into *a* plus *b*. Ans. $3x^2y^2 - 2cd(a + b)$.

2. The product of *m* square plus $2mn$ plus *n* square, and *a* square *b* cube *c* fourth. Ans. $(m^2 + 2mn + n^2)a^2b^3c^4$.

3. *A* plus the square root of *D* into *X* plus *Y*. Ans. $A + \sqrt{D}(X + Y)$.

4. *A* plus the square root of the quantity *D* into *X* plus *Y*. Ans. $A + \sqrt{D(X + Y)}$.

5. Ten *x* plus *y* minus 7 times the difference between *x* and *y* divided by 4, plus the quotient of *x* square minus *y* square divided by two *cd*. Ans. $10x + y - 7\left(x - \frac{y}{4}\right) + \frac{x^2 - y^2}{2cd}$.

When $a = 6$, $b = 5$ and $c = 4$, find the numerical values of

6. $a^2 + 2ab + b^2$. Ans. $6^2 + 2 \times 6 \times 5 + 5^2 = 121$.

7. $2a^2 + 3bc - 5.$ Ans. $72 + 60 - 5 = 127.$

8. $2ac^5 - a^2(a + b).$ Ans. 11,892.

9. $abc^2 + ab^2c - a^2bc.$ Ans. 360.

When $x = 8$ and $y = 6$, what do the following equal?

10. $(x+y)(x-y) - \sqrt{\dfrac{x+y^2}{11}}$?

Ans. $(8+6)(8-6) - \sqrt{\dfrac{8+6^2}{11}} = 26.$

11. $\sqrt{(x+y^2)(x^2+y)} - (x-y)(\sqrt[3]{x}+y)$? Ans. 39.5.

12. $\dfrac{x^2y^2}{x+y} + \dfrac{x^2y(x+y^2)}{\sqrt{3xy}}$? Ans. 1,572.57.

384. Thus far, algebra has been shown to differ from arithmetic in the following points:

1. In the use of letters as well as figures to represent numbers, and in the fact that letters are general in their significance, while figures are not.
2. In the use of equations.
3. In the omission of the multiplication sign when letters are used and multiplication is indicated.
4. In the recognition of certain quantities as positive and others as negative, when they are opposed in character. This is a very important difference. It will appear later that the use of negative quantities greatly modifies the arithmetical rules of addition, subtraction, multiplication, division, etc.

385. Another important difference will now be evident, which is, namely, that when letters are used, the various operations, as multiplication and involution, for example, cannot always be performed as in arithmetic, and must be simply indicated. Thus, in $4 \times 2 = 8$, the multiplication is actually performed. If, however, the letter a should represent 4, and b represent 2, their multiplication could be indicated only, by writing a and b together, as ab, which is equivalent to writing 4×2 without performing the operation. Again, $4^2 = 16$, 16 being obtained by actually squaring 4; but if $4 = a$, the square of a can only be indicated by a^2, which corresponds to 4^2.

ADDITION.

386. **Addition,** in algebra, is the process of finding an expression, called the **sum,** whose value shall equal the *combined value* of the quantities to be added.

387. Either positive or negative quantities may be added, or positive quantities may be added to negative quantities. For example, the sum of \$5 and \$3 is \$8. Letting the letter a = \$1, this would be stated in algebra as the sum of $5a$ and $3a$, which is $8a$. In like manner, $a + a = 2a$, $7a + 5a = 12a$, etc.

Again, if a person has two debts, one of \$5 and one of \$3, he is in debt to the amount of \$8. Remembering the meaning of negative quantities, this would be stated in algebra as the sum of $-5a$ and $-3a$, or $-8a$, the letter a representing \$1, as before. In like manner the sum of $-a$ and $-a$ is $-2a$; of $-5a$ and $-5a$ is $-10a$, etc.

Should a person have \$5, however, and owe \$3, his property would amount to but \$2; or if he owed \$7 he would be in debt to the amount of \$2. In arithmetic, the amount of his property or debt is found by subtraction. In algebra, however, where negative quantities are recognized, we should say that the sum of \$5 in money and \$3 in debt, that is, the sum or *combined value* (Art. **386**) of \$5 and $-$\$3, is \$2; while the sum or combined value of \$5 and $-$\$7 is $-$\$2. In the first case the $-$\$3 neutralized three of the $+$\$5, and in the second case the $+$\$5 neutralized five of the $-$\$7. By the same reasoning, the sum of $5a$ and $-3a$ is $2a$; of $5a$ and $-7a$ is $-2a$; of $3a$ and $-3a$ is 0, etc.

388. From the foregoing we see that addition does not always imply a numerical increase. This is because addition, in algebra, is finding the *combined value* of two or more quantities. If two of the quantities are opposite in character, one being positive and the other negative, they tend to neutralize or destroy each other, which reduces their combined value. The sum in algebra is sometimes called the *algebraic sum*, to distinguish it from the arithmetical sum.

389. *The sum of two or more quantities may be indicated by connecting them by their respective signs.* For example, the sum of mn, $5xy^2$ and $-2x$ may be indicated by connecting each term by the plus sign, thus: $mn + 5xy^2 + (-2x)$. (Art. **358.**) But since the quantity $2x$ is negative, it tends to neutralize the positive quantities, and reduces their value by the amount of $2x$. Hence, the sum, or combined value, will be indicated by $mn + 5xy^2 - 2x$.

ADDITION OF MONOMIALS.

There are two cases, according as the terms to be added are like or unlike. (Art. **367.**)

390. Like Terms.—To add like terms having the same sign:

Rule I.—*Add the coefficients, prefix the common sign to their sum, and annex the common symbols.*

To add like terms, some positive and some negative:

Rule II.—*Add the coefficients of the positive and negative terms separately, and subtract the less sum from the greater. Prefix the sign of the greater sum to the result, and annex the common symbols.*

EXAMPLE.—Find the sum of $-2abxy$, $-abxy$, $-3abxy$ and $-6abxy$.

SOLUTION.—The sum of the coefficients is 12 (remember that the coefficient of $-abxy$ is 1), and the common sign is $-$. The common symbols $abxy$ annexed to these give, as the result, $-12abxy$. (Rule I.)

EXAMPLE.—Combine xy^2, $-2xy^2$, $8xy^2$ and $-4xy^2$.

SOLUTION.—The sum of the coefficients of the positive terms is 9, and of the negative terms, 6. Their difference is 3, and the sign of the greater sum is $+$. The common symbols xy^2 annexed to these give, as the result, $3xy^2$. (Rule II.)

391. Unlike Terms.—In arithmetic, unlike numbers, as 5 books and 6 dollars, cannot be added. So, in algebra, unlike terms, as $3ab^2$, $3a^2b$, $-4xy$, etc., cannot be combined into one term, and their sum can only be indicated by connecting the terms by their respective signs. (Art. **389.**) This is another illustration of the fact stated in Art. **385**, that operations with letters cannot always be performed. Expressions in algebra are composed of quantities between which operations of addition, multiplication, etc., are indicated. The trinomial $m^2 - 2mn + n^2$, for example, is the indicated sum of $m^2, -mn$ and n^2, *but it is to be considered as one quantity in the same way that an arithmetical sum, obtained by actually performing the addition, is considered.*

Example.—What does $7cd^3 - 8cx - cd^3 + 6adx + 2cd^3$ equal?

Solution.—In this case part of the terms are like and part unlike. Combining like terms, $7cd^3 + 2cd^3 - cd^3 = 8cd^3$. Connecting the unlike terms with this result by their respective signs, we have, as the final result, $8cd^3 - 8cx + 6adx$.

EXAMPLES FOR PRACTICE.

392. Find the sum of the following:

1. $-6a^2, 2a^2, -5a^2, 4a^2, -3a^2$ and a^2. Ans. $-7a^2$.
2. $2a^2b, -a^2b, 11a^2b, -5a^2b, 4a^2b$ and $-9a^2b$. Ans. $2a^2b$.
3. $2x^2, 3xy, -x^2, 8y^2, -5xy$ and $-7y^2$. Ans. $x^2 - 2xy + y^2$.

Note.—Combine like terms and connect with respective signs.

4. $a^2bc, -2ab^2c, 3abc^2, -4a^2bc$ and $5ab^2c$. Ans. $3ab^2c - 3a^2bc + 3abc^2$.

What do the following equal?

5. $mn^2 + 2mn - mn^2 - 8mn + m^2n$? Ans. $m^2n - 6mn$.
6. $5cd^2 - 5cd^3 + xy - 2cd^2 + 3cd^3 + 4cd^2 - 6xy$? Ans. $7cd^2 - 2cd^3 - 5xy$.

ADDITION OF POLYNOMIALS.

393. Rule.—*To add polynomials, write them one underneath the other, with like terms in the same vertical column. Add each column separately and connect the results by their respective signs.*

Example.—Find the sum of $5a^2 + 6ac - 3b^2 - 2xy$, $7ac - 3a^2 + 4b^2 + 3xy$, and $4xy - 5b^2 + 8ac - a^2$.

Solution.—Writing like terms in the same vertical column, we have

$$\begin{array}{rrrrr} & 5a^2 + & 6ac & - 3b^2 & - 2xy \\ & -3a^2 + & 7ac & + 4b^2 & + 3xy \\ & -\ \ a^2 + & 8ac & - 5b^2 & + 4xy \\ \hline \textit{sum} = & a^2 + & 21ac & - 4b^2 & + 5xy. \end{array} \quad \text{Ans.}$$

Example.—Find the sum of $a^2x - ax^2 - x^2$, $ax - x^2 - a^2$, $-2a^2 - 2a^2x - 2ax^2$, and $3a^2 - 3a^2x + 3ax^2$.

Solution.—

$$\begin{array}{rrrrrr}
 & a^2x & -\,ax^2 & -\,x^2 & & \\
 & & & -\,x^2 & -\,a^2 & +\,ax \\
 & -\,2a^2x & -\,2ax^2 & & -\,2a^2 & \\
 & -\,3a^2x & +\,3ax^2 & & +\,3a^2 & \\
\hline
sum = & -\,4a^2x & +\,0 & -\,2x^2 & +\,0 & +\,ax =
\end{array}$$

$ax - 4a^2x - 2x^2$. Ans. (Art. **383.**)

EXAMPLES FOR PRACTICE.

394. Find the sum of the following:

1. $ax + 2bx + 4by - 3ay$, $2ax + bx + 2ay - by$, and $4ax + 3by$. Ans. $7ax + 3bx + 6by - ay$.
2. $a - x + 4y - 3z + w$, $z + 3a - 2x - y - w$, and $x + y + z$. Ans. $4a - 2x + 4y - z$.
3. $2a - 3b + 4d$, $2b - 3d + 4c$, $2d - 3c + 4a$, and $2c - 3a + 4b$. Ans. $3a + 3b + 3c + 3d$.
4. $6x - 3y + 7m$, $2n - x + y$, $2y - 4x - 5m$, and $m + n - y$. Ans. $x - y + 3m + 3n$.
5. $2x - 5y - z + 7$, $3y - 2 - 6x + 8z$, $z + 3x - 4$, and $1 + 2y - 5z$. Ans. $3z - x + 2$.

SUBTRACTION.

395. Subtraction is the process of finding the *difference* between two quantities; it is the same thing as finding the arithmetic ratio between two quantities. The quantity to be subtracted is called the **subtrahend,** and that from which it is taken, the **minuend.**

396. The **difference** must evidently be a quantity which, when added to the subtrahend, will produce the minuend. Thus, the difference between 10 and 6 is 4, 4 being the number that must be added to the subtrahend 6 to produce the minuend 10. In algebra, when negative quantities occur, it is convenient to make use of this principle, as illustrated in the following six cases, which cover all those that are met with:

1. If $3a$ be subtracted from $5a$, the difference must be a quantity which, when added to $3a$, will produce $5a$. This is evidently $2a$.

2. If $5a$ be subtracted from $3a$, the difference must be a quantity which, when added to $5a$, will produce $3a$. By Art. **390,** this is $-2a$.

In like manner,

3. Subtracting $-3a$ from $5a$ gives a difference of $8a$, since $-3a + 8a = 5a$.

4. Subtracting $5a$ from $-3a$ gives a difference of $-8a$, since $5a + (-8a) = -3a$. (Art. **390.**)

5. Subtracting $-3a$ from $-5a$ gives a difference of $-2a$, since $-3a + (-2a) = -5a$.

6. Subtracting $-5a$ from $-3a$ gives a difference of $2a$, since $-5a + 2a = -3a$.

397. Now, if the sign of the subtrahend had been changed in each case and the resulting expression *added* (algebraically) to the minuend, the results would have been exactly the same. Thus, in the first case, $3a$ with its sign changed becomes $-3a$, which, added to $5a$, equals $5a + (-3a)$, or $2a$; in the second case, $5a$ with its sign changed becomes $-5a$, which added to $3a$, equals $-5a + 3a$, or $-2a$; in the third case, $-3a$ with its sign changed becomes $3a$, which, added to $5a$, equals $5a + 3a$, or $8a$, and so on. *Hence the difference, or the quantity to be added to the subtrahend to produce the minuend, may be found by changing the sign of the subtrahend and adding it to the minuend.*

398. From the foregoing it is evident that subtraction does not always imply a numerical decrease, for note the result in the 3d case. This is because negative quantities are considered. The difference in the value of a man's property, for example, when he *has* \$3 and when he *owes* \$5, is \$8.

SUBTRACTION OF MONOMIALS.

399. From Art. **397,** we have the following:

Rule.—*To subtract one term from another, change the sign of the subtrahend and proceed as in addition.*

EXAMPLE.—From $-3ab^2x$ take $7ab^2x$.

SOLUTION.—Changing the sign of the subtrahend, $7ab^2x$, and *adding*, we have $-3ab^2x - 7ab^2x = -10ab^2x$. Ans.

EXAMPLE.—From $8c^2 - 3c^2$ take $4a - 6a$.

SOLUTION.—Combining like terms, $8c^2 - 3c^2 = 5c^2$ and $4a - 6a = -2a$. Changing the sign of $2a$ and adding, we have as the difference, $5c^2 + 2a$. (Art. **391.**)

EXAMPLES FOR PRACTICE.

400. Solve the following:

1. From $17a$ take $-11a$. Ans. $28a$.
2. From $-11a$ take $17a$. Ans. $-28a$.
3. Subtract $5cd$ from $-4cd$. Ans. $-9cd$.
4. Subtract $-10b^2$ from $-10b^2$. Ans. 0.
5. From $13y$ take $4y$. Ans. $9y$.
6. From $x - 2x$ take 4. Ans. $-x - 4$.
7. Subtract $6mn - mn$ from $mn - 6mn$. Ans. $-10mn$.
8. What quantity added to $10xy$ will produce $-12xy$? Ans. $-22xy$.
9. What, then, does $10xy$ subtracted from $-12xy$ equal? Ans. $-22xy$.

SUBTRACTION OF POLYNOMIALS.

401. To subtract one polynomial from another:

Rule.—*Write the subtrahend underneath the minuend, with like terms in the same vertical column. Change the sign of each term of the subtrahend, and add the result to the minuend.*

EXAMPLE.—From $3ac - 2b$ subtract $ac - b - d$.

SOLUTION.—

$$\begin{array}{rrrrl} & 3ac & -2b & & \\ & -ac & +b & +d & \text{signs changed.} \\ \hline \textit{difference} = & 2ac & -b & +d & \text{Ans.} \end{array}$$

EXAMPLE.—From $2x^3 - 3x^2y + 2xy^2$ subtract $x^3 - xy^2 + y^3$.

SOLUTION.—

$$\begin{array}{rrrrrl} & 2x^3 & -3x^2y & +2xy^2 & & \\ & -x^3 & & +xy^2 & -y^3 & \text{signs changed.} \\ \hline \textit{difference} = & x^3 & -3x^2y & +3xy^2 & -y^3 & \text{Ans.} \end{array}$$

EXAMPLES FOR PRACTICE.

402. Solve the following:

1. From $7a + 5b - 3c$ take $a - 7b + 5c - 4$. Ans. $6a + 12b - 8c + 4$.
2. From $3m - 5n + r - 2s$ take $2r + 3n - m - 5s$. Ans. $4m - 8n - r + 3s$.
3. Subtract $2x - 2y + 2$ from $y - x$. Ans. $3y - 3x - 2$.
4. Subtract $3x^3 + 4x^2y - 7xy^2 + y^3 - xy^3$ from $5x^3 + x^2y - 6xy^2 + y^3$. Ans. $2x^3 - 3x^2y + xy^2 + xy^3$.
5. From $x^2 + 2xy + y^2$ take $x^2 - 2xy + y^2 + 4$. Ans. $4xy - 4$.

403. Before proceeding further, the student should make sure that he fully understands the use of the signs + and −. It has been seen that they indicate the character of the quantities they precede as well as addition and subtraction. In $10a - 7a$, for example, if the minus sign is supposed to indicate the *character* of $7a$, the expression would mean $10a + (-7a)$, or that the *negative* quantity $-7a$ is to be *added* to $10a$; if the minus sign is supposed to indicate subtraction, however, the expression would mean $10a - (+7a)$, or that the *positive* quantity $7a$ is to be *subtracted* from $10a$. Thus, $10a - 7a$ may indicate either the algebraic sum of $+10a$ and $-7a$, as explained in Art. **389**, or the difference (arithmetic ratio) between $+10a$ and $+7a$. The result in either case is the same. Thus,

$$\begin{array}{lrlr} & +10a & & +10a \\ \text{adding} & \underline{-\ 7a} & \text{subtracting} & \underline{+\ 7a} \\ \textit{sum} & = +\ 3a & \textit{difference} & = +\ 3a \end{array}$$

But the minus sign is usually considered to indicate the character of the quantity which the sign precedes, unless the minus sign is followed by a symbol of aggregation. The arithmetic ratio between $-a$ and $-b$ must always be expressed as $-a-(-b)$, the second minus sign indicating subtraction, and the third minus sign, the character of b, i. e., whether positive or negative.

SYMBOLS OF AGGREGATION.

404. Parentheses, brackets, braces, etc., are frequently used to enclose expressions containing one or more terms, when it is desired to *indicate* the addition or subtraction of the expressions so enclosed. To *actually perform* the addition or subtraction, the parenthesis or other symbol must be removed, which requires a due regard for the signs.

405. *When a parenthesis or like symbol is preceded by a minus sign, it may be removed if the signs of all the enclosed terms be changed from + to − or from − to +.*

The reason for this is that the minus sign indicates subtraction, the entire expression within the parenthesis being the subtrahend, and when the subtraction is performed by removing the parenthesis, the signs of the subtrahend must be changed. (Art. **399.**)

406. *When a parenthesis or like symbol is preceded by a plus sign, it may be removed without changing the signs of the enclosed terms.* This is evident from the fact that the plus sign indicates addition ; in addition the signs are not changed.

EXAMPLE.—Remove the parentheses from $4c - (3a + 4ab - d)$.

SOLUTION.—Changing the sign of each enclosed term, and remembering that the sign of $3a$ is $+$, understood, we have, as the result, $4c - 3a - 4ab + d$. Ans.

EXAMPLE.—Remove the parentheses from $4a - 5x - (a - 4x) + (x - 8a)$.

SOLUTION.—$4a - 5x - (a - 4x) + (x - 8a) = 4a - 5x - a + 4x + x - 8a$. Adding the like terms, we have

$$\begin{array}{r}
4a - 5x \\
-\ a + 4x \\
\underline{-8a + \ \ x} \\
-5a + 0
\end{array} = -5a. \quad \text{Ans.}$$

407. Symbols of aggregation will often be found enclosing others. In such cases they may be removed in succession by the preceding rules, *always beginning with the innermost pair.*

EXAMPLE.—Remove the parenthesis, etc., from $6a - \{b - [7cd - 4a + (2cd - \overline{a - b})]\}$

SOLUTION.—We first remove the vinculum. This being in effect the same as the parenthesis, the minus sign before the a indicates that $+a$ and $-b$ are to be subtracted.

Hence, we have

$$6a - \{b - [7cd - 4a + (2cd - a + b)]\}.$$

Removing the parenthesis we have

$$6a - \{b - [7cd - 4a + 2cd - a + b]\}.$$

This, with the brackets removed, equals

$$6a - \{b - 7cd + 4a - 2cd + a - b\},$$

which equals $\quad 6a - b + 7cd - 4a + 2cd - a + b.$

Combining like terms,

$$6a - 4a - a - b + b + 7cd + 2cd = a + 9cd. \quad \text{Ans.}$$

408. From the foregoing, it is evident that an expression may be placed within a parenthesis preceded by a minus sign by changing all its signs, or within a parenthesis preceded by a plus sign without changing its signs.

Example.—Place within a parenthesis the last three terms of $4xy + 2bc - 8x - 5 + 2b$, indicating that they are to be subtracted from the first two.

Solution.—

$$4xy + 2bc - 8x - 5 + 2b = 4xy + 2bc - (8x + 5 - 2b). \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

409. Remove the parentheses from the following:

1. $-(2mn - m^2 - n^2)$. Ans. $m^2 - 2mn + n^2$.
2. $1 - (-b + c + 3)$. Ans. $b - c - 2$.
3. $5a - 4b + 3c - (-3a + 2b - c)$. Ans. $8a - 6b + 4c$.
4. $3x - (2x - 5) + (7 - x)$. Ans. 12.

Remove the parentheses, etc., from the following:

5. $m - [4n - k - (m + n - 2k)]$. Ans. $2m - 3n - k$.
6. $5x - (2x - 3y) - (x + 5y)$. Ans. $2x - 2y$.
7. $3a - [7a - (5a - b - a)] - (-a - 4b)$. Ans. $a + 3b$.
8. $3x + \{2y - [5x - (3y + \overline{x - 4y})]\}$. Ans. $y - x$.
9. $100x - \{200x - [500x - (-100x) - 300x] - 400x\}$. Ans. $600x$.
10. $7cx - \{4cy - [(4cx + 3cy) + cy - cx]\}$. Ans. $10cx$.

Note.—Observe that the sign before the inner parenthesis is +, understood.

11. Place the 2d, 3d, and 4th terms of the expression $2cd - 8m + 5x - 2y + x - 4a$, within a parenthesis preceded by a minus sign.
Ans. $2cd - (8m - 5x + 2y) + x - 4a$.

12. Indicate the addition of $-b^2 + cd$ to 75. Ans. $75 + (-b^2 + cd)$.

13. Enclose the whole expression $x^2 + 2xy + y^2 - (a - b)$ within brackets having a minus sign prefixed.
Ans. $-[-x^2 - 2xy - y^2 + (a - b)]$.

Note.—Changing the sign before the inner parenthesis is in effect the same as changing the sign of both a and b; hence, the signs of a and $-b$ remain as before.

MULTIPLICATION.

410. **Multiplication** is the process of taking one quantity as many times as there are units in another quantity.

411. The quantity that is taken, or multiplied, is called the **multiplicand,** and the quantity by which we multiply is called the **multiplier.** The result of multiplication is called the **product.**

412. The **product** is obtained by taking the multiplicand a certain number of times, as indicated by the multiplier. *The product, therefore, is the same kind of a quantity as the multiplicand.* For example, 5 dollars multiplied by 10 is 50 *dollars;* 5 units × 10 = 50 *units;* 5 pounds × 10 = 50 *pounds.* In each case the multiplicand of 5 dollars, 5 units, or 5 pounds is taken ten times to form a product of the same kind.

413. The same rule holds with regard to positive and negative quantities. Thus, the positive quantity $+5a$ multiplied by 10 is the positive quantity $+50a$, while the negative quantity $-5a$ multiplied by 10 is the negative quantity $-50a$.

414. The **multiplier** shows, 1st, how many times the multiplicand is to be taken to form the product, and 2d, whether the product is to be added or subtracted when taken in connection with other quantities, a multiplier having the plus sign indicating that the product is to be added, and a multiplier with the minus sign, that it is to be subtracted.

415. There are four cases in multiplication that cover, in principle, all that are met with; they may be illustrated as follows:

1. $+5$ multiplied by $+3 = +15$. Since the multiplicand 5 is positive, the product 15 must be positive. (Art. **413.**) The plus sign of the multiplier indicates that the product is to be added. Hence, $(+5) \times (+3) = +(+15)$, which, by Art. **406,** $= +15$.

2. $+5$ multiplied by $-3 = -15$. Since the multiplicand is positive, the product must be positive. The minus sign of the multiplier indicates that the product is to be subtracted. Hence, $(+5) \times (-3) = -(+15)$, which, by Art. **405,** $= -15$.

By similar reasoning it can be shown that

3. $(-5) \times (+3) = +(-15) = -15$.
4. $(-5) \times (-3) = -(-15) = +15$.

416. From these examples, it will be observed that when the multiplicand and multiplier have like signs, the product has the plus sign, and when they have unlike signs, the product has the minus sign. Hence, in finding the product of two quantities, *like signs produce plus and unlike signs minus.*

417. We know from arithmetic that the product of two numbers is the same in whatever order they are taken. Thus, 3×4 and 4×3 are each equal to 12. Similarly, in Algebra, the factors will give the same results whatever their order. Thus, $5a \times 4b = 5 \times a \times 4 \times b = 5 \times 4 \times a \times b = 20ab$. Hence, in finding the product of two quantities, *the coefficients are multiplied together and prefixed to the literal factors.*

418. Let it be required to multiply a^3 by a^2. By Art. **362,** a^3 means $a \times a \times a$, and a^2 means $a \times a$. Hence, $a^3 \times a^2 = a \times a \times a \times a \times a = a^5 = a^{3+2}$. Therefore, *the exponent of a letter in the product is equal to the sum of its exponents in the factors.*

419. Particular notice should be taken of the way that coefficients and exponents are treated in multiplication. *Coefficients are multiplied, while exponents are added.*

MULTIPLICATION OF MONOMIALS.

420. From the foregoing principles, we have, when there are two factors, the following:

Rule.—*Multiply the coefficients together; annex the letters of both monomials to the result, giving to each letter an exponent equal to the sum of its exponents in the factors. Make the sign of the product plus when the two factors have like signs, and minus when they have unlike signs.*

EXAMPLE.—Multiply $4a^2b$ by $-5a^3bc$.

SOLUTION.—The product of the coefficients is 20, and the letters to be annexed are a, b, and c. The new exponent of a is 5, and of b, 2, since $a^{2+3} = a^5$, and $b^{1+1} = b^2$. The sign of the product is minus, since the two factors have different signs. Hence, $4a^2b \times -5a^3bc = -20a^5b^2c$. Ans.

421. When there are more than two factors, we have simply three or more examples in multiplication to do in succession, each to be performed by the foregoing rule.

Example.—Find the continued product of $6x^2yz^3$, $-9x^2y^2z^2$, and $-3x^4yz$.

Solution.— $6x^2yz^3 \times -9x^2y^2z^2 = -54x^{2+2}y^{1+2}z^{3+2}$, or $-54x^4y^3z^5$. Now, multiplying this product by $-3x^4yz$, we have $-54x^4y^3z^5 \times -3x^4yz = 162x^8y^4z^6$. Ans.

EXAMPLES FOR PRACTICE.

422. Find the product of

1. a^3b^2 and $-5abd$. Ans. $-5a^4b^3d$.
2. $-7xy$ and $-7x^2y^2$. Ans. $49x^3y^3$.
3. $-15m^5n^6$ and $3mn$. Ans. $-45m^6n^7$.
4. $3a(x-y)^2$ and $2a^2(x-y)$. Ans. $6a^3(x-y)^3$.

Note.—Treat the $(x-y)$ as though it were a single letter.

5. Find the continued product of $2a^3m^2x$, $-3a^2mx^3$ and $4am^3x^2$. Ans. $-24a^6m^6x^6$.
6. What does $-a^2bn \times -2cdn \times -3bdc^2 \times -2acn^2$ equal? Ans. $12a^3b^2c^4d^2n^4$.

MULTIPLICATION OF POLYNOMIALS.

423. When one of the factors is a monomial:

Rule.—*Multiply each term of the polynomial by the monomial, remembering that like signs produce plus, and unlike signs produce minus.*

Example.—Find the product of $-9a^5+3a^3b^2-4a^2b^3-b^5$ and $-3ab^4$.

Solution.—

$$\begin{array}{ll} -9a^5+3a^3b^2-4a^2b^3-b^5 & \\ -3ab^4 & \\ \hline 27a^6b^4-9a^4b^6+12a^3b^7+3ab^9 & \text{Ans.} \end{array}$$

424. When both factors are polynomials:

Rule.—*Multiply each term of one polynomial by each term of the other, and add the partial products.*

Example.—Multiply $6a-4b$ by $4a-2b$.

Solution.—Write the multiplier under the multiplicand, and begin to multiply *at the left* instead of at the right, as in arithmetic, since polynomials are always written and read from the left, and there are no numbers to carry.

$$\begin{array}{lll} & 6a - 4b & (1) \\ & 4a - 2b & \\ \cline{2-2} \text{Multiplying (1) by } 4a \text{ gives} & 24a^2-16ab & (2) \\ \text{Multiplying (1) by } -2b \text{ gives} & \qquad -12ab+8b^2 & (3) \\ \cline{2-2} \text{Adding (2) and (3) gives} & 24a^2-28ab+8b^2 & \text{Ans.} \end{array}$$

It will be noticed that the like terms, $-16ab$ and $-12ab$, are written under each other so that it will be easier to add them.

EXAMPLE.—Multiply $x^3 - x + 1 + x^2$ by $1 - x^2 + x$.

SOLUTION.—With a view to bringing like terms in the same columns, arrange both multiplicand and multiplier either according to the increasing or decreasing powers of the same letter. (Art. **372.**) Arranging in this case according to the increasing powers of x, we have

$$\begin{array}{lllllllll}
 & 1 & -x & +x^2 & +x^3 & & & (1) \\
 & 1 & +x & -x^2 & & & & \\ \hline
\text{Multiplying (1) by 1 gives} & 1 & -x & +x^2 & +x^3 & & & (2) \\
\text{Multiplying (1) by } +x \text{ gives} & & x & -x^2 & +x^3 & +x^4 & & (3) \\
\text{Multiplying (1) by } -x^2 \text{ gives} & & & -x^2 & +x^3 & -x^4 & -x^5 & (4) \\ \hline
\text{Adding (2), (3), and (4) gives} & 1 & & -x^2 & +3x^3 & & -x^5 & \text{Ans.}
\end{array}$$

EXAMPLE.—Multiply $2a + 1 - 3a^2 + a^4$ by $a^3 - 2a - 2$.

SOLUTION.—Arranging according to the decreasing powers of a,

$$\begin{array}{lllllllll}
 & a^4 & & -3a^2 & +2a & +1 & & & (1) \\
 & a^3 & & -2a & -2 & & & & \\ \hline
\text{Multiplying (1) by } a^3 \text{ gives} & a^7 & -3a^5 & +2a^4 & +a^3 & & & & \\
\text{Multiplying (1) by } -2a \text{ gives} & & -2a^5 & & +6a^3 & -4a^2 & -2a & & \\
\text{Multiplying (1) by } -2 \text{ gives} & & & -2a^4 & & +6a^2 & -4a & -2 & \\ \hline
\text{Adding} & a^7 & -5a^5 & & +7a^3 & +2a^2 & -6a & -2 &
\end{array}$$

[Ans.

425. The product of two or more polynomials is often indicated by enclosing each of the factors in a parenthesis, and writing one after the other. When the indicated multiplication is performed, the expression is said to be expanded.

EXAMPLES FOR PRACTICE.

426. Multiply the following:

1. $x^2 + 2xy + y^2$ by $x + y$. Ans. $x^3 + 3x^2y + 3xy^2 + y^3$.
2. $3ab^2m^3 + 4a^2b - 2$ by $a^6b^7m^8$. Ans. $3a^7b^9m^{11} + 4a^8b^8m^8 - 2a^6b^7m^8$.
3. $c^2 - d^2$ by $c^2 + d^2$. Ans. $c^4 - d^4$.
4. $x^4 + x^2y^2 + y^4$ by $x^2 - y^2$. Ans. $x^6 - y^6$.
5. $x^2 + 1 - x^3 - x$ by $x + 1$. Ans. $1 - x^4$.
6. $3a^2 - 7a + 4$ by $2a^2 + 9a - 5$. Ans. $6a^4 + 13a^3 - 70a^2 + 71a - 20$.
7. $5m^2 - 3mn + 4n^2$ by $6m - 5n$. Ans. $30m^3 - 43m^2n + 39mn^2 - 20n^3$.

Expand the following:

8. $(2a - 3c)(4 - 3a)$. Ans. $8a - 12c - 6a^2 + 9ac$.
9. $(x + 2)(x - 2)(x^2 + 4)$. Ans. $x^4 - 16$.
10. $[x(x^2 - y^2) - 2][x(x^2 + y^2) + 2]$.

NOTE.—The expressions in the brackets reduce to $x^3 - xy^2 - 2$ and $x^3 + xy^2 + 2$. The product of these is $x^6 - x^2y^4 - 4xy^2 - 4$.

THREE IMPORTANT EXAMPLES.

427. The first two examples are:

1. To find the square of the *sum* of two quantities.

2. To find the square of the *difference* of two quantities.

Let a represent one quantity and b the other quantity. Their sum would then be represented by $a + b$ and their difference by $a - b$. Squaring these by multiplying each by itself, we have:

1.
$$\begin{array}{rrr} a & +b & \\ a & +b & \\ \hline a^2 & +ab & \\ & ab & +b^2 \\ \hline a^2 & +2ab & +b^2 \end{array}$$

2.
$$\begin{array}{rrr} a & -b & \\ a & -b & \\ \hline a^2 & -ab & \\ & -ab & +b^2 \\ \hline a^2 & -2ab & +b^2 \end{array}$$

428. By examining the products it will be evident that, since a and b represent any two quantities, *the square of the sum of two quantities equals the square of the first, plus twice the product of the first by the second, plus the square of the second.*

429. Also, that

The square of the difference of two quantities equals the square of the first, minus twice the product of the first by the second, plus the square of the second.

430. The third example is:

3. To find the product of the sum and difference of two quantities. By multiplication, we have

$$\begin{array}{rrr} a & +b & \\ a & -b & \\ \hline a^2 & +ab & \\ & -ab & -b^2 \\ \hline a^2 & & -b^2 \end{array}$$

431. That is, *the product of the sum and difference of two quantities equals the difference of their squares.*

432. These three principles are very important, and should be committed to memory. They may also be expressed by formulas in the same way that the rule in Art.

353 was expressed by a formula. Thus, letting a represent one quantity and b the other quantity,

$$(a+b)^2 = a^2 + 2ab + b^2. \qquad \textbf{(1.)}$$
$$(a-b)^2 = a^2 - 2ab + b^2. \qquad \textbf{(2.)}$$
$$(a+b)(a-b) = a^2 - b^2. \qquad \textbf{(3.)}$$

The meaning of these formulas will be made clear by supposing, for example, that $a = 10$ and $b = 2$. By formula **1**, the square of the sum of 10 and 2, or the square of 12, should be $10^2 + 2 \times 10 \times 2 + 2^2 = 100 + 40 + 4 = 144$, which we know to be actually the case.

433. By the use of these principles and formulas the *square* of the *sum or difference* of any two quantities, or the *product* of the *sum and difference* of any two quantities may be found, without actually performing the multiplication. *The student should practise until he can readily apply them.*

EXAMPLE.—Square $3x^2 + 5$.

SOLUTION.—The square of the first term is $3x^2 \times 3x^2 = 9x^4$, twice the product of the terms is $30x^2$, and the square of the last term is 25. Hence, by formula **1**, letting $a = 3x^2$ and $b = 5$,

$$(3x^2+5)^2 = 9x^4 + 30x^2 + 25. \quad \text{Ans.}$$

EXAMPLE.—Square $4cd - x$.

SOLUTION.—The square of the first term is $16c^2d^2$, twice the product of the first by the second is $8cdx$, and the square of the last term is x^2. Hence, by formula **2**, letting $a = 4cd$ and $b = x$,

$$(4cd - x)^2 = 16c^2d^2 - 8cdx + x^2. \quad \text{Ans.}$$

EXAMPLE.—Expand $(x^2+3)(x^2-3)$.

SOLUTION.—The square of the first term is x^4, and of the second, 9. Hence, by formula **3**, letting $a = x^2$ and $b = 3$,

$$(x^2+3)(x^2-3) = x^4 - 9. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

434. Square the following :

1. $m + n$. Ans. $m^2 + 2mn + n^2$.
2. $4x + 2$. Ans. $16x^2 + 16x + 4$.
3. $3a - 5b$. Ans. $9a^2 - 30ab + 25b^2$.
4. $1 - 2c^3$. Ans. $1 - 4c^3 + 4c^6$.
5. $3x^2 - 2y^3$. Ans. $9x^4 - 12x^2y^3 + 4y^6$.
6. $ab^2c^3 + 1$. Ans. $a^2b^4c^6 + 2ab^2c^3 + 1$.

Expand the following :

7. $(m+1)(m-1)$. Ans. $m^2 - 1$.

8. $(x^2 + y^2)(x^2 - y^2)$. Ans. $x^4 - y^4$.
9. $(3x^2y + 2)(3x^2y - 2)$. Ans. $9x^4y^2 - 4$.
10. $(4a + 4b^2)(4a - 4b^2)$. Ans. $16a^2 - 16b^4$.
11. Square $2c^2 - c + d$.

SOLUTION to 11.—By Art. **408,** this may be written $2c^2 - (c - d)$. Then, in squaring, the binomial $c - d$ should be used as one quantity. (See Art. **391.**) Thus, $(2c^2)^2 = 4c^4$; $(c - d)^2 = c^2 - 2cd + d^2$. Hence, by formula **2,** letting $a = 2c^2$ and $b = c - d$, we have $(2c^2 - c + d)^2 = 4c^4 - 4c^2(c - d) + c^2 - 2cd + d^2 = 4c^4 - 4c^3 + 4c^2d + c^2 - 2cd + d^2$. Ans. This method is sometimes, though not always, more convenient than direct multiplication.

DIVISION.

435. Division, in Algebra, is that process by which, when a product and one of its factors are given, the other factor may be found. The product of the two factors is called the **dividend,** the given factor the **divisor,** and the required factor the **quotient.**

From these definitions, it is clear that the quotient multiplied by the divisor produces the dividend. Division, therefore, is the converse of multiplication, and the following principles may be proved directly from those given in Arts. **416-419:**

436. *If the dividend and divisor have like signs, the quotient will have the plus sign; if they have unlike signs, the quotient will have the minus sign.*

437. *The coefficient of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.*

438. *The exponent of a letter in the quotient is equal to its exponent in the dividend, minus its exponent in the divisor.*

439. Let it be required to divide a^2 by a^2. We have to obtain a quotient which, when multiplied by the divisor a^2, will produce the dividend a^2. The quotient is evidently 1. By Art. **438,** however, we know that $a^2 \div a^2 = a^{2-2} = a^0$. Hence, *any quantity whose exponent is 0 is equal to 1.*

From the foregoing principles, the rules for division are obtained.

DIVISION OF MONOMIALS.

440. Rule.—*Divide the coefficient of the dividend by that of the divisor. Annex to the result the letters of the dividend, each with an exponent equal to its exponent in the dividend, minus its exponent in the divisor, omitting all letters whose exponents become zero.*

Make the sign of the quotient plus when the dividend and divisor have like signs, and minus when they have unlike signs.

Example.—Divide $6a^5b^4c^3$ by $-3a^2bc^3$.

Solution.—The quotient of $6 \div 3$ is 2. The letters to be annexed, and their exponents, are $a^{5-2} = a^3$, and $b^{4-1} = b^3$. The c has an exponent of $3 - 3 = 0$, so that it becomes equal to 1, and is omitted. The sign of the quotient is minus. Hence, $6a^5b^4c^3 \div -3a^2bc^3 = -2a^3b^3$. Ans. Proof: $-3a^2bc^3 \times -2a^3b^3 = 6a^5b^4c^3$.

Example.—Divide $-10a^6b^3c^2d$ by $-2ab^3c$.

Solution.— $-10a^6b^3c^2d \div -2ab^3c = 5a^{6-1}b^{3-3}c^{2-1}d = 5a^5cd$. Ans.

EXAMPLES FOR PRACTICE.

441. Divide the following:

1. $12m^2n$ by $4n$. Ans. $3m^2$.
2. $30x^6y^5bc^3$ by $-6x^5y^5c^2$. Ans. $-5xbc$.
3. $-44a^3b^3c^3$ by $-11ab^2c^3$. Ans. $4a^2b$.
4. $-100x^4y^3z^2$ by x^3y^2. Ans. $-100xyz^2$.
5. $75pq^2x^3m^4$ by $75x^3$. Ans. pq^2m^4.

DIVISION OF POLYNOMIALS.

442. When the divisor is a monomial:

Rule.—*Divide each term of the dividend by the divisor, remembering that like signs produce plus and unlike signs minus.*

Example.—Divide $12a^2b^4-9ab^3+6a^3b^4$ by $3ab^3$.

Solution.—

$$\begin{array}{r|l} 3ab^3 & 12a^2b^4 - 9ab^3 + 6a^3b^4 \\ \hline & 4ab \quad - 3 \quad + 2a^2b \end{array}$$ *quotient.* Ans.

EXAMPLES FOR PRACTICE.

443. Divide the following:

1. $64m^2n^3 - 32mn^2 + 8m^2n$ by $8mn$. Ans. $8mn^2 - 4n + m$.

2. $27x^3y^2z - 9x^3yz^2 - 333x^3y^2z^2$ by $-3x^3yz$. Ans. $-9y + 3z + 111yz$.

3. $10(x+y)^2 - 5a(x+y) + 5a^2(x+y)$ by $5(x+y)$.
Ans. $2(x+y) - a + a^2$.

444. When the divisor is a polynomial:

Rule.—*Arrange both dividend and divisor according to the ascending or descending powers of some letter.*

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Multiply the whole divisor by the first term of the quotient, and subtract the product from the dividend.

Regard the remainder as a new dividend, and divide its first term by the first term of the divisor, for the second term of the quotient. Multiply the whole divisor by the second term of the quotient, and subtract the product from the first remainder.

So proceed until a remainder of zero is found, or a remainder whose first term cannot be divided by the first term of the divisor.

Example.—Divide $x^4 + x^3 - 9x^2 - 16x - 4$ by $x^2 + 4x + 4$.

Solution.—

$$\begin{array}{r|l}
x^2+4x+4\ ) & x^4+x^3-9x^2-16x-4\ (\ x^2-3x-1 \textit{ quotient. } \text{Ans.} \\
 & \underline{x^4+4x^3+4x^2} \\
 & -3x^3-13x^2-16x \\
 & \underline{-3x^3-12x^2-12x} \\
 & -x^2-4x-4 \\
 & \underline{-x^2-4x-4} \\
 & 0 \quad 0 \quad 0
\end{array}$$

The first term x^2 of the divisor is contained in x^4, the first term of the dividend, x^2 times; hence, x^2 is the first term of the quotient. The whole divisor multiplied by this term gives $x^4 + 4x^3 + 4x^2$ as a product, which, subtracted from the dividend, gives as a remainder $-3x^3 - 13x^2 - 16x - 4$. It is not necessary here to bring down the -4, since only three terms are required to contain the divisor.

The first term x^2 of the divisor is contained in $-3x^3$, the first term of the new dividend, $-3x$ times. Multiplying the divisor by this new term of the quotient, we have $-3x^3 - 12x^2 - 12x$. Subtracting this from the first remainder, we obtain $-x^2 - 4x - 4$ for a new remainder, the -4 being brought down from the original dividend. The first term of the divisor is contained in the first term of the new remainder or dividend -1 time. Multiplying the divisor by this, we get $-x^2 - 4x - 4$, which, subtracted from $-x^2 - 4x - 4$, the last remainder, leaves a difference of zero. The work ends here, since there are no more terms in the dividend to be brought down.

Example.—Divide $9x^2y^2 + x^4 - 4y^4 - 6x^3y$ by $x^2 + 2y^2 - 3xy$.

Solution.—First arrange the dividend and divisor according to the descending powers of x.

$$\begin{array}{llllll}
x^2 - 3xy + 2y^2) & x^4 - 6x^3y & + 9x^2y^2 & & - 4y^4 & (x^2 - 3xy - 2y^2 \quad \text{Ans.} \\
& \underline{x^4 - 3x^3y + 2x^2y^2} & & & & \\
& \quad - 3x^3y & + 7x^2y^2 & & - 4y^4 & \\
& \quad \underline{- 3x^3y} & \underline{+ 9x^2y^2} & \underline{- 6xy^3} & & \\
& & - 2x^2y^2 & + 6xy^3 & - 4y^4 & \\
& & \underline{- 2x^2y^2} & \underline{+ 6xy^3} & \underline{- 4y^4} & \\
& & \quad 0 & \quad 0 & \quad 0 &
\end{array}$$

EXAMPLES FOR PRACTICE.

445. Divide the following:

1. $x^2 - 7x + 12$ by $x - 3$. Ans. $x - 4$.
2. $x^2 + x - 72$ by $x + 9$. Ans. $x - 8$.
3. $2x^3 - x^2 + 3x - 9$ by $2x - 3$. Ans. $x^2 + x + 3$.
4. $x^4 + 11x^2 - 12x - 5x^3 + 6$ by $3 + x^2 - 3x$. Ans. $x^2 - 2x + 2$.
5. $x^4 - 6xy - 9x^2 - y^2$ by $x^2 + y + 3x$. Ans. $x^2 - 3x - y$.
6. $x^6 - 1$ by $x - 1$. Ans. $x^5 + x^4 + x^3 + x^2 + x + 1$.

FACTORING.

446. In multiplication, two or more factors are multiplied together to form a *product*. **Factoring** is the process of resolving a product into its **factors**.

447. The factors of a quantity are those quantities which multiplied together will produce the quantity. Thus, $6a^2$ and b^3; $2a^2$ and $3b^3$; a^2 and $6b^3$; $2a$ and $3ab^3$; 2, 3, a, a, b, b and b, etc., are all factors of $6a^2b^3$, since $6a^2 \times b^3 = 6a^2b^3$, $2a^2 \times 3b^3 = 6a^2b^3$, $a^2 \times 6b^3 = 6a^2b^3$, $2a \times 3ab^3 = 6a^2b^3$ and $2 \times 3 \times a \times a \times b \times b \times b = 6a^2b^3$.

In solving algebraic problems, it is frequently very necessary to be able to recognize a factor of one or more algebraic expressions, and Arts. **446-471** give some of the simplest methods of discovering factors.

448. A quantity is a **perfect square** when it has two equal factors. One of the equal factors is called the **square root** of the quantity. (Art. **363.**)

449. Equal factors are those whose letters have the same exponents and coefficients, and which have the same signs. Thus, $3(4cd^2-y)$ and $3(4cd^2-y)$ are equal factors of $3(4cd^2-y)\times 3(4cd^2-y)=9(4cd^2-y)^2$; but $3(4cd^2-y)$ and $-3(4cd^2-y)$ are unequal factors, since the signs of the coefficient 3 are not the same. The factor $-3(4cd^2-y)$ is equivalent to $3(y-4cd^2)$, an expression which will have in general a different value from $3(4cd^2-y)$, when the numerical values are substituted for the letters.

450. A quantity is a **perfect cube** when it has three equal factors. One of the equal factors is called the **cube root** of the quantity.

451. In factoring, it is important to be able to easily distinguish quantities that are perfect squares and cubes, and to determine their roots. By definition, $9a^2b^2$ is a perfect square because $3ab\times 3ab=9a^2b^2$, and $3ab$ is its square root. Also, $8a^6$ is a perfect cube because $2a^2\times 2a^2\times 2a^2=8a^6$, and $2a^2$ is its cube root. In each of these cases the coefficients are multiplied together, and the exponents are *added*, to produce the coefficients and exponents of the power, according to the rules of multiplication. *Hence, a quantity is a perfect square when its coefficient is a perfect square, and the exponents of all its letters can be divided by 2;* it will also be shown later that a perfect square must be a positive quantity. *A quantity is a perfect cube when its coefficient is a perfect cube and the exponents of all its letters can be divided by 3.* For example, $36x^{10}$, $49b^2c^4d^8$, $16a^6b^{12}$ and 1 are all perfect squares, whose roots are $6x^5$, $7bc^2d^4$, $4a^3b^6$ and 1, respectively; $27x^{12}$, $-64b^3c^9d^6$, $8a^{15}b^{18}$ and 1 are all perfect cubes, whose roots are $3x^4$, $-4bc^3d^2$, $2a^5b^6$ and 1, respectively.

CASE I.

452. *When all the terms of an expression have a common factor, the expression may be resolved into two factors by dividing each term by the common factor.*

EXAMPLE.—Factor $16x^2y^2 + 4x^3y^2 - 12xy^4$.

SOLUTION.—It is evident that each term contains the common factor $4xy^2$. Dividing the expression by $4xy^2$, we obtain as a quotient $4x + x^2 - 3y^2$. The two factors, therefore, are $4xy^2$ and $4x + x^2 - 3y^2$. Hence, by Art. **452**, we have

$$16x^2y^2 + 4x^3y^2 - 12xy^4 = 4xy^2(4x + x^2 - 3y^2). \quad \text{Ans.}$$

453. When examining a polynomial for a monomial factor, first ascertain if the numerical coefficients have a common factor. This is readily done by dividing the polynomial by the smallest coefficient. If it will not divide each term exactly, factor this coefficient, and divide by each factor. If none of the factors will divide each term of the polynomial without a remainder, the polynomial has no numerical factor. Having ascertained that the coefficients have a common factor, reserve this factor and examine the polynomial to see if each term has a common letter. If so, divide each term by this letter, affecting it with an exponent corresponding to the lowest exponent of the letter in the polynomial, and multiply the numerical factor (if any) previously found by the letter. So proceed with the remaining letters.

EXAMPLE.—Ascertain if $12ab^2c^3 - 18a^3c^2y + 24a^2c^4 - 36a^4bc^5y^2$ has a monomial factor.

SOLUTION.—12 is the lowest coefficient, and will divide each term except the second. Resolving 12 into the factors 2 and 6, 6 will divide the second term; hence, 6 is a numerical factor of all the terms of the polynomial. The letters a and c are common to all of the terms, and the lowest powers of a and c are a and c^2. Multiplying together 6, a, and c^2, $6ac^2$ is a factor of the polynomial. Dividing the polynomial by $6ac^2$, the quotient is $2b^2c - 3a^2y + 4ac^2 - 6a^3bc^3y^2$; consequently, the factors are $6ac^2$ and $2b^2c - 3a^2y + 4ac^2 - 6a^3bc^3y^2$. Ans.

EXAMPLES FOR PRACTICE.

454. Factor the following expressions:

1. $a^4 + ax$. Ans. $a(a^3 + x)$.
2. $12a^5 - 2a^3 + 4a^4$. Ans. $2a^3(6a^2 - 1 + 2a)$.

3. $30m^4n^2 - 6n^3$. Ans. $6n^2(5m^4 - n)$.

4. $16x^2y^3 - 8x^5 + 8$. Ans. $8(2x^2y^3 - x^5 + 1)$.

5. $4x^3y - 12x^2y^2 + 8xy^3$. Ans. $4xy(x^2 - 3xy + 2y^2)$.

6. $49a^2b^3c^4 - 63a^3b^2c^4 + 7a^4b^2c^3$. Ans. $7a^2b^2c^3(7bc - 9ac + a^2)$.

CASE II.

455. *To factor a trinomial which is a perfect square:*

This case is simply the reverse of Arts. **428** and **429,** and may be expressed by the following formulas, which we have from **1** and **2** in Art. **432:**

$$a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2. \qquad \textbf{(4.)}$$

$$a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2. \qquad \textbf{(5.)}$$

The two trinomials $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ each have two equal factors, and are, therefore, perfect squares; moreover, since a may represent one quantity and b any other quantity, it is evident that *any* trinomial having the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ is a perfect square. Hence,

456. *Any trinomial is a perfect square when the first and last terms are perfect squares and positive, and the second term is twice the product of their square roots.*

457. By examining the foregoing trinomials and their factors, the following rule for obtaining one of the equal factors will be evident.

Rule.—*Extract the square roots of the first and last terms of the trinomial, and connect the results by the sign of the second term.*

EXAMPLE.—Factor $x^2 + 2xy + y^2$.

SOLUTION.—The square root of the first term is x; of the last term, y; twice this product equals the second term; the sign of the second term is plus. Hence, one of the equal factors is $x + y$. Therefore, by formula **4,** Art. **455,** letting $a = x$ and $b = y$,

$$x^2 + 2xy + y^2 = (x + y)(x + y) = (x + y)^2. \quad \text{Ans.}$$

EXAMPLE.—Factor $36m^2 - 24mn + 4n^2$.

SOLUTION.—The square root of the first term is $6m$; of the last term, $2n$; twice their product equals the second term; the sign of the second term is minus. Hence, one of the equal factors is $6m - 2n$. Therefore, by formula **5,** letting $a = 6m$ and $b = 2n$,

$$36m^2 - 24mn + 4n^2 = (6m - 2n)(6m - 2n) = (6m - 2n)^2. \quad \text{Ans.}$$

458. When the terms of a trinomial which is a perfect square do not come in their proper order, as indicated in Art. **456**, they should be rearranged to bring them so.

EXAMPLE.—Factor $6a^3bc + 9a^6b^2c^2 + 1$.

SOLUTION.—By Art. **451**, the terms $9a^6b^2c^2$ and 1 are perfect squares, and $6a^3bc$ is equal to twice the product of their square roots, or $6a^3bc = 2 \times 3a^3bc \times 1$. Hence, arranged in their proper order, $9a^6b^2c^2$ and 1 should be the first and last terms, and $6a^3bc$ the second term, as follows: $9a^6b^2c^2 + 6a^3bc + 1$. Factoring this expression, we have $3a^3bc + 1$ as one of the factors. Therefore,

$$6a^3bc + 9a^6b^2c^2 + 1 = (3a^3bc + 1)(3a^3bc + 1) = (3a^3bc + 1)^2. \quad \text{Ans.}$$

459. It is quite as important to be able to distinguish trinomials which are perfect squares from those that are not, as to be able to factor them. This can always be done by the aid of the principle stated in Art. **456**. For example, to find whether $9a^4b^6 + 4 - 6a^2b^3$ is a perfect square, we first arrange the terms so that the first and last are the perfect squares, and have $9a^4b^6 - 6a^2b^3 + 4$. Now, it will be observed that the second term is not equal to twice the product of the square roots of the first and last terms, or $2 \times 3a^2b^3 \times 2$ does not equal $6a^2b^3$, so that the trinomial is not a perfect square. Again, $4a^4 + 4a^2b^2 - b^4$ is not a perfect square, because the last term, b^4, is negative. It will be noticed that the second term, $4a^2b^2$, is a perfect square in this case. It must be placed second, however, because it is the only term that is equal to twice the product of the square roots of the other two.

460. In ascertaining whether a trinomial is a perfect square, first find whether the numerical coefficients of two of the terms have like signs and are perfect squares. Then, see if the coefficient of the remaining term is equal to twice the product of the square roots of the coefficients of the other two terms. Lastly, extract the square root of those two terms whose coefficients are perfect squares, and multiply the two results. If twice the product equals the remaining term, the trinomial is a perfect square.

EXAMPLE.—Ascertain whether $4a^2 + 16c^4 - 16ac^2$ is a perfect square.

SOLUTION.—Here all of the coefficients are perfect squares, and the first two terms have like signs. $\sqrt{4} = 2$, and $\sqrt{16} = 4$. Twice the product of 2 and 4 = (2 × 4) × 2 = 16 = the coefficient of the remaining term. Lastly, $\sqrt{4a^2} = 2a$ and $\sqrt{16c^4} = 4c^2$, and $(2a \times 4c^2) \times 2 = 16ac^2$ = the remaining term; hence, the trinomial is a perfect square and is equal to $(2a - 4c^2)(2a - 4c^2)$, or to $(4c^2 - 2a)(4c^2 - 2a)$. Ans.

EXAMPLE.—Ascertain if $b^2 - 6bd - 9d^2$ is a perfect square.

SOLUTION.—The two terms having like signs are $6bd$ and $9d^2$. Since $6bd$ is not a perfect square, the trinomial is not a perfect square. Ans.

EXAMPLE.—Ascertain if $20acd - 4a^2c^2 - 25d^2$ is a perfect square.

SOLUTION.—The two terms having like signs are $-4a^2c^2$ and $-25d^2$. According to Art. **456,** these two terms must be positive. Therefore, dividing the trinomial by -1, i. e., changing all the signs, it becomes $-(4a^2c^2 + 25d^2 - 20acd)$. The two terms having like signs are now perfect squares, and twice the product of their square roots is $(2ac \times 5d) \times 2 = 20acd$ = the remaining term. Consequently, the trinomial is a perfect square and is equal to $-(2ac - 5d)^2$, or $-(5d - 2ac)^2$. Ans.

EXAMPLE.—Ascertain if $4m^4n^2 - 3m^2np + 4p^2$ is a perfect square.

SOLUTION.—The first and last terms have like signs and their coefficients are perfect squares, but the coefficient of the remaining term is not equal to twice the product of the square roots of the coefficients of the other two terms. Therefore, the trinomial is not a perfect square. Ans.

EXAMPLES FOR PRACTICE.

461. Determine which of the following trinomials are perfect squares:

1. $x^2 + xy + y^2$.
2. $a^6 - 2a^3x^2 + x^4$.
3. $m^6 + 16 + 8m^3$.
4. $100 + 22y + y^2$.
5. $x^2y^2 + 30xy + 256$.
6. $16x^2 - 8xyz + y^2z^2$.
7. $m^2 + 2mn - n^2$.

Ans. The 2d, 3d, and 6th.

Factor the following trinomials:

8. $x^2 - 16x + 64$. Ans. $(x - 8)^2$.
9. $n^6 - 26n^3 + 169$. Ans. $(n^3 - 13)^2$.
10. $25x^2 + 70xyz + 49y^2z^2$. Ans. $(5x + 7yz)^2$.
11. $16c^2 + b^2 - 8bc$. Ans. $(4c - b)^2$.
12. $2mx + m^2 + x^2$. Ans. $(m + x)^2$.
13. $a^2b^4c^6 - 2ab^2c^3 + 1$. Ans. $(ab^2c^3 - 1)^2$.

CASE III.

462. *To factor an expression which is the difference between two perfect squares:*

This case is the reverse of Art. **431,** and may be expressed by the formula

$$a^2 - b^2 = (a + b)(a - b), \qquad \textbf{(6.)}$$

which we have from **3,** Art. **432.**

463. Since a may represent one quantity, and b any other quantity, it is evident from formula **6** that any expression which is the difference between two perfect squares may be factored by the following:

Rule.—*Extract the square roots of the first and last terms. Add these roots for the first factor, and subtract the second from the first for the second factor.*

Example.—Factor $9x^8y^6 - 4$.

Solution.—The square roots of the first and last terms are $3x^4y^3$ and 2. The sum of these roots is $3x^4y^3 + 2$ and the second subtracted from the first is $3x^4y^3 - 2$. Hence, by formula **6**, letting $a = 3x^4y^3$ and $b = 2$,

$$9x^8y^6 - 4 = (3x^4y^3 + 2)(3x^4y^3 - 2). \quad \text{Ans.}$$

Example.—Factor $(a + b)^2 - m^2n^2$.

Solution.—The square roots of the first and last terms are $a + b$ and mn. The sum of these roots is $a + b + mn$, and the second subtracted from the first is $a + b - mn$. Hence, by formula **6**, letting $a = a + b$ and $b = mn$,

$$(a + b)^2 - m^2n^2 = (a + b + mn)(a + b - mn). \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

464. Factor the following expressions:

1.	$a^2 - 16$.	Ans. $(a + 4)(a - 4)$.
2.	$a^2 - 49c^8$.	Ans. $(a + 7c^4)(a - 7c^4)$.
3.	$81x^6y^4 - 1$.	Ans. $(9x^3y^2 + 1)(9x^3y^2 - 1)$.
4.	$(ax + by)^2 - 1$.	Ans. $(ax + by + 1)(ax + by - 1)$.
5.	$25x^4y^2 - (bx + 1)^2$.	Ans. $[5x^2y + (bx + 1)][5x^2y - (bx + 1)] =$ $(5x^2y + bx + 1)(5x^2y - bx - 1)$.
6.	$1 - 169x^2y^4z^6$.	Ans. $(1 + 13xy^2z^3)(1 - 13xy^2z^3)$.

465. In example 5, the expression $(bx + 1)^2$ should be regarded as a single term; in fact, any number of terms may be regarded as a single term by enclosing them in parenthesis and operating on them as though they were a single letter.

When solving any examples requiring the application of the rules in Art. **463** or **466**, first ascertain if the numerical coefficients of the two terms are perfect squares or perfect cubes; if not, there is no use of examining further.

CASE IV.

466. *To factor an expression which is the sum or difference of two perfect cubes:*

Letting a represent one quantity and b some other quantity, the sum and difference of two perfect cubes will be represented by $a^3 + b^3$ and $a^3 - b^3$. By actual division it may be shown that

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2, \text{ and}$$
$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$

Hence, any expression which is the sum or difference of two perfect cubes may be factored as follows:

Rule.—*Extract the cube root of each term. Connect the results by the sign of the second term for the first factor, and obtain the second factor by division.*

It is to be noticed that the second factor will not be a perfect square, because its second term will not be twice the product of the square roots of the other two.

Example.—Factor $8x^6 - 27y^9$.

Solution.—The cube root of the first term is $2x^2$, and of the second term $3y^3$; the sign of the second term is minus. Consequently, the first factor is $2x^2 - 3y^3$. The second factor we find by division to be $4x^4 + 6x^2y^3 + 9y^6$. Hence, the factors are $2x^2 - 3y^3$ and $4x^4 + 6x^2y^3 + 9y^6$. Ans.

EXAMPLES FOR PRACTICE.

467. Factor the following expressions:

1. $x^3 - y^3$. Ans. $(x - y)(x^2 + xy + y^2)$.
2. $m^3 + 64n^6$. Ans. $(m + 4n^2)(m^2 - 4mn^2 + 16n^4)$.
3. $27a^3 - 8x^3$. Ans. $(3a - 2x)(9a^2 + 6ax + 4x^2)$.
4. $1{,}000 - 27a^6b^3$. Ans. $(10 - 3a^2b)(100 + 30a^2b + 9a^4b^2)$.
5. $1 + 729m^{12}n^{15}$. Ans. $(1 + 9m^4n^5)(1 - 9m^4n^5 + 81m^8n^{10})$.
6. $512a^3 - 64b^3$. Ans. $(8a - 4b)(64a^2 + 32ab + 16b^2)$.

CASE V.

468. Sometimes expressions may be resolved into two or more factors by the application of more than one of the given rules. The student should make himself so familiar with the first four cases that he will be able to determine readily when any of them may be applied.

When Case I is to be used in connection with other cases, it should be applied first.

Example.—Factor $3mx^2y^3 - 12my^7$.

Solution.—By Case I, $3mx^2y^3 - 12my^7 = 3my^3(x^2 - 4y^4)$. Factoring the expression in the parenthesis by Case III, $x^2 - 4y^4 = (x + 2y^2)(x - 2y^2)$. Hence, $3mx^2y^3 - 12my^7 = 3my^3(x + 2y^2)(x - 2y^2)$. Ans.

Example.—Factor $80a^2x^2 - 40ax^2 + 5x^2$.

Solution.—By Case I, $80a^2x^2 - 40ax^2 + 5x^2 = 5x^2(16a^2 - 8a + 1)$. Factoring the expression in the parenthesis by Case II, $16a^2 - 8a + 1 = (4a - 1)^2$. Hence, $80a^2x^2 - 40ax^2 + 5x^2 = 5x^2(4a - 1)^2$. Ans.

Example.—Factor $2mn + 1 - m^2 - n^2$.

Solution.—Arrange the expression as follows: $1 - m^2 + 2mn - n^2 = 1 - (m^2 - 2mn + n^2)$. (Art. **408.**) By Case II, this equals $1 - (m - n)^2$. By Case III, this equals $[1 + (m - n)]\,[1 - (m - n)] = (1 + m - n)(1 - m + n)$. Ans.

Example.—Factor $a^6 - b^6$.

Solution.—By Case III, $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$. By Case IV, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. Hence, $a^6 - b^6 = (a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2)$. Ans.

Example.—Factor $4a^2 + x^4 - c^2 + 2cd + 4ax^2 - d^2$.

Solution.—This may be arranged as follows: $4a^2 + 4ax^2 + x^4 - c^2 + 2cd - d^2 = 4a^2 + 4ax^2 + x^4 - (c^2 - 2cd + d^2)$.

By Case II, this equals $(2a + x^2)^2 - (c - d)^2$. Hence, by Case III, $4a^2 + x^4 - c^2 + 2cd + 4ax^2 - d^2 = (2a + x^2 + c - d)(2a + x^2 - c + d)$. [Ans.

Example.—Factor $ac - bc + ad - bd$.

Solution.—We observe that, if the first two and last two terms be factored by Case I, they will each show the same binomial factor, $a - b$. Thus, $ac - bc + ad - bd = (ac - bc) + (ad - bd) = c\,(a - b) + d\,(a - b)$. Applying Case I, again, we have (dividing by $a - b$) the factors $(a - b)$ and $(c + d)$. Ans.

Example.—Factor $x^2 + ax - bx - ab$.

Solution.—This example is like the last. Hence, $x^2 + ax - bx - ab = (x^2 + ax) - (bx + ab) = x\,(x + a) - b\,(x + a) = (x + a)(x - b)$. Ans.

469. When factoring polynomials which come under Case V, first ascertain whether there is a monomial factor in the expression. If there is one, divide it out and reserve it. If the remaining terms cannot apparently be factored by Cases II, III, and IV, endeavor to so arrange the various terms that they may be factored by the application of some of the preceding rules. No fixed rules can be given which will cover all of the different expressions which fall under Case V, and the results depend entirely upon the ingenuity of the student, who must have considerable practice before he can factor polynomials successfully. It is important, however, that he should have some knowledge of the process. The explanations to the following examples are more full than those given above, and will probably afford some assistance to understanding the solutions given under Case V:

EXAMPLE.—Factor $ax^6 - ay^6 + b^2x^6 - b^2y^6$.

SOLUTION.—It is readily seen that a is a factor of the first two terms, and b^2 a factor of the last two. Enclosing the first two and last two terms in parentheses, the polynomial becomes $(ax^6 - ay^6) + (b^2x^6 - b^2y^6)$, which, of course, equals $a(x^6 - y^6) + b^2(x^6 - y^6)$. It is now seen that both terms of this *binomial* have the common factor $(x^6 - y^6)$. Dividing it out, the quotient is $a + b^2$. Hence, the required factors are $(a + b^2)$ and $(x^6 - y^6)$. But since x^6 and y^6 are perfect squares, the quantity $x^6 - y^6$ may be factored by Case III. Thus, $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$. Both of the factors last obtained may be factored by Case IV. Thus, $x^3 + y^3 = (x^2 - xy + y^2)(x + y)$ and $x^3 - y^3 = (x^2 + xy + y^2)(x - y)$. Therefore, since it is impossible to factor any further, $ax^6 - ay^6 + b^2x^6 - b^2y^6 = (a + b^2)(x^2 - xy + y^2)(x^2 + xy + y^2)(x + y)(x - y)$. Ans.

EXAMPLE.—Factor $4 - 9m^2 - n^2 + 6mn$.

SOLUTION.—Apparently, none of the rules will apply here; hence, the chief dependence must be placed upon the proper arrangement of the terms. Noticing that the terms $9m^2$ and n^2 are both perfect squares and have like signs, and that the term $6mn$ is twice the product of the square roots of $9m^2$ and n^2, the last three terms are enclosed in parenthesis, and the expression becomes $4 - (9m^2 + n^2 - 6mn)$. The second term of this binomial is a perfect square, according to Art. **456,** and the binomial may be written $4 - (3m - n)^2$, since $(3m - n)^2 = 9m^2 - 6mn + n^2$. The binomial $4 - (3m - n)^2$ may now be factored by Case III, since both terms are perfect squares. Therefore, $4 - (3m - n)^2 =$
$[2 + (3m - n)]\ [2 - (3m - n)] = (2 + 3m - n)\ (2 - 3m + n)$. Ans.

If the student will carefully study Art. **470,** in connection with Arts. **446—471,** he should experience no great difficulty in factoring. Until he has become accustomed to factoring, the student should prove his work by multiplying the factors together, and comparing the result with the original expression.

EXAMPLES FOR PRACTICE.

470. Factor the following expressions:

1. $x^4 - y^4$. Apply Case III, twice. Ans. $(x^2 + y^2)(x + y)(x - y)$.
2. $3abx^2 + 3ay^2b + 6axyb$. Apply Cases I and II. Ans. $3ab(x + y)^2$.
3. $a^4b^2 - ab^5$. Apply Cases I and IV. Ans. $ab^2(a - b)(a^2 + ab + b^2)$.
4. $2bc - b^2 - c^2 + 4$. Ans. $(2 + b - c)(2 - b + c)$.
5. $16m^2 - 25d^4 + 4n^2 + 16mn$. Ans. $(4m + 2n + 5d^2)(4m + 2n - 5d^2)$.
6. $y^2 - ay + by - ab$. Ans. $(y - a)(y + b)$.
7. $c^2 - 1 + 4x - 4x^2 - 2cd^2 + d^4$. Apply Cases II and III, after arranging the terms as follows: $(c^2 - 2cd^2 + d^4) - (4x^2 - 4x + 1)$.
Ans. $(c - d^2 + 2x - 1)(c - d^2 - 2x + 1)$.
8. $a^2 - x^2 - 1 + 2x$. Apply Cases II and III.
Ans. $(a + 1 - x)(a - 1 + x)$.
9. $4b^3 - 16ab^3 + 16a^2b^3$. Apply Cases I and II. Ans. $4b^3(1 - 2a)^2$.
10. $x^8 - m^8$. Ans. $(x^4 + m^4)(x^2 + m^2)(x + m)(x - m)$.

CASE VI.

471. Expressions of the form $a^n \pm b^n$ frequently occur, in which n is an integer (whole number). The sign ± is read *plus or minus*, and means that either sign may be used. *One of the factors will be $a + b$, when n is an* **even** *number (2, 4, 6, etc.), and the connecting sign is $-$, or when n is an odd number (3, 5, 7, etc.,) and the connecting sign is $+$. When the connecting sign is $-$, $a - b$ is* ***always*** *a factor.* $a^n + b^n$ cannot be factored when n is an even number, unless n has a value, $2p$, p being any number greater than 1.

Thus, $x^4 - y^4$ may be divided by $x + y$, and also by $x - y$; $x^4 + y^4$ cannot be factored; $x^5 + y^5$ may be divided by $x + y$; $x^5 - y^5$ may be divided by $x - y$. $x^6 + y^6$ can be factored, since it equals $x^{2 \times 3} + y^{2 \times 3}$; it is divisible by $x^2 + y^2$.

LEAST COMMON MULTIPLE.

472. The **least common multiple** (L. C. M.) of two or more quantities is the least quantity that may be divided by each without a remainder. When the quantities have no common factor, the L. C. M. will be their product; but when they have a common factor, a quantity less than their product may be found that each will exactly divide.

473. To obtain the L. C. M. of two or more quantities:

Rule.—*Find all the factors of each quantity. Select the smallest number of these necessary to form a product that each quantity will divide without a remainder. The product of the factors selected will be the L. C. M.*

EXAMPLE.—Find the L. C. M. of $x^2+2xy+y^2$, x^2-y^2, and $x-y$.

SOLUTION.—Factoring each quantity,

$$\begin{aligned} x^2+2xy+y^2 &= (x+y)(x+y). && (1)\\ x^2-y^2 &= (x+y)(x-y). && (2)\\ x-y &= x-y. && (3) \end{aligned}$$

To be divisible by (1), the L. C. M. must evidently contain the factors $(x+y)(x+y)$; hence, we select these for two factors. To be divisible by (2), it must contain the factors $(x+y)(x-y)$; but, as a factor $x+y$ was taken before, it is necessary to select only the $x-y$. To be divisible by (3), the L. C. M. must contain $x-y$; but as a factor $x-y$ has already been taken, this is not to be selected. Now, expressing the product of the factors selected, we have as the L. C. M. $(x+y)(x+y)(x-y)=(x+y)^2(x-y)$, which is the least quantity that each of the other quantities will exactly divide. Ans.

EXAMPLE.—Find the L. C. M. of $36ab^2$, $12a^2b^2$, and $50abc$.

SOLUTION.—Factoring,

$$\begin{aligned} 36ab^2 &= 3\times 3\times 2\times 2\times a\times b\times b. && (1)\\ 12a^2b^2 &= 3\times 2\times 2\times a\times a\times b\times b. && (2)\\ 50abc &= 5\times 5\times 2\times a\times b\times c. && (3) \end{aligned}$$

To be divisible by (1), the L. C. M. must contain all the factors of (1); hence, we select these. To be divisible by (2), it must contain, in addition to those already selected, the factor a, which we select. To be divisible by (3), it must contain, in addition to the factors taken, the factors 5, 5, and c, which we select. The product of the factors selected is $3\times 3\times 2\times 2\times a\times b\times b\times a\times 5\times 5\times c=900a^2b^2c$, the L. C. M. [Ans.

474. The following method of finding the L. C. M. of several quantities is, perhaps, easier to understand and apply than that given in the above. It will be explained by means of examples:

EXAMPLE.—Find the L. C. M. of (x^3-1), (x^2-1), and $(x+1)$.

SOLUTION.—Factor each quantity as follows:

$x^3-1=(x^2+x+1)(x-1)$; $x^2-1=(x+1)(x-1)$, and $x+1=x+1$. Arrange these in a row, separating the different quantities by commas, thus,

$$\begin{array}{r|lcr} x-1 & (x^2+x+1)(x-1), & (x+1)(x-1), & x+1 \\ \hline x+1 & x^2+x+1, & x+1 & x+1 \\ \hline & x^2+x+1, & 1, & 1 \end{array}$$

Now, select some factor and divide each quantity by it, if possible; if any quantity cannot be thus divided, bring it down. Selecting $x - 1$ and dividing each quantity by it, the result is $x^2 + x + 1$, $x + 1$ and $x + 1$. Dividing the remaining quantities by another factor, as $x + 1$, the result is $x^2 + x + 1$. Multiplying this last remainder and the two preceding divisors together, the result is $(x^2 + x + 1)(x + 1)(x - 1)$, the L. C. M. Ans.

Applying the above method to the first example in Art. **473**,

$$\begin{array}{r|lll}
x+y & (x+y)(x+y), & (x+y)(x-y), & x-y \\
\hline
x-y & x+y, & x-y, & x-y \\
\hline
 & x+y, & 1, & 1
\end{array}$$

Hence, the L. C. M. is $(x + y)(x - y)(x + y)$. Ans.

Applying this method to the second example in Art. **473**,

$$\begin{array}{r|lll}
2ab & 36ab^2, & 12a^2b^2, & 50abc \\
\hline
6b & 18b, & 6ab, & 25c \\
\hline
3 & 3, & a, & 25c \\
\hline
a & 1, & a, & 25c \\
\hline
 & 1, & 1, & 25c
\end{array}$$

Hence, the L. C. M. $= 2ab \times 6b \times 3 \times a \times 25c = 900a^2b^2c$. Ans.

475. The L. C. M. may often be found by inspection.

Example.—Find the L. C. M. of $x^3 - y^3$ and $x - y$.

Solution.—The least quantity that $x^3 - y^3$ will exactly divide is $x^3 - y^3$. As $x - y$ will also divide this without a remainder, $x^3 - y^3$ is evidently the L. C. M.

EXAMPLES FOR PRACTICE.

476. Find the L. C. M. of the following:

1. $9a^2b^4$, $18a^2b^2$, and $15a^3b$. Ans. $90a^3b^4$.
2. $2x + 1$ and $2(4x^2 - 1)$. Ans. $2(4x^2 - 1)$.
3. $a^2 - b^2$ and $a^3 - b^3$. Ans. $(a + b)(a^3 - b^3)$.
4. $2a - 1$, $4a^2 - 1$, and $4a^2 + 1$. Ans. $16a^4 - 1$.
5. $2a^2 + 2ab$, $3ab - 3b^2$, and $4a^2c - 4b^2c$. Ans. $12abc(a^2 - b^2)$.

FRACTIONS.

477. A **fraction,** in Algebra, is considered as an expression indicating division. The sign ÷ is seldom used, it being more convenient to write the dividend, or quantity to be divided, above a horizontal line, with the divisor below it, in the form of a fraction.

478. Thus, the fraction $\frac{a+b}{c-d}$ means that $a+b$ is to be divided by $c-d$, and is the same as $(a+b) \div (c-d)$. It is read "$a+b$ divided by $c-d$," or "$a+b$ *over* $c-d$." All fractions are read in this way in Algebra, except simple numerical fractions, as $\frac{1}{2}$, $\frac{25}{32}$, etc., which are read as in arithmetic.

479. The dividend, or quantity above the line, is called the **numerator,** and the divisor, or quantity below the line, the **denominator.** The numerator and denominator are called the **terms** of a fraction.

480. Any whole number or quantity may be taken as a fraction with a denominator of 1, and in this treatment of fractions it will be considered as such. Thus, 24 is the same as $\frac{24}{1}$, since $24 \div 1 = 24$. The student should observe the difference between this and the definition of a reciprocal which follows.

481. A **reciprocal** of a quantity is 1 divided by that quantity; that is, a reciprocal is a fraction having the number 1 for its numerator. The reciprocal of 5 is $1 \div 5 = \frac{1}{5}$; the reciprocal of x^2+a is $\frac{1}{x^2+a}$. The reciprocal of a number may be found by dividing 1 by that number; or a number may be found from its reciprocal by dividing 1 by the reciprocal. For example, the reciprocal of $250 = 1 \div 250 = .004$; the number whose reciprocal is .004 is $1 \div .004 = \frac{1}{.004} = 250$.

482. The **signs of a fraction** are three in number: namely, the sign before the dividing line, which indicates whether the fraction is to be added or subtracted (if this sign is +, it is always omitted when the fraction stands alone), and the signs of the numerator and denominator. *Any two signs of a fraction may be changed without altering its value, but if any one, or all three be changed, the value of the fraction will be changed from + to − or from − to +.* These principles can be shown to follow from Art. **436.** If the numerator or denominator is a polynomial, its sign belongs to the entire numerator or denominator; and *when changing signs, care must be taken to change the sign before each term* in the numerator or denominator. Thus, with the signs before the dividing line and numerator changed,

$-\frac{a-b}{c-d} = \frac{-a+b}{c-d}$, not $\frac{-a-b}{c-d}$.

That $-\frac{a-b}{c-d} = -\frac{-a+b}{-c+d}, + \frac{-a+b}{c-d}$ or $+\frac{a-b}{-c+d}$ can be readily shown. It is evident, from the laws of fractions, that, if both numerator and denominator be multiplied by the same quantity, the *value* of the fraction will not be changed. Multiplying both terms of the fraction $-\frac{a-b}{c-d}$ by -1, we get $-\frac{-(a-b)}{-(c-d)} = -\frac{-a+b}{-c+d}$. Again, multiplying *and* dividing a quantity by another quantity does not alter the value of the quantity. For instance, $2 = 2 \times \frac{4}{4}$. Now, the sign before the dividing line of a fraction applies to the whole fraction. Hence, $-\frac{a-b}{c-d} = -\left(\frac{a-b}{c-d}\right)$. Multiplying this last expression by -1, it becomes $+\left(\frac{a-b}{c-d}\right) = \frac{a-b}{c-d}$. To divide a fraction by any quantity, we may divide the numerator or multiply the denominator. Dividing the numerator of the last expression by -1, the fraction becomes $\frac{-(a-b)}{c-d} = \frac{-a+b}{c-d}$; multiplying the denominator, it becomes $\frac{a-b}{-(c-d)}=\frac{a-b}{-c+d}$. Therefore, $-\frac{a-b}{c-d}=$ $-\frac{-a+b}{-c+d}=\frac{-a+b}{c-d}=\frac{a-b}{-c+d}$.

If the above demonstration is not satisfactory, let the student substitute numerical values for a, b, c, and d.

REDUCTION OF FRACTIONS.

483. To reduce a fraction is to change its form without changing its value. Thus, $\frac{10x}{5}$ and $\frac{20x}{10}$ have different forms, but like values, since $10x \div 5$ and $20x \div 10$ are each equal to $2x$.

In reducing fractions we have the general rule that the numerator and denominator may both be multiplied, or both divided, by the same quantity without changing the value of the fraction.

484. To reduce a fraction to its simplest form:

Rule.—*Resolve the numerator and denominator into their factors and cancel those that are common to both.*

This is in effect the same as dividing both numerator and denominator by the same quantity, and does not change the value of the fraction.

485. *In reducing, or performing other operations upon fractions, the student must learn to use polynomial factors, wherever they occur, as though they were one quantity—like monomial factors.* This is illustrated in the following examples, where there are polynomial factors in both numerator and denominator that can be canceled. (See Art. **391.**)

Example.—Reduce $\frac{x^2+2xy+y^2}{x^2-y^2}$ to its simplest form.

Solution.—Factoring both numerator and denominator,

$$\frac{x^2+2xy+y^2}{x^2-y^2}=\frac{(x+y)(x+y)}{(x+y)(x-y)}.$$

Canceling the common factor $x + y$ from both, gives, as the result,

$$\frac{\cancel{(x+y)}(x+y)}{\cancel{(x+y)}(x-y)} = \frac{x+y}{x-y}. \quad \text{Ans.}$$

Example.—Reduce $\frac{3x^5 - 6x^4y}{6x^2y^2 - 12xy^3}$ to its simplest form.

Solution.— $\frac{3x^5 - 6x^4y}{6x^2y^2 - 12xy^3} = \frac{3x^4(x - 2y)}{6xy^2(x - 2y)}$, when factored.

Canceling the common factors, we have, as the result,

$$\frac{\overset{x^3}{\cancel{3x^4}}\cancel{(x-2y)}}{\underset{2}{\cancel{6x}}y^2\cancel{(x-2y)}} = \frac{x^3}{2y^2}. \quad \text{Ans.}$$

486. Sometimes the whole numerator is contained in the denominator, or the denominator in the numerator. The numerator or denominator will then reduce to the number 1.

Example.—Reduce $\frac{b + 3c^2}{2b^2 + 6bc^2}$ to its simplest form.

Solution.— $\frac{b + 3c^2}{2b^2 + 6bc^2} = \frac{\cancel{b + 3c^2}}{2b(\cancel{b + 3c^2})} = \frac{1}{2b}$. Ans.

Example.—Reduce $\frac{x^6 - 1}{x^3 - 1}$ to its simplest form.

Solution.— $\frac{x^6 - 1}{x^3 - 1} = \frac{(x^3 + 1)\cancel{(x^3 - 1)}}{\cancel{x^3 - 1}} = \frac{x^3 + 1}{1} = x^3 + 1$. Ans. (Art. **480.**)

487. From the last example it will be seen that division may sometimes be performed by cancelation. Thus, $\frac{x^6 - 1}{x^3 - 1}$ means $(x^6 - 1) \div (x^3 - 1)$, and the divisor $x^3 - 1$ canceled from the dividend $x^6 - 1$ gives the quotient $x^3 + 1$. A factor must be common to each term of the numerator and to each term of the denominator in order to be canceled. Thus, the factor x cannot be canceled from $\frac{3ax}{x + 4m}$, because it is not common to both terms of the denominator.

EXAMPLES FOR PRACTICE.

488. Reduce the following to their simplest form:

1. $\frac{m^3 - n^3}{m(m^2 + mn + n^2)}$. Ans. $\frac{m - n}{m}$.
2. $\frac{3a + 3b}{a^2 - b^2}$. Ans. $\frac{3}{a - b}$.
3. $\frac{x^4 - y^4}{x^2 - y^2}$. Ans. $x^2 + y^2$.
4. $\frac{54a^3b^5c^2}{72a^2b^2c}$. Ans. $\frac{3ab^3c}{4}$.
5. $\frac{12a^2x^3}{36a^3x^5}$. Ans. $\frac{1}{3ax^2}$.
6. $\frac{n^3 - 2n^2}{n^2 - 4n + 4}$. Ans. $\frac{n^2}{n - 2}$.

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489. When fractions are to be added or subtracted, it is necessary to so reduce them that all the denominators will be alike. This is called **reducing them to a common denominator.** The common denominator may be any multiple of the given denominators, but it is always better that it should be the *least* common multiple, also called the *least common denominator.*

490. To reduce fractions to a common denominator:

Rule.—*Find the L. C. M. of the given denominators. Divide this by each denominator. Multiply the corresponding numerators by each quotient for the new numerators, and write the results over the common denominators.*

This is in effect the same as multiplying both numerator and denominator by the same quantity, which, by Art. **483,** does not change the value of the fraction. Before applying the rule, all fractions should be reduced to their simplest form.

EXAMPLE.—Reduce $\frac{1}{1 - b}$, $\frac{4}{(1 - b)^2}$, and $\frac{3}{(1 - b)^3}$ to a common denominator.

SOLUTION.—The L. C. M. of the denominators is $(1 - b)^3$, since this is the least quantity that each denominator will divide without a remainder. Dividing this by $1 - b$, the first denominator (use the method of Art. **487**), the quotient is $(1 - b)^2$; dividing it by $(1 - b)^2$, the second denominator, the quotient is $(1 - b)$; dividing it by the third denominator, the quotient is 1. Multiplying the corresponding numerators by these quotients, we obtain as the new numerators, $1 \times (1-b)^2 = (1-b)^2$, $4 \times (1-b) = 4(1-b)$ and $3 \times 1 = 3$. Hence the reduced fractions are $\frac{(1-b)^2}{(1-b)^3}$, $\frac{4(1-b)}{(1-b)^3}$ and $\frac{3}{(1-b)^3}$. Ans.

EXAMPLES FOR PRACTICE.

491. Reduce the following to a common denominator:

1. $\frac{3yz}{2x}$, $\frac{4xz}{3y}$ and $\frac{5xy}{4z}$. Ans. $\frac{18y^2z^2}{12xyz}$, $\frac{16x^2z^2}{12xyz}$ and $\frac{15x^2y^2}{12xyz}$.

2. $\frac{x^2y}{10}$, $\frac{xyz}{15}$ and $\frac{7yz^2}{30}$. Ans. $\frac{3x^2y}{30}$, $\frac{2xyz}{30}$ and $\frac{7yz^2}{30}$.

3. $\frac{2}{a^3x^3}$, $\frac{3}{ax^3}$, and $\frac{4}{a^2x}$. Ans. $\frac{2}{a^3x^3}$, $\frac{3a^2}{a^3x^3}$ and $\frac{4ax^2}{a^3x^3}$.

4. $\frac{m+n}{m-n}$ and $\frac{m-n}{m+n}$. Ans. $\frac{m^2+2mn+n^2}{m^2-n^2}$ and $\frac{m^2-2mn+n^2}{m^2-n^2}$.

5. $\frac{2}{x}$, $\frac{3}{2x-1}$ and $\frac{2x-1}{4x^2-1}$.

Ans. $\frac{2(4x^2-1)}{x(4x^2-1)}$, $\frac{3x(2x+1)}{x(4x^2-1)}$ and $\frac{x(2x-1)}{x(4x^2-1)}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

492. To add or subtract fractions:

Rule.—*Reduce them, if necessary, to a common denominator. Add or subtract the numerators, and write the result over the common denominator.*

Example.—Find the sum of $\frac{2a-b}{5}$ and $\frac{a+b}{4}$.

Solution.—$\frac{2a-b}{5}$ and $\frac{a+b}{4}$ reduced to a common denominator become $\frac{4(2a-b)}{20}$ and $\frac{5(a+b)}{20}$, which equal $\frac{8a-4b}{20}$ and $\frac{5a+5b}{20}$. Adding the numerators, we have $8a - 4b + 5a + 5b = 13a + b$. The result written over the common denominator gives, as the sum, $\frac{13a+b}{20}$. The work is written as follows:

$$\frac{2a-b}{5} + \frac{a+b}{4} = \frac{8a-4b}{20} + \frac{5a+5b}{20} =$$

$$\frac{8a-4b+5a+5b}{20} = \frac{13a+b}{20}. \quad \text{Ans.}$$

Example.—Subtract $\frac{6b-2}{3b}$ from $\frac{4a-1}{2a}$.

Solution.— $\frac{4a-1}{2a} - \frac{6b-2}{3b} = \frac{12ab-3b}{6ab} - \frac{12ab-4a}{6ab}$, when reduced to a common denominator. Subtracting the second numerator from the first, and writing the result over the common denominator, we have $\frac{12ab-3b}{6ab} - \frac{12ab-4a}{6ab} = \frac{(12ab-3b)-(12ab-4a)}{6ab} = \frac{12ab-3b-12ab+4a}{6ab}$, with the parentheses removed. Combining like terms in the numerator gives, as the result, $\frac{4a-3b}{6ab}$. Ans.

493. *If, as in the last example, the numerator of the fraction to be subtracted has more than one term, care must be taken to change the sign of every term before combining.* It will usually be convenient to inclose the whole numerator in a parenthesis before combining. The parenthesis may then be removed by the rules of Arts. **405** and **406.**

Example.—Simplify $\frac{x^3}{x-1} - \frac{x^2}{x+1} - \frac{x}{x-1} + \frac{1}{x+1}$.

Solution.—Reducing to the common denominator, x^2-1,

$$\frac{x^3}{x-1} - \frac{x^2}{x+1} - \frac{x}{x-1} + \frac{1}{x+1} = \frac{x^4+x^3}{x^2-1} - \frac{x^3-x^2}{x^2-1} - \frac{x^2+x}{x^2-1} + \frac{x-1}{x^2-1}.$$

Adding or subtracting the numerators as required, we have

$$\frac{(x^4+x^3)-(x^3-x^2)-(x^2+x)+(x-1)}{x^2-1},$$

which, with the parentheses removed, =

$$\frac{x^4+x^3-x^3+x^2-x^2-x+x-1}{x^2-1}.$$

Combining like terms we have, as the result,

$$\frac{x^4-1}{x^2-1} = x^2+1. \quad \text{Ans.}$$

Example.—Simplify $\frac{1}{(x-2)^2} + \frac{1}{2-x}$.

Solution.—If the denominator of the second fraction were written $x-2$ instead of $2-x$, $(x-2)^2$ would be the common denominator. By Art. **482,** the signs of the denominator and the sign before the fraction $\frac{1}{2-x}$ may be changed, giving $-\frac{1}{-2+x} = -\frac{1}{x-2}$. (Art. **373.**) Hence, we have $\frac{1}{(x-2)^2} + \frac{1}{2-x} = \frac{1}{(x-2)^2} - \frac{1}{x-2}$, which, when reduced to a common denominator, =

$$\frac{1}{(x-2)^2} - \frac{x-2}{(x-2)^2} = \frac{1-(x-2)}{(x-2)^2} = \frac{1-x+2}{(x-2)^2} = \frac{3-x}{(x-2)^2}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

494. Simplify the following:

1. $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}$. Ans. $\frac{47x}{60}$.

2. $\frac{4x-3}{5}+\frac{7x+1}{3}+\frac{3x}{2}$. Ans. $\frac{139x-8}{30}$.

3. $\frac{1}{x-y}-\frac{1}{x^2-y^2}$. Ans. $\frac{x+y-1}{x^2-y^2}$.

4. $\frac{a^2+b^2}{2}-\frac{(a+b)^2}{4}$. Ans. $\frac{2(a^2+b^2)-(a^2+2ab+b^2)}{4}=\frac{(a-b)^2}{4}$,

after removing parentheses and combining.

5. $\frac{a^2}{a^2-1}+\frac{a}{a-1}-\frac{a}{a+1}$. Ans. $\frac{a^2+2a}{a^2-1}$.

6. $\frac{4m^2+1}{4m^2}-\frac{3m-1}{12m^3}+\frac{1-12n}{12n}$. Ans. $\frac{n+m^3}{12m^3n}$.

7. $\frac{y}{(x+y)^2}+\frac{y}{x^2-y^2}-\frac{1}{x+y}$.

Ans. $\frac{y(x-y)+y(x+y)-(x^2-y^2)}{(x+y)^2(x-y)}=\frac{2xy-x^2+y^2}{x^3+x^2y-xy^2-y^3}$.

8. $\frac{x}{x+1}+\frac{x}{1-x}+\frac{3x}{x^2-1}$. Ans. $\frac{x}{x^2-1}$.

MULTIPLICATION OF FRACTIONS.

495. Multiplication, in fractions, is the process of finding a fractional part of a fraction. Thus, $\frac{3}{4}\times\frac{1}{2}$ means $\frac{1}{2}$ of $\frac{3}{4}$. One-half of $\frac{3}{4}$ inch, for example, is $\frac{3}{8}$ inch; $\frac{2}{3}$ of $\frac{3}{4}$ inch is $\frac{1}{2}$ or $\frac{6}{12}$ inch. The result in each case is the same as that which would be obtained by finding the product of the numerators, and writing it over the product of the denominators.

496. Hence, to multiply fractions:

Rule.—*Multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product.*

497. Two or more fractions may be multiplied together. Common factors in the numerators and denominators should be canceled before performing the multiplication, and the product should be reduced to its simplest form.

Example.—Find the product of $\frac{6a^2}{5}$, $\frac{2ab}{3c}$ and $\frac{2ac}{b^2}$.

Solution.—The product of the numerators is $6a^2 \times 2ab \times 2ac = 24a^4bc$, and of the denominators, $5 \times 3c \times b^2 = 15b^2c$. Writing $24a^4bc$ over $15b^2c$, we have, for the product, $\frac{24a^4bc}{15b^2c} = \frac{8a^4}{5b}$ when reduced to its lowest terms. The work is written as follows:

$$\frac{\overset{2}{\cancel{6}}a^2}{5} \times \frac{2a\cancel{b}}{\cancel{3c}} \times \frac{2a\cancel{c}}{\underset{b}{\cancel{b^2}}} = \frac{8a^4}{5b}. \quad \text{Ans.}$$

Example.—Find the product of $3m^3n^4$ and $\frac{11x^2}{x+4mn}$.

Solution.—By Art **480**, $3m^3n^4 = \frac{3m^3n^4}{1}$. The product of the numerators is $3m^3n^4 \times 11x^2 = 33m^3n^4x^2$, and of the denominators, $1 \times (x+4mn) = x+4mn$. Hence,

$$3m^3n^4 \times \frac{11x^2}{x+4mn} = \frac{3 \times 11m^3n^4x^2}{x+4mn} = \frac{33m^3n^4x^2}{x+4mn}. \quad \text{Ans.}$$

498. When the numerators or denominators consist of more than one term they should be factored, if possible, to aid in canceling the common factors of the result.

Example.—Find the product of $\frac{x^2+2x}{(x-1)^2}$, $\frac{x^2-1}{x-2}$, and $\frac{x^2-4x+4}{x+2}$.

Solution.—Factoring the numerators and denominators of the fractions, and writing the factors of the numerators together over the factors of the denominators, we have $\frac{x^2+2x}{(x-1)^2} \times \frac{x^2-1}{x-2} \times \frac{x^2-4x+4}{x+2} =$

$$\frac{x\cancel{(x+2)}(x+1)\cancel{(x-1)}\cancel{(x-2)}(x-2)}{\cancel{(x-1)}(x-1)\cancel{(x-2)}\cancel{(x+2)}} = \frac{x(x+1)(x-2)}{x-1}. \quad \text{Ans.}$$

499. To multiply expressions in which addition or subtraction is indicated, first perform the addition or subtraction.

Example.—Find the product of $\frac{1}{a^3} - \frac{4c^2}{a}$ and $\frac{a^3}{1+2ac}$.

Solution.—Performing the subtraction, $\frac{1}{a^3} - \frac{4c^2}{a} = \frac{1-4a^2c^2}{a^3}$.

$$\text{Multiplying, } \frac{1-4a^2c^2}{a^3} \times \frac{a^3}{1+2ac} = \frac{\overset{1-2ac}{\cancel{(1-4a^2c^2)}}\cancel{a^3}}{\cancel{(1+2ac)}\cancel{a^3}} = 1-2ac. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

500. Multiply the following:

1. $\frac{3a^2bc}{5abc^2}$ by $\frac{10ab^2c}{3abc}$. Ans. $\frac{2ab}{c}$.

2. $\frac{5x^2y}{7x}$ by $21xy$. Ans. $15x^2y^2$.

Find the product of

3. $\frac{3x^2y}{4xz^2}$, $\frac{5y^2z}{6xy}$ and $\frac{-12x^2}{2xy^2}$. Ans. $\frac{15x}{4z}$.

4. $\frac{x^2-y^2}{c^3-d^3}$, $\frac{c-d}{(x+y)^2}$ and $\frac{x^3+y^3}{x-y}$. Ans. $\frac{x^2-xy+y^2}{c^2+cd+d^2}$.

5. $\frac{4y}{x}-\frac{16}{xy}$ and $\frac{1}{2y+4}$. Ans. $\frac{2y-4}{xy}$.

6. $\frac{a+b}{2}+\frac{a-b}{4}$ and $\frac{4}{9a^2+6ab+b^2}$. Ans. $\frac{1}{3a+b}$.

DIVISION OF FRACTIONS.

501. Division, in fractions, is the process of finding how many times one fraction is contained in another. For example, $\frac{1}{4}$ inch is contained in $\frac{3}{4}$ inch 3 times; $\frac{3}{4}$ inch is contained in $\frac{6}{4}$ inches 2 times. The result in each case is the same as that which would be obtained by multiplying the denominator of the first fraction by the numerator of the second for the numerator of the quotient, and the numerator of the first fraction by the denominator of the second for the denominator of the quotient.

502. Hence, to divide by a fraction:

Rule.—*Invert the divisor, and proceed as in multiplication.*

EXAMPLE.—Divide $\frac{3a^2b}{5x^3y}$ by $\frac{9ab^3}{10x^4y^2}$.

SOLUTION.—The divisor inverted $=\frac{10x^4y^2}{9ab^3}$.

$$\text{Hence, } \frac{3a^2b}{5x^3y}\div\frac{9ab^3}{10x^4y^2}=\frac{3a^2b}{5x^3y}\times\frac{10x^4y^2}{9ab^3}=\frac{\overset{2a\ \ x\ y}{\cancel{3}\times\cancel{10}\cancel{a^2}\cancel{b}\cancel{x^4}\cancel{y^2}}}{\underset{3\ \ b^2}{\cancel{5}\times\cancel{9}\cancel{a}\cancel{b^3}\cancel{x^3}\cancel{y}}}=\frac{2axy}{3b^2}.\text{ Ans.}$$

EXAMPLE.—Divide x^2+2x+1 by $\frac{x+1}{x-1}$.

SOLUTION.—By Art. **480,** $(x^2+2x+1)\div\frac{x+1}{x-1}=\frac{x^2+2x+1}{1}\times\frac{x-1}{x+1}$

$$=\frac{\cancel{(x+1)}\,(x+1)\,(x-1)}{\cancel{x+1}}=x^2-1.\quad\text{Ans.}$$

EXAMPLES FOR PRACTICE.

503. Divide the following:

1. $\frac{9x^3-3x^4}{24}$ by $\frac{3x}{8}$. Ans. $\frac{3x^2-x^3}{3}$.
2. $\frac{ab-bx}{a+z}$ by $\frac{ac-cx}{a+z}$. Ans. $\frac{b}{c}$.
3. $\frac{1-8b^2+16b^4}{1+2b}$ by $\frac{1-4b^2}{3a}$. Ans. $3a(1-2b)$.
4. $6a^2cd-6abcd$ by $\frac{6acd}{a^2+ab+b^2}$. Ans. a^3-b^3.

MIXED QUANTITIES AND COMPLEX FRACTIONS.

504. A **mixed quantity** is an expression containing both integral and fractional parts, as $2a^2-\frac{c+d}{4}$. Considering the integral part as a fraction with a denominator of 1 (Art. **480**), a mixed quantity becomes simply the indicated addition or subtraction of two fractions; thus, $2a^2-\frac{c+d}{4}=\frac{2a^2}{1}-\frac{c+d}{4}$. By *integral part* is meant any expression which does not contain fractions or negative exponents.

505. Any fraction may be reduced to either an entire or mixed quantity by dividing the numerator by the denominator. It frequently happens that by performing the indicated division, the fraction will be reduced to a simpler form. The case of reducing a fraction to an entire quantity was taken up in Art. **486**.

EXAMPLE.—Simplify $\frac{4x^2+12x-1}{2x+3}$.

SOLUTION.—Performing the indicated division,

$$\begin{array}{r l}
2x+3\,)\,4x^2+12x-1\,(& 2x+3-\frac{10}{2x+3} \quad \text{Ans.} \\
\underline{4x^2+6x\phantom{{}-1}} & \\
6x-1 & \\
\underline{6x+9} & \\
-10 &
\end{array}$$

When any operation, as multiplication or division, is to be performed with a mixed quantity, it is sometimes easier to first reduce it to a fraction.

506. To reduce a mixed quantity to a fraction:

Rule.—*Consider the integral part to be a fraction with a denominator of 1, and perform the indicated addition or subtraction.*

EXAMPLE.—Reduce $x^2 + xy + y^2 - \frac{b}{x-y}$ to a fraction.

SOLUTION.— $x^2 + xy + y^2 - \frac{b}{x-y} = \frac{x^2 + xy + y^2}{1} - \frac{b}{x-y}$; subtracting the second fraction from the first gives

$$\frac{(x^2 + xy + y^2)(x-y) - b}{x-y} = \frac{x^3 - y^3 - b}{x-y}. \quad \text{Ans.}$$

EXAMPLE.—Multiply $\frac{a}{a+x} + \frac{a}{a-x}$ by $1 + \frac{a^2 + x^2}{2ax}$.

Performing the additions,

SOLUTION.—$\frac{a}{a+x} + \frac{a}{a-x} = \frac{2a^2}{a^2 - x^2}$; $1 + \frac{a^2 + x^2}{2ax} = \frac{a^2 + 2ax + x^2}{2ax}$.

Multiplying, $\frac{\overset{a}{\cancel{2a^2}}}{\underset{a-x}{\cancel{a^2 - x^2}}} \times \frac{\overset{a+x}{\cancel{a^2 + 2ax + x^2}}}{\cancel{2a}x} = \frac{a(a+x)}{x(a-x)}$. Ans.

EXAMPLES FOR PRACTICE.

507. Solve the following:

1. Reduce $\frac{a^2c + b^2}{c}$ to a mixed quantity. Ans. $a^2 + \frac{b^2}{c}$.
2. Simplify $\frac{x^2 + 4xy + 5y^2 - 3x}{x + 2y}$ Ans. $x + 2y - 3 + \frac{y^2 + 6y}{x + 2y}$.
3. Reduce $x + 3 - \frac{7x + 3}{2x + 1}$ to a fraction. Ans. $\frac{2x^2}{2x + 1}$.
4. From $3a + \frac{a+b}{d}$ subtract $a - \frac{a-b}{d}$.
   Ans. $2a + \frac{2a}{d} = \frac{2a(d+1)}{d}$.
5. Divide $m + n - \frac{2n}{m-n}$ by $m - n - \frac{2n}{m+n}$. Ans. $\frac{m+n}{m-n}$.

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508. A **complex fraction** is one having a fraction in its numerator, or denominator, or both. Since any fraction is an expression of division, a complex fraction may be simplified by dividing the part above the line by the part below it. Thus, the complex fraction $\frac{\frac{5}{8}}{\frac{3}{4}} = \frac{5}{8} \div \frac{3}{4}$; inverting the divisor and multiplying, $\frac{5}{8} \times \frac{4}{3} = \frac{5}{6}$.

The same result would have been obtained if both terms had been multiplied by the least common denominator of the denominators of the fractions in the numerator and denominator. Thus, $\frac{\frac{5}{8}\times 8}{\frac{3}{4}\times 8}=\frac{5}{6}$.

The latter is generally the simpler method to use.

509. Hence, to simplify a complex fraction:

Rule.—*Multiply both terms by the least common denominator of the denominators of the fractional parts.*

Example.—Simplify $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{1}{y}-\dfrac{1}{x}}$.

Solution.—The least common denominator of the denominators is xy. Multiplying each term by this, we have

$$\frac{\dfrac{x}{y}\times xy-\dfrac{y}{x}\times xy}{\dfrac{1}{y}\times xy-\dfrac{1}{x}\times xy}=\frac{x^2-y^2}{x-y}=x+y. \quad \text{Ans.}$$

The multiplication can generally be performed mentally, without writing the least common denominator, at the same time canceling common factors.

Example.—Simplify $\dfrac{x^2+\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

Solution.—The L. C. D. is x. Multiplying each term by this, we have

$$\frac{x^3+1}{x+1}=x^2-x+1. \quad \text{Ans.} \quad (\text{Art. } \mathbf{466.})$$

Example.—Simplify $\dfrac{1}{1+\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}}$.

Solution.—This is the case of a complex fraction in which the denominator is itself a complex fraction.

First, consider the part $\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}$.

Multiplying both terms by $1-a$, we have

$$\frac{a(1-a)}{(1+a)(1-a)+2a^2}=\frac{a-a^2}{1-a^2+2a^2}=\frac{a-a^2}{1+a^2}.$$

The fraction thus becomes $\dfrac{1}{1+\dfrac{a-a^2}{1+a^2}}$.

Multiplying both terms by $1+a^2$, the L. C. D., we have

$$\frac{1+a^2}{1+a^2+a-a^2}=\frac{1+a^2}{1+a}.\quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

510. Simplify the following:

1. $\dfrac{\dfrac{3ac^2}{16}}{24}$. Ans. $\dfrac{ac^2}{128}$.

2. $\dfrac{1+\dfrac{a}{c}}{c-\dfrac{a^2}{c}}$. Ans. $\dfrac{1}{c-a}$.

3. $\dfrac{2\frac{7}{8}}{8-2x+\dfrac{x^2}{8}}$. Ans. $\dfrac{23}{(8-x)^2}$.

NOTE.—$2\frac{7}{8}$ means $2+\frac{7}{8}$. Hence for the numerator multiply 2 by the least common denominator 8, and add 7.

4. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$. Ans. $\dfrac{4}{3x+3}$.

5. $\dfrac{\dfrac{a}{b}-\dfrac{b^2}{a^2}}{a^2+ab+b^2}$. Ans. $\dfrac{a-b}{a^2b}$.

INVOLUTION.

511. Involution is the process of raising a quantity to any required power (Art. **362**), by taking it as many times as a factor as there are units in the exponent of the power.

The rules for raising a monomial to any power follow directly from the rules of multiplication. For example, let it be required to raise $3a^2$ to the 4th power. Writing $3a^2$ four times as a factor and performing the multiplication, we have, by Art. **420**,

$$(3a^2)^4=3a^2\times 3a^2\times 3a^2\times 3a^2=81a^8.$$

Here the power is $81a^8$, and the exponent of the power is 4. It will be observed that the exponent of a in the power, produced by adding together the exponents of a in the factors, is the product of 4, the exponent of the power, and 2, the exponent of a in $3a^2$, or $4 \times 2 = 8$.

512. Again, since like signs produce plus, it is evident that *any* power of a *positive* quantity will be *positive*, and by multiplying a negative quantity by itself a number of times it will appear that any *odd* power of a *negative* quantity will be *negative*, and any *even* power *positive*. Thus, $(-a)^2 = (-a) \times (-a) = +a^2$; $(-a)^3 = (+a^2)(-a) = -a^3$; $(-a)^4 = (-a^3)(-a) = +a^4$, etc.

513. From the preceding, we have, to raise a monomial to any power:

Rule.—*Raise the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the power. If the power is even, make its sign plus; if odd, make its sign the same as that of its root.*

EXAMPLE.—Find the value of $(-4bc^2d^3)^3$.

SOLUTION.—Four raised to the required power equals $4 \times 4 \times 4$, or 64; the exponent of b in the power is $3 \times 1 = 3$; of c, $3 \times 2 = 6$, and of d, $3 \times 3 = 9$. Since the power is *odd*, the sign is the same as that of its root, or minus. Hence, $(-4bc^2d^3)^3 = -64b^3c^6d^9$. Ans.

514. A fraction may be raised to any power by raising both numerator and denominator to the required power.

EXAMPLE.—Find the value of $\left(-\frac{2x^3}{3y^4}\right)^4$.

SOLUTION.— $(2x^3)^4 = 16x^{12}$, for the numerator of the power; $(3y^4)^4 = 81y^{16}$ for the denominator of the power. Since the power is even, its sign will be plus. Hence,

$$\left(-\frac{2x^3}{3y^4}\right)^4 = \frac{16x^{12}}{81y^{16}}. \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

515. Find the values of the following:

1. $(2a^2b^4)^4$. Ans. $16a^8b^{16}$.
2. $(-5ax^3y^2)^4$. Ans. $625a^4x^{12}y^8$.
3. $(-m^3n^4)^7$. Ans. $-m^{21}n^{28}$.
4. $\left(-\frac{a^3b^3}{2}\right)^4$. Ans. $\frac{a^{12}b^{12}}{16}$.
5. $(\frac{2}{3}ax^2)^6$. Ans. $\frac{64}{729}a^6x^{12}$.
6. $\left(\frac{m^4n^5x}{-6c^2d^3}\right)^3$. Ans. $-\frac{m^{12}n^{15}x^3}{216c^6d^9}$.

INVOLUTION OF POLYNOMIALS.

516. Since involution consists of successive multiplications, a polynomial may be raised to any power by multiplying it by itself until it has been taken as many times as a factor as there are units in the exponent of the power.

When finding the square of a binomial, however, the method of Arts. **428** and **429** should be used, for it saves actual multiplication. If a polynomial of more than two terms is to be involved, it may sometimes be divided into two parts, thus forming a binomial which can be squared by this method, as explained in example 11, Art. **434**.

This method may be extended to include any power, by first raising the binomial $a + b$ to that power by multiplication, and then substituting for a and b in the result the two terms of the binomial that it is required to raise to the given power. The student need not use this method, however, for any except the second power.

EXAMPLES FOR PRACTICE.

517. Find the values of the following:

1. $(2a - x - 4y)^2$. Ans. $4a^2 - 4ax - 16ay + x^2 + 8xy + 16y^2$.
2. $(-3a^2x - 2ax^2 + x^3)^2$. Ans. $9a^4x^2 + 12a^3x^3 - 2a^2x^4 - 4ax^5 + x^6$.
3. $(x^2 - ax + a^2)^3$.
Ans. $x^6 - 3ax^5 + 6a^2x^4 - 7a^3x^3 + 6a^4x^2 - 3a^5x + a^6$.

EVOLUTION.

518. **Evolution** is the process of extracting any required root of a quantity. It is exactly the reverse of involution. Thus, by involution, $(2a)^4 = 2a \times 2a \times 2a \times 2a = 16a^4$; by evolution, $\sqrt[4]{16a^4} = 2a$.

519. Since every *even power* of both positive and negative quantities is positive (Art. **512**), it follows that the *even roots* of a positive quantity may be either positive or negative. Thus, since both $(+a)^4$ and $(-a)^4 = +a^4$, $\sqrt[4]{+a^4}$ equals either $+a$ or $-a$. When it is desired to indicate that a quantity is either positive or negative, the double sign ± is used. Thus, $\pm a$ means that a is to be taken as either $+a$ or $-a$, and is read "plus or minus a."

520. From Art. **512** it follows, also, that there can be no even root of a negative quantity. A *negative even* power is impossible; hence, an *even root* of a *negative* quantity is impossible.

521. Since evolution is the reverse of involution, we have the following for extracting any root of a monomial:

Rule.—*Extract the required root of the numerical coefficient, and divide the exponent of each letter by the index of the root. Make the sign of every even root of a positive quantity* ±, *and the sign of every odd root of any quantity the same as that of the quantity.*

Example.—Find the value of $\sqrt[4]{256a^4b^{12}c^8}$.

Solution.—The 4th root of 256 is 4. The exponent of a in the root is $4 \div 4 = 1$; of b, $12 \div 4 = 3$; and of c, $8 \div 4 = 2$. As this is an even root of a positive quantity, the sign should be ±. Hence, $\sqrt[4]{256a^4b^{12}c^8} = \pm 4ab^3c^2$. Ans.

Example.—Find the value of $\sqrt[3]{\frac{27m^3x^9}{a^9b^6c^{12}}}$.

Solution.— $\sqrt[3]{27m^3x^9} = 3mx^3$; $\sqrt[3]{a^9b^6c^{12}} = a^3b^2c^4$. The quantity is positive, and, as this is an odd root, its sign must be the same, or positive.

$$\text{Hence, } \sqrt[3]{\frac{27m^3x^9}{a^9b^6c^{12}}} = \frac{3mx^3}{a^3b^2c^4}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

522. Find the values of the following:

1. $\sqrt[3]{-125x^3y^6z^9}$. Ans. $-5xy^2z^3$.
2. $\sqrt[4]{10{,}000a^{16}b^{20}c^8}$. Ans. $\pm 10a^4b^5c^2$.
3. $\sqrt[5]{243m^{15}n^{20}}$. Ans. $3m^3n^4$.
4. $\sqrt{-4a^2b^2}$. Impossible.
5. $\sqrt[5]{\frac{-x^5y^{10}z^{15}}{a^{20}b^{15}c^{10}d^5}}$. Ans. $-\frac{xy^2z^3}{a^4b^3c^2d}$.
6. $\sqrt[6]{64\frac{a^6d^{12}}{b^6d^6}}$. Ans. $\pm 2\frac{ad^2}{bd}$.

SQUARE ROOT OF POLYNOMIALS.

523. In Art. **455,** the method of extracting the square root of a trinomial, which is a perfect square, was explained. The method of extracting the square root of *any* polynomial which is a perfect square, is as follows:

Rule.—*Arrange the terms according to the powers of some letter.*

Write the square root of the first term as the first term of the required root, and subtract its square from the given polynomial.

For a trial divisor, divide the first term of the remainder by twice the part of the root already found; annex the result to the root, and also to the divisor.

Multiply the divisor as it now stands by the term of the root last obtained, and subtract the product from the remainder.

If there is still a remainder, use twice the part of the root already found, for a trial divisor, and continue as directed above.

Example.—Find the square root of the expression $1 + 10x^2 + 25x^4 + 16x^6 - 24x^5 - 20x^3 - 4x$.

Solution.—First, arrange the expression according to the decreasing powers of x:

$$\begin{array}{r|l}
 & \overset{\text{root}}{4x^3 - 3x^2 + 2x - 1}. \text{ Ans.} \\
 & 16x^6 - 24x^5 + 25x^4 - 20x^3 + 10x^2 - 4x + 1 \\
 & 16x^6 \\
8x^3 - 3x^2 & -24x^5 + 25x^4 \\
 & -24x^5 + 9x^4 \\
8x^3 - 6x^2 + 2x & 16x^4 - 20x^3 + 10x^2 \\
 & 16x^4 - 12x^3 + 4x^2 \\
8x^3 - 6x^2 + 4x - 1 & -8x^3 + 6x^2 - 4x + 1 \\
 & -8x^3 + 6x^2 - 4x + 1
\end{array}$$

The square root of the first term, $16x^6$, is $4x^3$, which is the first term of the root. This, squared and subtracted from the given polynomial, leaves a remainder of $-24x^5 + 25x^4$, plus the terms that were not brought down. Doubling $4x^3$, the part of the root already found, gives $8x^3$; $-24x^5$ divided by $8x^3 = -3x^2$, which is the second term of the root.

Annexing this to the $8x^2$, the trial divisor, we have for our complete divisor $8x^3 \times 3x^2$, which, multiplied by $-3x^2$, $= -24x^5 + 9x^4$. Subtracting this, and bringing down the next two terms, we have $16x^4 - 20x^3 + 10x^2$. For a new trial divisor, double the root already found, and obtain $2(4x^3 - 3x^2) = 8x^3 - 6x^2$, which is contained in the above dividend $+2x$ times, the third term of the root. In like manner, the last term is found to be -1. Since this is an even root, it would have the double sign $\pm$. (Art. **519.**) In such examples, however, it is not necessary to consider the minus sign.

EXAMPLE.—What is the square root of $x^6 + 8x^4y^2 - 4x^5y - 4xy^5 + 8x^2y^4 - 10x^3y^3 + y^6$?

SOLUTION.— $\underline{|x^3 - 2x^2y + 2xy^2 - y^3.}$ Ans.

$$\begin{array}{r|l}
 & x^6 - 4x^5y + 8x^4y^2 - 10x^3y^3 + 8x^2y^4 - 4xy^5 + y^6 \\
 & \underline{x^6} \\
2x^3 - 2x^2y & -4x^5y + 8x^4y^2 \\
 & \underline{-4x^5y + 4x^4y^2} \\
2x^3 - 4x^2y + 2xy^2 & 4x^4y^2 - 10x^3y^3 + 8x^2y^4 \\
 & \underline{4x^4y^2 - 8x^3y^3 + 4x^2y^4} \\
2x^3 - 4x^2y + 4xy^2 - y^3 & -2x^3y^3 + 4x^2y^4 - 4xy^5 + y^6 \\
 & \underline{-2x^3y^3 + 4x^2y^4 - 4xy^5 + y^6}
\end{array}$$

EXAMPLES FOR PRACTICE.

524. Extract the square root of the following polynomials:

1. $4x^4 + 9 - 30x - 20x^3 + 37x^2$. Ans. $2x^2 - 5x + 3$.
2. $16y^4 + 24y^3 + 89y^2 + 60y + 100$. Ans. $4y^2 + 3y + 10$.
3. $-2x - 12x^5 + 10x^4 + 5x^2 - 10x^3 + 9x^6 + 1$. Ans. $3x^3 - 2x^2 + x - 1$.

NOTE.—Arrange according to the decreasing powers of x.

ALGEBRA.

(CONTINUED.)

EXPONENTS.

525. In Art. **361,** an **exponent** was defined as a figure written to the right and above a quantity to show how many times the latter is taken as a factor. This definition is not complete, because it applies only to positive integral exponents. Fractional and negative exponents sometimes occur, and require a more extended definition. The rules for positive integral exponents are used for fractional and negative exponents, however, and will be repeated here:

526. I. In multiplication, exponents of like quantities are added. (Art. **418.**)

II. In division, the exponents of quantities in the divisor are subtracted from the exponents of like quantities in the dividend. (Art. **438.**)

III. When raising a monomial to any power, its exponents are multiplied by the exponent of the power. (Art. **511.**)

IV. When extracting the root of a monomial, its exponents are divided by the index of the root. (Art. **521.**)

527. Since letters may represent numbers, they may be used for exponents, the same as figures. Thus, a^n means that a is to be taken as many times as a factor as there are units in n, or $a \times a \times a$, etc., to n factors. Such exponents are called **literal exponents.** Fractional, negative and literal exponents are all read by using the word exponent. Thus, $a^{\frac{2}{3}}$, a^{-4}, $a^{\frac{n}{m}}$, and a^n, are read, "a, exponent $\frac{2}{3}$," "a, exponent, minus 4," "a, exponent $n \div m$," and "a. exponent n," respectively.

528. The meaning of **fractional exponents** will be evident from the following illustration: By Art. **526,** IV, the square root of a^6 is a^3, obtained by dividing the exponent 6 of the power, by the index 2 of the root. Since, however, a fraction is an expression of division (Art. **477**), the square root of a^6 might be *indicated* by writing the exponent of a, as $\frac{6}{2}$, thus *indicating* the division of 6 by 2. Thus, $a^{\frac{6}{2}}$, means the square root of a^6. Hence,

The numerator of a fractional exponent denotes a power, and the denominator, a root.

For example, $a^{\frac{1}{2}} = \sqrt{a}$; $c^{\frac{2}{3}} = \sqrt[3]{c^2}$; $x^{\frac{m}{n}} = \sqrt[n]{x^m}$; $2c^{\frac{m}{5}} = 2\sqrt[5]{c^m}$, the exponent $\frac{m}{5}$ applying only to the c.

Since $\frac{6}{2} = \frac{1}{2} \times 6$, $a^{\frac{6}{2}} = a^{\frac{1}{2} \times 6}$; in other words, $a^{\frac{6}{2}}$ may also be read as the sixth power of the square root of a.

529. The meaning of **negative exponents** may be illustrated as follows: Let it be required to divide a^4 by a^2. By Art. **526,** II, we have $\frac{a^4}{a^2} = a^{4-2} = a^2$; also, $a^2 \div a^2 = \frac{a^2}{a^2} = a^{2-2} = a^0$, and $a^0 \div a^2 = \frac{a^0}{a^2} = a^{0-2} = a^{-2}$, etc. From this, it will be seen that $\frac{a^0}{a^2} = a^{-2}$; but, by Art. **439,** $a^0 = 1$, so that 1 may be written for a^0, in this expression, thus: $\frac{a^0}{a^2} = \frac{1}{a^2} = a^{-2}$. Hence,

A quantity affected with a negative exponent denotes the reciprocal of the same quantity affected with an equal positive exponent. (Art. **481.**)

530. Also, since in $\frac{1}{a^2} = a^{-2}$, the a^{-2} changes to a^2 when placed in the denominator, we may state that,

A factor may be changed from the numerator to the denominator, or from the denominator to the numerator, if the sign of its exponent be changed.

For example, $\frac{n^{-3}}{ab} = \frac{1}{abn^3}$; $\frac{n}{ab^{-\frac{1}{3}}} = \frac{nb^{\frac{1}{3}}}{a}$; $\frac{x^{-\frac{2}{3}}}{5y^{-1}} = \frac{y}{5x^{\frac{2}{3}}}$, etc. In the last, the positive exponent 1 of the y is not written.

EXAMPLE.—Express, with positive exponents,

$$a^{-1}b^{-2}c^3 + a^{-2}b^{-\frac{3}{4}}c^{-\frac{1}{3}} + a^3b^{-2}.$$

SOLUTION.—Since these terms may be taken as fractions, with one for the denominators, we have, by transferring the letters with negative exponents to the denominators,

$$a^{-1}b^{-2}c^3 + a^{-2}b^{-\frac{3}{4}}c^{-\frac{1}{3}} + a^3b^{-2} = \frac{c^3}{ab^2} + \frac{1}{a^2b^{\frac{3}{4}}c^{\frac{1}{3}}} + \frac{a^3}{b^2}. \quad \text{Ans.}$$

531. It should be observed that only *factors* of the *whole* denominator or numerator can be changed, and that they must become factors of the *whole* numerator or denominator to which they are transferred. Thus, in $\frac{a}{bc^{-2}+d}$, the c^{-2} cannot be transferred to the numerator, by merely changing the signs of the exponent, because it is not a factor of the *whole* denominator; but the exponent may be made positive in the following manner:

$$\frac{a}{bc^{-2}+d} = \frac{a}{\frac{b}{c^2}+d} = \frac{a}{\frac{b+c^2d}{c^2}} = \frac{a\,c^2}{b+c^2d}.$$

In $\frac{ac^{-2}}{b+d}$, the c^{-2}, when transferred, must become a factor of the *whole* denominator, thus, $\frac{a}{c^2(b+d)}$.

532. The method of dealing with fractional, negative, and literal exponents will be clearly shown by the examples which follow:

EXAMPLE.—Find the products of the following: a^3 and a^{-1}; n and $n^{-\frac{1}{4}}$; $c^{\frac{n}{m}}$ and $c^{\frac{2n}{m}}$; x^{-1} and $\sqrt[3]{x^2}$; $2c^{-\frac{2}{3}}$ and $\frac{1}{-3\sqrt[3]{c^2}}$. Write all the products with positive exponents.

SOLUTION.—Apply Art. **526,** I, in each case. The exponents of the first are 3 and -1; their sum is $3-1=2$. Hence, $a^3 \times a^{-1} = a^2$. Ans.

In like manner, $n \times n^{-\frac{1}{4}} = n^{1-\frac{1}{4}} = n^{\frac{3}{4}}$, Ans., the exponent $\frac{3}{4}$ being the sum of 1 and $-\frac{1}{4}$; $c^{\frac{n}{m}} \times c^{\frac{2n}{m}} = c^{\frac{3n}{m}}$. Ans.

To multiply the next, change $\sqrt[3]{x^2}$ to $x^{\frac{2}{3}}$ by Art. **528.** Then $x^{-1} \times x^{\frac{2}{3}} = x^{-\frac{1}{3}} = \frac{1}{x^{\frac{1}{3}}}$. Ans. (Art. **530.**) In the last one $2c^{-\frac{2}{3}} = \frac{2}{c^{\frac{2}{3}}}$ and $\frac{1}{-3\sqrt[3]{c^2}} = \frac{1}{-3c^{\frac{2}{3}}}$; whence, $\frac{2}{c^{\frac{2}{3}}} \times \frac{1}{-3c^{\frac{2}{3}}} = \frac{2}{-3c^{\frac{4}{3}}} = -\frac{2}{3c^{\frac{4}{3}}}$. Ans.

EXAMPLE.—Find the quotients of the following: $a^3 \div a^{-1}$; $n^{-\frac{1}{4}} \div n^{-\frac{1}{2}}$; $c^{\frac{n}{m}} \div c^{\frac{2n}{m}}$; $x^2 \div \sqrt[4]{x^2}$.

SOLUTION.—Apply Art. **526,** II, in each case. The exponents of the first are 3 and -1; subtracting, $3-(-1)=4$. Hence, $a^3 \div a^{-1} = a^4$. Ans.

In like manner, $n^{-\frac{1}{4}} \div n^{-\frac{1}{2}} = n^{-\frac{1}{4}+\frac{1}{2}} = n^{\frac{1}{4}}$. Ans.; $c^{\frac{n}{m}} \div c^{\frac{2n}{m}} = c^{\frac{n}{m}-\frac{2n}{m}} = c^{-\frac{n}{m}}$. Ans.; $x^2 \div \sqrt[4]{x^2} = x^2 \div x^{\frac{1}{2}} = x^{\frac{3}{2}}$. Ans.

EXAMPLE.—Find the values of the following: $(a^{-1})^{-\frac{1}{2}}$; $(cd^{-2})^{\frac{5}{2}}$; $(x^a)^{-b} \div (x^{-a})^{-b}$; $\sqrt{9m^{-2}x^{\frac{2}{3}}}$.

SOLUTION.—Apply Art. **526,** III. In the first, multiplying the exponents (see Art. **511**), $-1 \times -\frac{1}{2} = \frac{1}{2}$. Hence, $(a^{-1})^{-\frac{1}{2}} = a^{\frac{1}{2}}$, or $\sqrt{a}$. Ans. In like manner, $(cd^{-2})^{\frac{5}{2}} = c^{\frac{5}{2}}d^{-5}$, Ans., since $1 \times \frac{5}{2} = \frac{5}{2}$, and $-2 \times \frac{5}{2} = -5$. In the next one, $(x^a)^{-b} = x^{-ab}$ and $(x^{-a})^{-b} = x^{ab}$. Dividing, $x^{-ab} \div x^{ab} = x^{-ab-ab} = x^{-2ab}$. Ans. For the last example, apply Art. **526,** IV. Since this is the square root, divide the exponents by 2, thus: $-2 \div 2 = -1$, and $\frac{2}{3} \div 2 = \frac{1}{3}$. Hence, $\sqrt{9m^{-2}x^{\frac{2}{3}}} = 3m^{-1}x^{\frac{1}{3}}$. Ans.

EXAMPLES FOR PRACTICE.

533. Clear the following of negative exponents:

1. $x^2y^{-2}z^{-\frac{1}{3}}$. Ans. $\dfrac{x^2}{y^2z^{\frac{1}{3}}}$.
2. $3a^{-1}b + \dfrac{2a}{b^{-2}c^{-1}} + c^{-1}$. Ans. $\dfrac{3b}{a} + 2ab^2c + \dfrac{1}{c}$.
3. $\dfrac{4a^{-2}(c+d)}{2c+d}$. Ans. $\dfrac{4(c+d)}{a^2(2c+d)}$.

Express the following without radical signs:

4. $\sqrt[3]{b^{-2}}$. Ans. $(b^{-2})^{\frac{1}{3}}$ or $b^{-\frac{2}{3}}$.
5. $4a\sqrt{a^{-1}b^{-3}}$. Ans. $4a^1a^{-\frac{1}{2}}b^{-\frac{3}{2}} = 4a^{\frac{1}{2}}b^{-\frac{3}{2}}$.

Find the values of the following:

6. $m^{\frac{1}{2}} \times m^{-\frac{1}{6}}$. Ans. $m^{\frac{1}{3}}$.
7. $2ab^{\frac{1}{2}} \times a^{-\frac{3}{4}}b$. Ans. $2a^{\frac{1}{4}}b^{\frac{3}{2}}$.
8. $c^{\frac{n}{2}} + \sqrt{c^n}$. Ans. c^n.
9. $2x^{-2} \div (x^2)^{-\frac{1}{2}}$. Ans. $2x^{-1}$.
10. $\left(cd^{-\frac{n}{m}}\right)^{2m} \times \sqrt{d^{4n}}$. Ans. c^{2m}.

RADICALS.

534. A **radical** is a root indicated by a radical sign. (Art. **363.**)

535. An indicated root that can be exactly obtained is called a **rational** quantity; when the root cannot be exactly obtained, it is called a **surd.** Thus, $\sqrt{9}$ is a rational quantity, and $\sqrt{2}$ is a surd.

536. The **degree** of a radical is denoted by the index of the radical sign; thus, $\sqrt[3]{a+2}$ is of the third degree.

537. By Art. **511,** an expression like $(ab)^{\frac{1}{4}} = a^{\frac{1}{4}}b^{\frac{1}{4}}$, or $a^{\frac{1}{4}} \times b^{\frac{1}{4}}$. Hence, expressing the exponent $\frac{1}{4}$ by the radical sign (Art. **528**), $\sqrt[4]{ab} = \sqrt[4]{a} \times \sqrt[4]{b}$. That is, the product of two radicals of the same degree is equal to a radical of like degree, consisting of the product of the quantities under the radical sign.

538. In like manner, $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$. That is, *when the quantity under the radical sign has a factor whose indicated root can be extracted, the quantity may be placed outside the radical sign.*

Example.—Simplify the following: $\sqrt{48}$; $\sqrt{27a^5x^7}$; $\sqrt[3]{108}$; $3\sqrt[3]{4a^{12}}$.

Solution.—In each case, resolve the quantity under the radical sign into two factors, one of which is the greatest factor of which the indicated root can be extracted, and apply Art. **538.**

$\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}$. Ans.

$\sqrt{27a^5x^7} = \sqrt{9a^4x^6 \times 3ax} = \sqrt{9a^4x^6} \times \sqrt{3ax} = 3a^2x^3\sqrt{3ax}$. Ans.

$\sqrt[3]{108} = \sqrt[3]{27 \times 4} = \sqrt[3]{27} \times \sqrt[3]{4} = 3\sqrt[3]{4}$. Ans.

$3\sqrt[3]{4a^{12}} = 3\sqrt[3]{a^{12} \times 4} = 3\sqrt[3]{a^{12}} \times \sqrt[3]{4} = 3a^4\sqrt[3]{4}$. Ans.

539. Until the student becomes accustomed to the handling of radicals, it may, perhaps, be better for him to use the fractional exponent method of expressing them, in preference to the radical sign. Thus, in simplifying $\sqrt{48}$, write it $48^{\frac{1}{2}}$, and solve as follows: $48^{\frac{1}{2}} = (16 \times 3)^{\frac{1}{2}} = 16^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 4 \times 3^{\frac{1}{2}} = 4\sqrt{3}$. In a similar manner, $\sqrt{27a^5x^7} = 27^{\frac{1}{2}} \times a^{\frac{5}{2}} \times x^{\frac{7}{2}} = 9^{\frac{1}{2}} \times 3^{\frac{1}{2}} \times a^{\frac{4}{2}} \times a^{\frac{1}{2}} \times x^{\frac{6}{2}} \times x^{\frac{1}{2}} = 3 \times 3^{\frac{1}{2}} \times a^2 \times a^{\frac{1}{2}} \times x^3 \times x^{\frac{1}{2}} = 3a^2x^3 \times 3^{\frac{1}{2}}a^{\frac{1}{2}}x^{\frac{1}{2}} = 3a^2x^3\sqrt{3ax}$.

540. When the quantity under the radical sign is a fraction, *multiply both terms by such a quantity as will make the denominator a perfect power of the same degree as the radical.*

EXAMPLE.—Reduce $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{9}{8}}$ to their simplest forms.

SOLUTION.—$\sqrt{\frac{2}{3}} = \sqrt{\frac{2 \times 3}{3 \times 3}} = \sqrt{\frac{6}{9}}$. Factoring this expression, as was done in the last example, $\sqrt{\frac{6}{9}} = \sqrt{\frac{1}{9} \times 6} = \sqrt{\frac{1}{9}} \times \sqrt{6} = \frac{1}{3}\sqrt{6}$. Ans. (Art. **538.**)

$\sqrt{\frac{9}{8}} = \sqrt{\frac{18}{16}}$. Factoring, this equals $\sqrt{\frac{9}{16} \times 2} = \sqrt{\frac{9}{16}} \times \sqrt{2} = \frac{3}{4}\sqrt{2}$. Ans.

541. EXAMPLE.—Remove from under the radical sign the denominator of $\sqrt[4]{\frac{5a}{2b^3c^2}}$.

SOLUTION.—It is necessary to multiply both terms of the fraction by such a quantity as will make the exponents of the denominator all equal to 4. Hence, the denominator must be $2^4b^4c^4$. Dividing $2^4b^4c^4$ by $2b^3c^2$, the result is $2^3bc^2 = 8bc^2$. Multiplying both numerator and denominator by $8bc^2$, the radical becomes $\sqrt[4]{\frac{40abc^2}{2^4b^4c^4}} = \frac{1}{2bc}\sqrt[4]{40abc^2}$. Ans.

542. It follows, also, from Art. **538,** that a factor outside a radical sign may be placed under it by raising it to a power corresponding to the degree of the radical.

EXAMPLE.—In $3x\sqrt{2x}$, introduce the $3x$, and in $2a^2b\sqrt[3]{3ab}$, the $2a^2b$, under the radical signs.

SOLUTION.—Squaring $3x$, we have $9x^2$. Hence, since this is the reverse of Art. **538.**

$$3x\sqrt{2x} = \sqrt{9x^2} \times \sqrt{2x} = \sqrt{18x^3}. \quad \text{Ans.}$$

Since $(2a^2b)^3 = 8a^6b^3$, $2a^2b\sqrt[3]{3ab} = \sqrt[3]{8a^6b^3} \times \sqrt[3]{3ab} = \sqrt[3]{24a^7b^4}$. Ans.

EXAMPLES FOR PRACTICE.

543. Simplify the following:

1. $\sqrt{50}$. Ans. $5\sqrt{2}$.
2. $3\sqrt[3]{24}$. Ans. $6\sqrt[3]{3}$.
3. $2\sqrt[4]{80}$. Ans. $4\sqrt[4]{5}$.
4. $\sqrt{125a^4d^3}$. Ans. $5a^2d\sqrt{5d}$.
5. $\sqrt{\frac{5}{6}}$. Ans. $\frac{1}{6}\sqrt{30}$.
6. $\sqrt[3]{\frac{5}{9}}$. Ans. $\frac{1}{3}\sqrt[3]{15}$.
7. $\frac{a}{b}\sqrt{\frac{b}{2a^3}}$. Ans. $\frac{1}{2ab}\sqrt{2ab}$.

Introduce the coefficients of the radicals in the following, under the radical sign:

8. $5\sqrt{32}$. Ans. $\sqrt{800}$.
9. $x\sqrt[3]{x^2y^2}$. Ans. $\sqrt[3]{x^5y^2}$.
10. $2x\sqrt[5]{xy}$. Ans. $\sqrt[5]{32x^6y}$.

ADDITION AND SUBTRACTION OF RADICALS.

544. Rule.—*Reduce the radicals to their simplest forms, and proceed as in addition or subtraction, combining like radicals, and indicating the addition or subtraction of unlike terms.*

EXAMPLE.—Find the sum of $\sqrt{18}$, $\sqrt{27}$, and $\sqrt{\frac{1}{2}}$.

SOLUTION.—Simplifying, by Art. **538,**

$$\sqrt{18} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}.$$
$$\sqrt{27} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}.$$
$$\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{2}{4}} = \sqrt{\tfrac{1}{4}} \times \sqrt{2} = \tfrac{1}{2}\sqrt{2}.$$

Sum $= (3 + \frac{1}{2})\sqrt{2} + 3\sqrt{3} = \frac{7}{2}\sqrt{2} + 3\sqrt{3}$. Ans.

EXAMPLE.—From $2\sqrt[3]{48}$ take $\sqrt[3]{162}$.

SOLUTION.—Simplifying,

$$2\sqrt[3]{48} = 2\sqrt[3]{8} \times \sqrt[3]{6} = 4\sqrt[3]{6}.$$
$$\sqrt[3]{162} = \sqrt[3]{27} \times \sqrt[3]{6} = 3\sqrt[3]{6}.$$

Difference $= (4 - 3)\sqrt[3]{6} = \sqrt[3]{6}$. Ans.

545. In addition and subtraction of radicals, it is better to use the fractional form of expressing them in all cases, since the liabilities of making mistakes are then greatly reduced. Thus, in finding the sum of $\sqrt{18} + \sqrt{27} + \sqrt{\frac{1}{2}}$, $\sqrt{18} = 9^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 3 \times 2^{\frac{1}{2}}$, $\sqrt{27} = 9^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3 \times 3^{\frac{1}{2}}$ and $\sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2} = \frac{1}{2} \times 2^{\frac{1}{2}}$. Hence, $3 \times 2^{\frac{1}{2}} + 3 \times 3^{\frac{1}{2}} + \frac{1}{2} \times 2^{\frac{1}{2}} = (3 + \frac{1}{2}) \times 2^{\frac{1}{2}} + 3 \times 3^{\frac{1}{2}} = 3\frac{1}{2}\sqrt{2} + 3\sqrt{3}$. Here it is perfectly evident that it would not do to add $3 \times 3^{\frac{1}{2}}$ and $3 \times 2^{\frac{1}{2}}$, and get $6 \times 5^{\frac{1}{2}}$.

EXAMPLE.— $\sqrt{320} - \sqrt{80} = 64^{\frac{1}{2}} \times 5^{\frac{1}{2}} - 16^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 8 \times 5^{\frac{1}{2}} - 4 \times 5^{\frac{1}{2}} = (8 - 4)5^{\frac{1}{2}} = 4\sqrt{5}$. Ans.

EXAMPLES FOR PRACTICE.

546. Find the value of the following:

1. $\sqrt{50} + \sqrt{72}$. Ans. $11\sqrt{2}$.
2. $\sqrt{\frac{3}{5}} + \sqrt{\frac{1}{15}} + \sqrt{\frac{15}{49}}$. Ans. $\frac{43}{105}\sqrt{15}$.

3. $4\sqrt[3]{54} + 5\sqrt[3]{128} + \sqrt[3]{81}$. Ans. $32\sqrt[3]{2} + 3\sqrt[3]{3}$.

4. $\sqrt[3]{1029} - \sqrt[3]{192}$. Ans. $3\sqrt[3]{3}$.

5. $b\sqrt[3]{27a^6b} - \sqrt[3]{216a^6b^4}$. Ans. $-3a^2b\sqrt[3]{b}$.

MULTIPLICATION AND DIVISION OF RADICALS.

547. Rule.—*If the radicals are of the same degree, multiply or divide the quantities under the radical signs, and write the results under the common radical sign.*

EXAMPLE.—Find the product of $\sqrt{12}$ and $\sqrt{3}$.

SOLUTION.— $\sqrt{12} \times \sqrt{3} = \sqrt{12 \times 3} = \sqrt{36} = 6$. Ans.

EXAMPLE.—Divide $\sqrt[3]{9a^4}$ by $\sqrt[3]{3a}$.

SOLUTION.— $\sqrt[3]{9a^4} \div \sqrt[3]{3a} = \sqrt[3]{9a^4 \div 3a} = \sqrt[3]{3a^3} = a\sqrt[3]{3}$. Ans.

548. If the radicals are not of the same degree, they must be reduced to equivalent ones, all of which have the same degree, as follows: *Express the radicals with fractional exponents; reduce these fractions to a common denominator, and express the resulting fractional exponents with radical signs.*

EXAMPLE.—Multiply together $\sqrt{2}$, $\sqrt[3]{3}$ and $\sqrt[6]{5}$.

SOLUTION.—By Art. **528**, $\sqrt{2} = 2^{\frac{1}{2}}$; $\sqrt[3]{3} = 3^{\frac{1}{3}}$; $\sqrt[6]{5} = 5^{\frac{1}{6}}$; reducing the exponents to a common denominator, 6,

$$2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8}.$$
$$3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}.$$
$$5^{\frac{1}{6}} = \sqrt[6]{5}.$$

Multiplying, $\sqrt[6]{8} \times \sqrt[6]{9} \times \sqrt[6]{5} = \sqrt[6]{360}$. Ans.

549. EXAMPLE.—Multiply $7\sqrt{5}$, $2\sqrt[4]{6}$ and $\sqrt[6]{4}$.

SOLUTION.—Writing these surds with fractional exponents, and multiplying, the result is $(7 \times 5^{\frac{1}{2}}) \times (2 \times 6^{\frac{1}{4}}) \times 4^{\frac{1}{6}} = 14(5^{\frac{1}{2}} \times 6^{\frac{1}{4}} \times 4^{\frac{1}{6}})$. That part of the product included in parenthesis cannot be multiplied together by reason of the difference in the exponents. For example, $2^3 \times 5^2 = 8 \times 25 = 200$; it does not equal 10^5, since this equals 100,000. Hence, in order to multiply the three numbers included in parenthesis, they must all be affected with the same exponent. To effect this, the fractional exponents must first be reduced to a common denominator. The common denominator is 12; $5^{\frac{1}{2}} = 5^{\frac{6}{12}}$, $6^{\frac{1}{4}} = 6^{\frac{3}{12}}$ and $4^{\frac{1}{6}} = 4^{\frac{2}{12}}$. But $5^{\frac{6}{12}} = (5^6)^{\frac{1}{12}} = 15{,}625^{\frac{1}{12}}$, $6^{\frac{3}{12}} = (6^3)^{\frac{1}{12}} = 216^{\frac{1}{12}}$ and $4^{\frac{2}{12}} = (4^2)^{\frac{1}{12}} = 16^{\frac{1}{12}}$. Since the numbers now have the exponent, they may be multiplied together; thus, $15{,}625^{\frac{1}{12}} \times 216^{\frac{1}{12}} \times 16^{\frac{1}{12}} = (15{,}625 \times 216 \times 16)^{\frac{1}{12}} = (54{,}000{,}000)^{\frac{1}{12}}$. Hence, the product of $7\sqrt{5}$, $2\sqrt[4]{6}$ and $\sqrt[6]{4} =$

$$14 \times 54{,}000{,}000^{\frac{1}{12}} = 14\sqrt[12]{54{,}000{,}000}. \quad \text{Ans.}$$

To prove that the answer is correct, perform the indicated operations in both cases. If the results are the same, the work must be correct. Thus, in the last example, $7\sqrt{5} = 15.65248$; $2\sqrt[4]{6} = 3.13017$ and $\sqrt[6]{4} = 1.25992$. Therefore, $15.65248 \times 3.13017 \times 1.25992 = 61.7297$.

But $14\sqrt[12]{54{,}000{,}000}$ also equals 61.7297, showing that the work was correct.

In all doubtful cases arising from the use of surds and radicals, if the student will use fractional exponents instead of the radical sign, he should experience but little difficulty. In addition and subtraction, the surds (not the coefficients of the surds) must be alike; if they are not alike, the addition or subtraction must be indicated. For example, $3 \times 4^{\frac{1}{3}} + 14 \times 4^{\frac{1}{3}} = 17 \times 4^{\frac{1}{3}}$, for the same reason that $3a + 14a = 17a$; but $3 \times 4^{\frac{1}{3}} + 14 \times 5^{\frac{1}{3}} = 3 \times 4^{\frac{1}{3}} + 14 \times 5^{\frac{1}{3}}$, for the same reason that $3a^2 + 14b^2 = 3a^2 + 14b^2$. In multiplication and division, the surds need not be alike, but the fractional exponents of the surds must be alike, for the reason stated above.

550. If there are factors outside the radical signs, multiply or divide them separately.

EXAMPLE.—Divide $4\sqrt{3}$ by $2\sqrt[3]{3}$.

SOLUTION.— $4\sqrt{3} = 4 \times 3^{\frac{1}{2}} = 4 \times 3^{\frac{3}{6}} = 4\sqrt[6]{27}$.

$2\sqrt[3]{3} = 2 \times 3^{\frac{1}{3}} = 2 \times 3^{\frac{2}{6}} = 2\sqrt[6]{9}$.

$4 \div 2 = 2$ and $27 \div 9 = 3$; hence, $4\sqrt{3} \div 2\sqrt[3]{3} = 4\sqrt[6]{27} \div 2\sqrt[6]{9} = 2\sqrt[6]{3}$.

EXAMPLES FOR PRACTICE.

551. Multiply the following, reducing the answers to their simplest forms:

1. $5\sqrt{8}$ by $3\sqrt{5}$. Ans. $30\sqrt{10}$.
2. $\sqrt[3]{8}$ by $4\sqrt{48}$. Ans. $32\sqrt{3}$.
3. $\frac{1}{3}\sqrt[3]{4}$ by $\frac{3}{4}\sqrt[3]{12}$. Ans. $\frac{1}{2}\sqrt[3]{6}$.

Divide the following:

4. $3\sqrt{15}$ by $4\sqrt{5}$. Ans. $\frac{3}{4}\sqrt{3}$.

5. $\frac{1}{4}\sqrt{\frac{3}{5}}$ by $\frac{3}{7}\sqrt{\frac{3}{5}}$. Ans. $\frac{7}{12}$.

6. $\sqrt[5]{2}$ by $\sqrt[4]{3}$. Ans. $\sqrt[20]{\frac{16}{243}}$.

INVOLUTION AND EVOLUTION OF RADICALS.

552. A radical may be raised to any power, or any root of a monomial radical may be taken, by writing the radical with fractional exponents and proceeding as in Art. **532.** If there is a factor outside the radical it should be involved separately in involution, and in evolution it should first be introduced under the radical sign.

553. The most important case is that of raising a radical to the power corresponding to the degree of the radical, when the radical sign will disappear; thus, to raise $\sqrt[3]{4a}$ to the third power, $(\sqrt[3]{4a})^3 = [(4a)^{\frac{1}{3}}]^3 = (4a)^{\frac{3}{3}} = 4a^1$, or $4a$. Ans.

EQUATIONS.

554. As defined in Art. **354,** an **equation** is a statement of equality between two expressions, as $x + 6 = 14$.

555. Every equation has two parts, called the **first** and **second members.** The first member is the part on the left of the sign of equality, and the second member the part on the right of that sign. In $x + 6 = 14$, $x + 6$ is the first member, and 14 is the second member.

556. Equations usually consist of **known** and **unknown quantities;** that is, of quantities whose values are given, and of quantities whose values are not given, but are to be found. Thus, in $x + 6 = 14$, 6 and 14 are known quantities, and x is unknown; but since by the statement of the equation, $x + 6$ must equal 14, x must have such a value that when added to 6 the sum will be 14. Hence, the value of x is fixed for this particular case, and in a similar manner the value of a single unknown quantity in any equation is fixed by the relations that it bears to the known quantities, and this value can usually be found.

557. To solve an equation is to find the value of the unknown quantity. This is done by a series of **transformations** by which the first member becomes the unknown quantity, and the second member becomes a known quantity, which is, therefore, the value of the unknown quantity.

TRANSFORMATIONS.

558. In transforming an equation, the equality of its members must be preserved; otherwise, the existing relations between the known and unknown quantities will be destroyed. Transformations are based upon the following principles·

559. In any equation:

I. The same quantity may be added to both members. For example, if 2 be added to both members of $x^2 = 16$, the members of the resulting equation, $x^2 + 2 = 18$, will be equal.

II. The same quantity may be subtracted from both members. Thus, if $x^2 = 16$, then $x^2 - 2 = 14$.

III. Both members may be multiplied or both divided by the same quantity. Thus, if $x^2 = 16$, then $2x^2 = 32$ and $\frac{x^2}{2} = 8$.

IV. Both members may be raised to the same power. Thus, if $x^2 = 16$, then $x^4 = 256$.

V. Like roots of both members may be extracted. Thus, if $x^2 = 16$, then $x = 4$.

A little thought will show that none of these operations will destroy the equality of the members. In the equation $16 = 16$, for example, by I, $16 + 2 = 16 + 2$; by II, $16 - 2 = 16 - 2$; by III, $16 \times 2 = 16 \times 2$, etc. It is to be observed, however, that after any transformation, the *members* do not equal their original values.

560. Transposition.—In transforming an equation, it is frequently necessary to transpose, or change a term from one member to the other. For example, in the equation $3x + 5 = 12$, let it be required to transpose the $+ 5$ to the second member. This may be done by *subtracting* + 5 from both members, which, by Art. **559,** II, will not destroy the equality; thus,

Subtracting + 5 from both members,

$$\begin{array}{rcl} 3x + 5 &=& 12 \\ \underline{\ 5} && \underline{5} \\ 3x &=& 12 - 5 = 7. \end{array}$$

Again, let it be required to transpose the − 5 in $3x - 5 = 12$, to the second member. This may be done by *adding* + 5 to both members, which, by Art. **559,** I, will not destroy the equality; thus,

Adding + 5 to both members,

$$\begin{array}{rcl} 3x - 5 &=& 12 \\ \underline{\ 5} && \underline{5} \\ 3x &=& 12 + 5 = 17. \end{array}$$

Now, what was really accomplished in each case was that 5 was transposed from the first to the second member, *with its sign changed;* and in changing a term from the second to the first member, the same operation would be performed. Hence,

561. *Any term may be transposed from either member of an equation to the other, if its sign be changed.*

562. Cancelation.—*When the same term appears with the same sign in both members of an equation, it may be canceled from both.* For, in the equation $x + a = 6 + a$, we have, by transposing the a in the first member, to the second member, $x = 6 + a - a$; whence, the a's cancel, leaving $x = 6$. It must be observed that terms will not cancel from both members unless they have the *same* sign. Thus, in $x - a = 6 + a$, we have, by transposing the $-a$, $x = 6 + 2a$.

563. Changing Signs.—It is sometimes desirable to change the sign of a quantity in an equation from − to + or from + to −. To change it, we use the following principle: *the signs of all the terms of both members of an equation may be changed without destroying the equality.* For, in the equation $-x + 4 = 10 - a$, both members may be multiplied by -1 (Art. **559,** III), giving $x - 4 = -10 + a$, or $a - 10$.

564. Clearing of Fractions.—When an equation contains fractions it must be cleared of them in order to find the value of the unknown quantity.

Example.—Clear the equation $x + \frac{x}{2} + \frac{3x}{4} + \frac{2x}{6} = 100$ of fractions.

Solution.—The L. C. M. of the denominators 2, 4 and 6 is 12. By Art. **559,** III, both members may be multiplied by the same quantity. Hence, multiplying each term by 12, we have $12x + \frac{12x}{2} + \frac{36x}{4} + \frac{24x}{6} = 1{,}200$. Now, reducing each fraction to its simplest form, which will not alter its value (Art. **483**), and so will not destroy the equality of the members, we have $12x + 6x + 9x + 4x = 1{,}200$, the denominators to all the fractions having canceled.

565. *Hence, to clear an equation of fractions, multiply each term of the equation by the L. C. M. of the denominators.*

566. Instead of multiplying the numerators by the L. C. M. and then reducing the fractions to their simplest forms, it is easier to divide the L. C. M. by each denominator, and then multiply the corresponding numerators by the quotients, as was done in Art. **490.**

Example.—Clear the equation $\frac{2x}{x+2} = \frac{1}{2} - \frac{3x+2}{x^2-4}$ of fractions.

Solution.—The L. C. M. of the denominators is $2(x^2-4)$. Dividing this by $x+2$ and multiplying $2x$ by the quotient, $2(x-2)$, gives $4x(x-2)$, or $4x^2-8x$; dividing $2(x^2-4)$ by 2 and multiplying 1 by the quotient, x^2-4, gives x^2-4; and dividing $2(x^2-4)$ by x^2-4 and multiplying $-(3x+2)$ by the quotient, 2, gives $-6x-4$. Hence, the equation becomes $4x^2-8x = x^2-4-6x-4$, all the denominators having canceled in the process.

567. *Where a fraction is preceded by a minus sign, care must be taken to change the sign of every term of the numerator when clearing of fractions. See Art.* ***493.***

EXAMPLES FOR PRACTICE.

568. Clear the following equations of fractions:

1. $x + \frac{3x}{4} + \frac{5}{7} = 16 - \frac{2}{x}$. Ans. $28x^2 + 21x^2 + 20x = 448x - 56$.

2. $\frac{x}{4} - \frac{x-3}{2} = \frac{a}{6}$. Ans. $3x - 6x + 18 = 2a$.

3. $\frac{x}{a-b} - x = \frac{a-b}{a+b} - 1$.
Ans. $ax + bx - a^2x + b^2x = a^2 - 2ab + b^2 - a^2 + b^2$.

4. $\frac{1}{(a-b)} = \frac{x}{a-b} - \frac{a+b}{x}$. Ans. $x = x^2 - a^2 + b^2$.

SOLUTION OF SIMPLE EQUATIONS.

569. A **simple equation** is one containing only the first power of the unknown quantity, when cleared of radical and aggregation signs and fractions. It is also called an equation of the **first degree.**

570. The unknown quantity in a simple equation containing but one unknown quantity is usually represented by the letter x. Known quantities are represented by figures and by the *first* letters of the alphabet. Equations containing known quantities represented by letters are called **literal equations,** and if any literal equation be solved (Art. **557**), the value of the unknown quantity will usually contain one or more of the first letters of the alphabet.

571. To solve a simple equation:

Rule.—*Clear the equation of fractions, if it has any.*

Transpose the terms containing unknown quantities to the first member, and the known terms to the second member.

Combine the terms containing the unknown quantity into one term and reduce the second member to its simplest form.

Divide both members of the resulting equation by the coefficient of the unknown quantity (Art. **559,** *III), and the second member of this last equation will be the value of the unknown quantity.*

This rule does not hold absolutely in all cases, since special methods are often used, of which the student can learn only by practice.

572. To verify the result, substitute the value of the unknown quantity in the original equation, which should then reduce so that both members will be alike. When this occurs the equation is said to be **satisfied.**

573. In the following examples, the value of the unknown quantity x is to be determined. All the transformations used depend upon principles explained in Arts. **558-567.**

EXAMPLE.—Solve the equation $20 + 5x - 3x - 18 = 10$.

SOLUTION.—Transposing 20 and -18 to the second member,

$$5x - 3x = 10 - 20 + 18.$$

Combining like terms, $2x = 8$.

Dividing both members by 2 (Art. **559,** III),

$$x = 4. \quad \text{Ans.}$$

To verify the result, substitute 4 for x in the original equation. (Art **572.**) Thus,

$$20 + 5 \times 4 - 3 \times 4 - 18 = 10,$$

or $$20 + 20 - 12 - 18 = 10.$$

Combining, $$10 = 10, \text{ which proves the result.}$$

Example.—Solve the equation $5x - (10 - x) = 5x + 4(x - 1)$.

Solution.—Removing the parentheses,

$$5x - 10 + x = 5x + 4x - 4.$$

Transposing -10 to the second member and $5x + 4x$ to the first member, $$5x + x - 5x - 4x = 10 - 4.$$

Combining like terms, $$-3x = 6.$$

Changing signs to make the term containing x positive,

$$3x = -6. \quad \text{(Art. } \mathbf{563.}\text{)}$$

Dividing both members by 3, $x = -2$. Ans.

Proof: $5 \times -2 - (10 + 2) = 5 \times -2 + 4(-2 - 1)$,

or $$-10 - 10 - 2 = -10 - 8 - 4.$$

Combining, $$-22 = -22, \text{ which proves the result.}$$

Example.—Solve the equation

$$16 - x - \{7x - [8x - (9x - \overline{3x - 6x})]\} = 0.$$

Solution.—Removing the aggregation signs (Art. **407**),

$$16 - x - 7x + 8x - 9x + 3x - 6x = 0.$$

Transposing 16 to the second member,

$$-x - 7x + 8x - 9x + 3x - 6x = -16.$$

Combining like terms, $$-12x = -16.$$

Dividing both members by -12,

$$x = \tfrac{16}{12} = 1\tfrac{1}{3}. \quad \text{Ans.}$$

Example.—Solve the equation

$$\frac{2x + 2}{2} + \frac{1}{4} = \frac{8 - 6x}{5} + \frac{2(6x + 7)}{8}.$$

Solution.—Reducing the first term of the first member and the last term of the second member to a simpler form, the equation becomes

$$x + 1 + \frac{1}{4} = \frac{8 - 6x}{5} + \frac{6x + 7}{4}.$$

Clearing of fractions by multiplying each term of both members by 20, the L. C. M. of the denominators, we have

$$20x + 20 + 5 = 32 - 24x + 30x + 35.$$

Transposing terms,

$$20x + 24x - 30x = 32 + 35 - 20 - 5.$$

Uniting terms, $$14x = 42.$$

Dividing by 14, $$x = 3. \quad \text{Ans.}$$

Example.—Solve the equation $x + \frac{x+4}{2} - \frac{3x-4}{5} - \frac{x}{8} = 9$.

Solution.—Clearing of fractions by multiplying each term by 40, the L. C. M., and remembering that the sign of the second fraction is minus (Art. **567**), $40x + 20x + 80 - (24x - 32) - 5x = 360$.

Removing parenthesis and transposing terms,

$$40x + 20x - 24x - 5x = 360 - 80 - 32.$$

Uniting terms, $31x = 248$.

Dividing by 31, $x = 8$. Ans.

Example.—Solve the equation $\frac{3}{1-x} - \frac{2}{1+x} + \frac{1}{1-x^2} = 0$.

Solution.—Clearing of fractions by multiplying by $1 - x^2$, the L. C. M.,

$$3(1 + x) - 2(1 - x) + 1 = 0.$$

$$3 + 3x - 2 + 2x + 1 = 0.$$

Uniting terms, $5x = -2$.

$x = -.4$. Ans.

Note.—0 multiplied or divided by any number $= 0$.

574. If the denominators in a fractional equation are partly monomial and partly polynomial, it will be easier to clear of fractions at first partially, multiplying by the L. C. M. of the *monomial* denominators.

Example.—Solve the equation $\frac{8x+5}{14} = \frac{4x+6}{7} - \frac{7x-3}{6x+2}$.

Solution.—Clearing of fractions partially, by multiplying each term by 14, and noticing that 2 may be canceled from the denominator of the second fraction of the second member when multiplying by 14,

$$8x + 5 = 8x + 12 - \frac{49x - 21}{3x + 1}.$$

Transposing and uniting the terms (Art. **562**),

$$\frac{49x - 21}{3x + 1} = 7.$$

Clearing of fractions by multiplying each term by $3x + 1$,

$$49x - 21 = 21x + 7$$

$$28x = 28$$

$x = 1$. Ans.

Example.—Solve the equation $1 + \frac{3}{x-1} = \frac{3 + \frac{4-x}{1-x}}{3}$.

Solution.—Simplifying the second member by multiplying both numerator and denominator of the fraction by $1 - x$ (Art. **509**),

$$1 + \frac{3}{x-1} = \frac{3(1 - x) + 4 - x}{3(1 - x)}.$$

Changing the signs of the first fraction so as to bring $1 - x$ in the denominator (Art. **482**), and clearing of fractions by multiplying by $3(1 - x)$, $$3(1 - x) - 9 = 3(1 - x) + 4 - x.$$

Canceling $3(1 - x)$ from both members and transposing,
$$x = 13. \quad \text{Ans.}$$

575. When powers of the unknown quantity higher than the first appear in an equation, they will often cancel, the equation thus reducing to a simple one.

Example.—Solve the equation
$$(x + 3)^2 - 3x(4x + 1) = 5x^2 - (4x - 5)^2.$$

Solution.—Performing the operations indicated,
$$x^2 + 6x + 9 - 12x^2 - 3x = 5x^2 - (16x^2 - 40x + 25).$$

Removing the parentheses and transposing terms,
$$x^2 + 6x - 12x^2 - 3x - 5x^2 + 16x^2 - 40x = -25 - 9.$$

Combining like terms, $-37x = -34$.

Dividing by -37, $x = \frac{34}{37}$. Ans.

576. In **literal equations** (Art. **570**), the terms containing known or unknown quantities cannot always be combined into one. In solving, all terms containing unknown quantities must be brought into the first member without regard to whether they contain known quantities.

Example.—Solve the literal equation $2ax - 3b = x + c - 3ax$.

Solution.—Transposing the terms containing the unknown quantities to the first member and the remaining terms to the second member, and combining like terms, $5ax - x = 3b + c$.

Factoring $5ax - x$ with a view to bringing x alone in the first member, $$(5a - 1)x = 3b + c.$$

The coefficient of x is now $5a - 1$, this being considered as one quantity. (Art. **391.**)

Dividing by $5a - 1$, $x = \dfrac{3b + c}{5a - 1}$. Ans.

Proof.—Since the original equation is equivalent to $5ax - x = 3b + c$, it will be sufficient to satisfy this equation. Hence, substituting the value of x,
$$\frac{5a(3b + c)}{5a - 1} - \frac{3b + c}{5a - 1} = 3b + c, \quad \text{or} \quad \frac{(5a - 1)(3b + c)}{5a - 1} = 3b + c.$$

Canceling the $5a - 1$, $3b + c = 3b + c$.

Example.—Solve the equation
$$(x + a)(x - b) - (x - a)(x + b) = a^2 - b^2.$$

Solution.—Performing the operations indicated,
$$x^2 + ax - bx - ab - (x^2 - ax + bx - ab) = a^2 - b^2.$$

Combining like terms, $2ax - 2bx = a^2 - b^2$;

whence, $2(a - b)x = a^2 - b^2$,

or $x = \frac{a^2 - b^2}{2(a - b)} = \frac{(a + b)(a - b)}{2(a - b)} = \frac{(a + b)}{2}$. Ans.

Example.—Solve the equation $\frac{3x + 1}{x + 1} = \frac{3bx - 2a + c}{b(x + 1) - a}$.

Solution.—Clearing of fractions,

$$(3x + 1)[b(x + 1) - a] = (x + 1)(3bx - 2a + c), \text{ or}$$

$$3bx(x + 1) - 3ax + b(x + 1) - a = 3bx(x + 1) - (2a - c)(x + 1).$$

Canceling $3bx(x + 1)$ from both members,

$$-3ax + bx + b - a = -2ax + cx - 2a + c.$$

Transposing and uniting terms,

$$-ax + bx - cx = -a - b + c.$$

Changing signs and factoring,

$$(a - b + c)x = a + b - c;$$

whence, $x = \frac{a + b - c}{a - b + c}$. Ans.

EXAMPLES FOR PRACTICE.

577. Solve the following equations:

1. $16 - 3x = 13 - 6x$. Ans. $x = -1$.
2. $3(4x - 5) + 6 = 1 + 2x$. Ans. $x = 1$.
3. $6(5 - 2x) = 6 - 2(x - 2)$. Ans. $x = 2$.
4. $\frac{2x}{3} - \frac{4x}{3} = 5 - \frac{3x}{4}$. Ans. $x = 60$.
5. $\frac{x + 1}{3} - \frac{x + 4}{5} = 16 - \frac{x + 3}{4}$. Ans. $x = 41$.
6. $\frac{x}{3} - \frac{x^2 - 5x}{3x - 7} = \frac{2}{3}$. Ans. $x = -7$.
7. $\frac{5 - 2x}{x + 1} - \frac{3 - 2x}{x + 4} = 0$. Ans. $x = 4\frac{1}{4}$.
8. $2x - 4a = 3ax + a^2 - a^2x$. Ans. $x = \frac{a^2 + 4a}{a^2 - 3a + 2}$.
9. $\frac{ax + 2x}{5a} - \frac{a^2 + 4a + 4}{4b} = 0$. Ans. $x = \frac{5a^2 + 10a}{4b}$.

Suggestion.—Transposing the second term to the second member,

$$\frac{ax + 2x}{5a} = \frac{a^2 + 4a + 4}{4b} = \frac{(a + 2)^2}{4b}.$$

Multiplying both sides by $5a$,

$$ax + 2x = \frac{5a(a + 2)^2}{4b}.$$

Solving for x,

$$x = \frac{5a(a + 2)^2}{(a + 2)4b} = \frac{5a(a + 2)}{4b} = \frac{5a^2 + 10a}{4b}.$$

10. $\frac{a(c^2 + x^2)}{cx} = ab + \frac{ax}{c}$. Ans. $x = \frac{c}{b}$

SIMPLE EQUATIONS CONTAINING RADICALS.

578. To solve equations containing radicals, the radical signs must first be removed. For example, to solve $\sqrt{x} = 4$, we know from Art. **559,** IV, that both members may be raised to the same power. Hence, squaring both members, $\sqrt{x} \times \sqrt{x} = 4 \times 4$, or $x = 16$, since, by Art. **553,** any radical raised to a power corresponding to its index equals the quantity under the radical sign. Again, in $\sqrt[3]{x} = 4$, cubing both members, $\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x} = 4 \times 4 \times 4$, or $x = 64$.

579. Hence, to solve equations containing a single radical:

Rule.—*Transpose the terms of the given equation so that the radical will stand alone in one member; then raise both members to a power corresponding to the index of the radical.*

Example.—Solve the equation $\sqrt{x^2 - 5} - x = -1$.

Solution.—Transposing $-x$ so that the radical will stand alone,

$$\sqrt{x^2 - 5} = x - 1.$$

Squaring both members, since the index of the radical is 2,

$$x^2 - 5 = x^2 - 2x + 1.$$

Transposing and uniting terms,

$$2x = 6,$$

or $x = 3$. Ans.

580. Before clearing of the radicals the equation should be simplified as much as possible.

Example.—Solve the equation $\frac{4}{9}\sqrt{x - 18} = \frac{8}{3}$.

Solution.—Dividing by $\frac{4}{9}$, $\sqrt{x - 18} = \frac{8}{3} \times \frac{9}{4} = 6$.

Squaring $x - 18 = 36$,

or $x = 54$. Ans.

581. *If there are two or more radicals in the equation, remove each one separately, beginning with the most complex, as directed above.*

Example.—Solve the equation $\sqrt{4x + 21} = 2\sqrt{x} + 1$.

Solution.—The square of $\sqrt{4x + 21}$ is $4x + 21$, and the square of $2\sqrt{x} + 1$ is $4x + 4\sqrt{x} + 1$; hence, squaring,

$$4x + 21 = 4x + 4\sqrt{x} + 1.$$

Transposing and uniting, $4\sqrt{x} = 20$.

Dividing by 4, $\sqrt{x} = 5$.

Squaring again, $x = 25$. Ans.

Example.—Solve the equation

$$\sqrt{4x+\sqrt{16x+2}} = 1+2\sqrt{x}.$$

Solution.—Squaring, $4x+\sqrt{16x+2} = 1+4\sqrt{x}+4x$.

Canceling $4x$ and squaring again,

$$16x+2 = 1+8\sqrt{x}+16x.$$

Canceling $16x$ and transposing,

$$-8\sqrt{x} = -1.$$

Changing signs, $8\sqrt{x} = 1$,

or $\sqrt{x} = \frac{1}{8}$.

Squaring again, $x = \frac{1}{64}$. Ans.

582. When there are fractions with radicals in the denominators, it is usually better to clear of fractions before squaring.

Example.—Solve the equation $\sqrt{x}+\sqrt{x-24} = \frac{60}{\sqrt{x-24}}$.

Solution.—The product of $\sqrt{x}$ and $\sqrt{x-24}$ is $\sqrt{x^2-24x}$, and the product of $\sqrt{x-24}$ and $\sqrt{x-24}$ is $x-24$. Hence, clearing of fractions,

$$\sqrt{x^2-24x}+x-24 = 60.$$

Transposing and uniting,

$$\sqrt{x^2-24x} = 84-x.$$

Squaring, $x^2-24x = 7{,}056-168x+x^2$,

or $144x = 7{,}056$;

whence, $x = 49$. Ans.

Example.—Solve the equation $\frac{a}{b} = \frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}$.

Solution.—Clearing of fractions by multiplying a by $\sqrt{x}-\sqrt{b}$ and $\sqrt{x}+\sqrt{b}$ by b,

$$a\sqrt{x}-a\sqrt{b} = b\sqrt{x}+b\sqrt{b}.$$

Transposing, $-a\sqrt{b}$ and $b\sqrt{x}$,

$$a\sqrt{x}-b\sqrt{x} = a\sqrt{b}+b\sqrt{b}.$$

Factoring, $(a-b)\sqrt{x} = (a+b)\sqrt{b}$;

whence, $\sqrt{x} = \frac{(a+b)\sqrt{b}}{a-b}$.

Squaring, $x = \frac{(a+b)^2b}{(a-b)^2} = \left(\frac{a+b}{a-b}\right)^2b$. Ans.

Example.—Solve the equation $ax^{-\frac{1}{3}}+d = c$.

Solution.—Transposing, $ax^{-\frac{1}{3}} = c-d$,

or $\frac{a}{x^{\frac{1}{3}}} = c-d$. (Art. **530.**)

Clearing of fractions, $a = x^{\frac{1}{3}}(c-d)$,

or $x^{\frac{1}{3}}(c-d) = a$.

Dividing by $c-d$, $x^{\frac{1}{3}} = \frac{a}{c-d}$.

Cubing, $x = \left(\frac{a}{c-d}\right)^3$. Ans.

EXAMPLES FOR PRACTICE.

583. Solve the following equations:

1. $\sqrt{x+1} - 2 = 3$. Ans. $x = 24$.
2. $\sqrt{x-32} = 16 - \sqrt{x}$. Ans. $x = 81$.
3. $x^{-\frac{1}{2}} + 2x^{-\frac{1}{2}} + 3x^{-\frac{1}{2}} = 2$. Ans. $x = 9$.
4. $\sqrt{x} + \sqrt{a+x} = \frac{2a}{\sqrt{a+x}}$. Ans. $x = \frac{1}{3}a$.
5. $2(\sqrt{x} - \sqrt{x-3}) = \frac{4}{\sqrt{x}}$. Ans. $x = 4$.
6. $\frac{1}{(10x+35)^{-\frac{1}{3}}} - 3 = 2$. Ans. $x = 9$.

PROBLEMS LEADING TO SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

584. There are two steps in the solution of problems by Algebra:

First.—The relations which exist between the known and the unknown quantities—that is, between those whose values are given in the problem and those whose values are required—must be stated by one or more equations. This is called the **statement** of the problem.

Second.—The resulting equation or equations must be solved, giving the values of the required quantities.

585. It will thus be seen that by the algebraic method, the answer to a problem is used in the solution and operated upon as though it were a known quantity, which is one great advantage over the arithmetical method.

The ability to state a problem by means of an equation depends upon the ingenuity of the operator and his ability to reason, rather than upon his knowledge of Algebra. No definite rule can be given for making the statement, but in general, where there is only one unknown quantity in a problem:

586. *Decide what quantity it is whose value is to be found. This will be the unknown quantity, or the answer. Then represent the unknown quantity by x and form an equation that will indicate the relations between the known and the unknown quantities as stated in the problem.*

NOTE.—The equation will also indicate the operations that would be performed in proving the statement made in the problem, were the answer known. Hence, the equation may often be formed by noticing what operations would be performed upon the answer in proving.

EXAMPLE.—Find such a number that, when 14 is added to its double, the sum shall be 30.

SOLUTION.—The quantity whose value is required is the number itself. As this is the unknown quantity, let x = the number, whence $2x$ must be double the number. Now the problem states that when 14 is added to double the number the sum will be 30. In other words, when 14 is added to $2x$ the sum will be 30. Hence, the statement of the problem in the form of an equation is,

$$2x + 14 = 30; \text{ whence, solving,}$$
$$x = 8. \quad \text{Ans.}$$

EXAMPLE.—Find a number which, when multiplied by 4, will exceed 40 by as much as it is now below 40.

SOLUTION.—Let x = the required number, which, when multiplied by 4, becomes $4x$. According to the conditions of the problem, the amount by which 4 times the required number, or $4x$, exceeds 40 is equal to the amount that the number itself, or x, is below 40.

But $4x - 40$ is the amount by which $4x$ exceeds 40, and $40 - x$ is the amount by which x is below 40.

Hence, by the conditions, we have the statement

$$4x - 40 = 40 - x.$$

Transposing and uniting, $5x = 80$,

or $x = 16$. Ans.

EXAMPLE.—Two loads of brick together weigh 4,000 lb.; but if 500 lb. be transferred from the smaller to the larger load, the latter will weigh 7 times as much as the former. How much does each load weigh?

SOLUTION.—If the weights of the two loads were known and it was desired to prove the correctness of the example, we should add 500 lb. to the weight of the larger load and subtract 500 lb. from the weight of the smaller load, as stated in the example. The larger load should then weigh 7 times as much as the smaller. To obtain the equation the same thing is done by letting x = the weight of one load, whence $4,000 - x$ equals the weight of the other load.

Let x = the weight of the smaller load.

Then $4,000 - x$ = the weight of the larger load.

Also, $x - 500$ = the weight of the smaller load after transferring 500 lb.

And $4,000 - x + 500$ = the weight of the larger load after transferring 500 lb.

By the conditions, the larger load now weighs 7 times as much as the smaller.

Hence, $7(x - 500) = 4,000 - x + 500.$

Solving, $7x - 3,500 = 4,500 - x,$

or $8x = 8,000;$

$$\left.\begin{array}{ll}\text{whence,} & x = 1,000 \text{ lb.} = \text{weight of smaller load,} \\ \text{and} & 4,000 - x = 3,000 \text{ lb.} = \text{weight of larger load.}\end{array}\right\} \text{Ans.}$$

PROOF.—$1,000 - 500 = 500$ = weight of the smaller load, and $3,000 + 500 = 3,500$ = weight of the larger load after the 500 pounds have been transferred; $3,500 \div 500 = 7$.

Until the student has attained considerable proficiency in solving problems of this kind, it is a good plan to prove all problems.

EXAMPLE.—The circumference of the fore wheel of a carriage is 10 feet, and of the hind wheel 12 feet. What distance has the carriage traveled, when the fore wheel has made 8 more turns than the hind wheel?

SOLUTION.—In this example the distance traveled is not known, but is required to be found. Suppose that the distance is known, and that it equals x feet, and that we wish to see whether the statement is true that the fore wheel makes 8 more revolutions than the hind wheel in passing over x feet. The number of revolutions of the fore wheel is evidently $\frac{x}{10}$, and of the hind wheel, $\frac{x}{12}$. The example states that the difference between them equals 8.

Hence, $$\frac{x}{10} - \frac{x}{12} = 8. \qquad (1)$$

Solving for x, $12x - 10x = 960,$

or $2x = 960,$ and

$x = 480$ ft. Ans.

PROOF.— $\frac{480}{10} = 48$ = revolutions of fore wheel.

$\frac{480}{12} = 40$ = revolutions of hind wheel.

$48 - 40 = 8$. Compare this proof with (1).

EXAMPLE.—A water cistern connected with three pipes can be filled by one of them in 80 minutes, by another in 200 minutes, and by the third in 300 minutes. In what time will the cistern be filled when all three pipes are open at once?

SOLUTION.—Here the unknown quantity is the number of minutes required to fill the cistern by all three pipes together. Supposing this to be x minutes, the example may be proved by noticing that the sum of the quantities of water flowing through each pipe separately in a given length of time, as 1 minute, must be equal to the quantity flowing through all three together in the same length of time. According to the problem, the quantity discharged by the first pipe in one minute would be $\frac{1}{80}$, by the second $\frac{1}{200}$, and by the third $\frac{1}{300}$ of the contents of the cistern. In like manner the quantity discharged by all three at once in one minute would be $\frac{1}{x}$. Then, if the example is stated correctly, we must have $\frac{1}{80} + \frac{1}{200} + \frac{1}{300} = \frac{1}{x}$.

Clearing of fractions,

$$x(30 + 12 + 8) = 2{,}400,$$

$$\text{or} \qquad 50x = 2{,}400;$$

$$\text{whence,} \qquad x = 48 \text{ minutes.} \quad \text{Ans.}$$

EXAMPLE.—A man rows a boat a certain distance *with* the tide, at the rate of $6\frac{2}{3}$ miles an hour, and returns at the rate of $3\frac{1}{3}$ miles an hour, *against* a tide half as strong. If the man is pulling at a uniform rate, what is the velocity of the stronger tide?

SOLUTION.—If the following statement is not clear, the student should reason it out for himself in a manner similar to that used in the last three examples.

Let x = number of miles per hour that the stronger tide is running;

then, $\frac{x}{2}$ = number of miles per hour that the weaker tide is running.

Hence, $6\frac{2}{3} - x$ and $3\frac{1}{3} + \frac{x}{2}$ are expressions for the rate at which the man is pulling. But, as he is pulling at a constant rate all of the time, these expressions must be equal. Hence,

$$6\tfrac{2}{3} - x = 3\tfrac{1}{3} + \frac{x}{2},$$

$$\text{or} \qquad \frac{20}{3} - x = \frac{10}{3} + \frac{x}{2}.$$

$$\text{Clearing of fractions,} \qquad 40 - 6x = 20 + 3x,$$

$$\text{or} \qquad -9x = -20;$$

$$\text{whence,} \qquad x = 2\tfrac{2}{9} \text{ miles per hour.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

587. Solve the following examples:

1. The greater of two numbers is four times the lesser number, and their sum is 400; what are the numbers? Ans. 80 and 320.

2. A farmer has 108 animals, consisting of horses, sheep and cows. He has four times as many cows as horses, lacking 8, and five times as many sheep as horses, lacking 4; how many has he of each kind?

$$\text{Ans.} \begin{cases} 12 \text{ horses.} \\ 40 \text{ cows.} \\ 56 \text{ sheep.} \end{cases}$$

3. A can do a piece of work in 8 days, and B can do it in 10 days; in what time can they do it working together? Ans. $4\frac{4}{9}$ days.

4. Find five consecutive numbers whose sum is 150.
Ans. $28 + 29 + 30 + 31 + 32$.

5. A boat whose rate of sailing is 6 miles per hour in still water, moves down a stream which flows at the rate of 3 miles per hour, and returns, making the round trip in 8 hours; how far did it go down the stream? Ans. 18 mi.

QUADRATIC EQUATIONS.

588. A **quadratic equation** is one in which the *square* is the highest power of the unknown quantity, when simplified as stated in Art. **569.** It is also called an equation of the **second degree.**

589. A **pure quadratic equation** is one which contains the square only of the unknown quantity, as $x^2 + 2ab = 10$.

590. An **affected quadratic equation** is one containing both the square and the first power of the unknown quantity, as $x^2 + 2x = 6$.

591. By the processes used to reduce simple equations, any pure quadratic equation may be reduced to an equation having the square of the unknown quantity alone in the first member, and some known quantity in the second member, as in $x^2 = a$, where x^2 is the square of the unknown quantity and a is a known quantity. The value of the unknown quantity may then be found by extracting the square root of both members, which, by Art. **559,** V, will not destroy the equality of the equation. By referring to Art. **519,** it will be seen that after extracting the square root, each member should be written with the ± sign. Thus, extracting the square root of both members of $x^2 = a$, we have $\pm x = \pm \sqrt{a}$. This may be taken in four ways, namely, that

$$+x = +\sqrt{a}.$$
$$+x = -\sqrt{a}.$$
$$-x = -\sqrt{a}.$$
$$-x = +\sqrt{a}.$$

But by Art. **563,** the signs of both members of the last two equations may be changed, making $+x = +\sqrt{a}$ and

$+x = -\sqrt{a}$, the same as in the first two equations. Hence, the equation $x^2 = a$ has the two values,

$$x = +\sqrt{a} \text{ and}$$
$$x = -\sqrt{a},$$

and these may be expressed by writing x in the first member without any sign (plus, understood), and writing the square root of a in the second member with the ± sign, thus, $x = \pm\sqrt{a}.$

592. From the foregoing, we have the following rule for solving a pure quadratic equation:

Rule.—*Reduce the given equation to the form of* $x^2 = a$ (*Art.* ***591***), *and extract the square root of both members, writing the ± sign before the square root of the second member.*

Note.—The **root** of an equation is the value of the unknown quantity. From this it will be seen that a simple equation has one root, and a quadratic equation has two roots. In general, any equation has as many roots as there are units in the exponent of the unknown quantity.

Example.—Solve the equation $\frac{x^2}{16} - \frac{x^2-3}{5} = \frac{1}{20}.$

Solution.—Clearing of fractions by multiplying each term by 80,

$$5x^2 - 16(x^2-3) = 4.$$

Transposing and uniting, $-11x^2 = -44,$

or $x^2 = 4.$

Extracting the square root of both members,

$$x = \pm 2. \quad \text{Ans.}$$

Example.—Solve the equation

$$\frac{\sqrt{x-2}}{\sqrt{x+2}} + \frac{\sqrt{x+2}}{\sqrt{x-2}} = 4.$$

Solution.—Clearing of fractions by multiplying each term by $\sqrt{x+2} \times \sqrt{x-2}$,

$$x - 2 + x + 2 = 4\sqrt{x+2} \times \sqrt{x-2};$$

or, $2x = 4\sqrt{x^2-4}.$ (Art. **547.**)

Dividing by 2, $x = 2\sqrt{x^2-4}.$

Squaring, $x^2 = 4(x^2-4),$

or $x^2 = 4x^2 - 16;$

whence, $-3x^2 = -16,$

and $x^2 = \frac{16}{3}.$

Extracting the square root of both members,

$$x = \pm 4\sqrt{\tfrac{1}{3}}. \quad \text{Ans.}$$

By Art. **540,** the answer may also be written $\pm\frac{4}{3}\sqrt{3}$, a better form.

EXAMPLES FOR PRACTICE.

593. Solve the following equations:

1. $3x^2 - 57 - 4x^2 = -8x^2 + 6.$ Ans. $x = \pm 3.$

2. $\frac{1}{2x^2} + 7 = \frac{9}{4x^2}.$ Ans. $x = \pm \frac{1}{2}.$

3. $35 - \frac{x^2 + 50}{5} = x^2 - \frac{x^2 - 10}{3}.$ Ans. $x = \pm 5.$

4. $x\sqrt{6 + x^2} = 1 + x^2.$ Ans. $x = \pm \frac{1}{2}.$

5. $\frac{\sqrt{x+1}}{\sqrt{x-1}} + \frac{\sqrt{x-1}}{\sqrt{x+1}} = 3.$ Ans. $x = \pm 3\sqrt{\frac{1}{5}} = \pm \frac{3}{5}\sqrt{5}.$

6. $x\sqrt{a + x^2} = b + x^2.$

Note.—After squaring, transpose all terms containing x^2 into the first member and factor.

Ans. $x = \pm b\sqrt{\frac{1}{a - 2b}} = \pm \frac{b}{a - 2b}\sqrt{a - 2b}.$

AFFECTED QUADRATIC EQUATIONS.

594. The solution of affected quadratic equations depends upon the form of the trinomial obtained by squaring a binomial, as $x + a$ or $x - a$, where a and x represent known and unknown quantities, respectively. By Arts. **428** and **429**, $(x \pm a)^2 = x^2 \pm 2ax + a^2$. It is to be observed that the first term of this trinomial is the *square* of the *unknown quantity*, and that it is *positive;* the second term contains the *first power* of the *unknown quantity*, and the third term is a *known quantity*, which is a perfect square and positive.

595. Every affected quadratic equation can be reduced so that the *second member* will become a *known quantity*, and the *first member will consist of only two terms, corresponding to* x^2 *and* $\pm 2ax$, *as above.* When reduced to this form, it may be solved by adding a known quantity to both members, corresponding to the a^2 above, which will make a perfect square of the first member. Then, by extracting the square root of both members of the equation, the first member will be in the form of $x \pm a$, containing only the first power of the unknown quantity.

596. Inspection of the trinomial $x^2 \pm 2ax + a^2$ shows that the third term, or the quantity to be added to both members, is equal to the square of half the coefficient of x. Adding this quantity to both members is called **completing the square,** meaning that it makes a perfect square of the first member.

597. Hence, to solve an affected quadratic equation:

Rule.—*Reduce the equation to the form of* $x^2 \pm ax = b$*; that is, so that the second member consists of a known quantity, while the first member has but two terms, the first being the square of the unknown quantity and positive, its coefficient being 1, and the second containing the first power of the unknown quantity with any coefficient.*

Complete the square by adding the square of half the coefficient of x *to both members.*

Extract the square root of both members and solve for x.

Example 1.—Solve the equation, $4x^2 - 16x - 128 = 0$.

Solution.—Transposing the -128,

$$4x^2 - 16x = 128. \quad (1)$$

Dividing by 4, $x^2 - 4x = 32$.

The equation now fulfils the conditions of the rule; namely, the second member is a known quantity, 32, while the first member has but two terms, the first being x^2 alone, and positive, and the second containing the first power of x.

The coefficient of x divided by 2 is -2; the square of this is 4, the quantity to be added to both members. Hence, completing the square,

$$x^2 - 4x + 4 = 32 + 4 = 36.$$

Extracting the square root of both members,

$$x - 2 = \pm 6.$$

Transposing -2, $x = 2 + 6$, or $2 - 6$;

whence, $x = 8$, or -4. Ans.

Proof.—Substituting 8 for x in equation (1),

$$4(8)^2 - 16 \times 8 = 4 \times 64 - 128 = 128.$$

Substituting -4 for x, $4(-4)^2 - 16(-4) = 64 + 64 = 128$.

Example 2.—Solve the equation $-x^2 + 13x = -14$.

Solution.—Changing all the signs to make the first term positive,

$$x^2 - 13x = 14. \quad (1)$$

To complete the square, $(-\frac{13}{2})^2$ or $\frac{169}{4}$ must be added to both members. To do this, change 14 into fourths, giving $\frac{56}{4}$.

Completing the square,

$$x^2 - 13x + \frac{169}{4} = 14 + \frac{169}{4} = \frac{56}{4} + \frac{169}{4} = \frac{225}{4}.$$

Extracting the square root, $x - \frac{13}{2} = \pm \frac{15}{2}$;

whence, $x = \frac{28}{2}$ or $-\frac{2}{2} = 14$ or -1. Ans.

NOTE.—It is better, as a rule, to use common fractions than to use decimals, when solving affected quadratic equations. Until the student has attained considerable proficiency in solving these equations, he should prove the correctness of his result, as was done in example 1.

EXAMPLE 3.—Solve the equation, $-3x^2 - 7x = \frac{10}{3}$.

SOLUTION.—Dividing both members by -3 to make x^2 stand alone, and positive, $x^2 + \frac{7}{3}x = -\frac{10}{9}$.

Completing the square by adding $(\frac{7}{6})^2 = \frac{49}{36}$,

$$x^2 + \tfrac{7}{3}x + \tfrac{49}{36} = -\tfrac{10}{9} + \tfrac{49}{36} = \tfrac{9}{36}.$$

Extracting the square root, $x + \frac{7}{6} = \pm \frac{3}{6}$;

whence, $x = -\frac{2}{3}$ or $-\frac{5}{3}$. Ans.

EXAMPLE 4.—Solve the equation $x - \dfrac{x^3 - 8}{x^2 + 5} = 2$.

SOLUTION.—Clearing of fractions,

$$x^3 + 5x - x^3 + 8 = 2x^2 + 10.$$

Transposing and uniting terms,

$$-2x^2 + 5x = 2.$$

Dividing by -2, $x^2 - \frac{5}{2}x = -1$.

Completing the square, $x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{9}{16}$.

Extracting the square root,

$$x - \tfrac{5}{4} = \pm \tfrac{3}{4};$$

whence, $x = 2$ or $\frac{1}{2}$. Ans.

EXAMPLE 5.—Solve the literal equation, $acx^2 - bcx + adx = bd$.

SOLUTION.—Reducing the equation so that the first member will contain two terms, one with x^2 and one with x,

$$acx^2 - (bc - ad)x = bd.$$

Here, $bc - ad$ is the coefficient of x. Dividing by ac to make the first term a perfect square,

$$x^2 - \frac{bc - ad}{ac}x = \frac{bd}{ac}.$$

Completing the square,

$$x^2 - \frac{bc - ad}{ac}x + \frac{(bc - ad)^2}{4a^2c^2} = \frac{bd}{ac} + \frac{b^2c^2 - 2abcd + a^2d^2}{4a^2c^2} = \frac{b^2c^2 + 2abcd + a^2d^2}{4a^2c^2}.$$

Extracting the square root,

$$x - \frac{bc - ad}{2ac} = \pm \frac{bc + ad}{2ac}.$$

Transposing and uniting,

$$x = \frac{bc - ad}{2ac} \pm \frac{bc + ad}{2ac};$$

whence, $x = \dfrac{b}{a}$ or $-\dfrac{d}{c}$. Ans.

598. An affected quadratic equation always contains three terms: the first, containing the square of the unknown quantity; the second, the first power of the unknown quantity, and the third, containing no power of the unknown quantity. Every such equation may be solved by means of the rule given in Art. **597.**

EXAMPLE.—Solve the equation $(2m + n)x^2 - mx = 4m^2n^2$.

SOLUTION.—According to the rule, the coefficient of x^2 must be 1; hence, dividing both sides of the equation by the coefficient of x^2 (which will not alter the equality), the equation becomes $x^2 - \frac{m}{2m + n}x = \frac{4m^2n^2}{2m + n}$. According to the rule, the square of one-half of the coefficient of x (the first power of the unknown quantity) must be added to both members of the equation. In this example, the coefficient of x is $\frac{m}{2m + n}$, and the square of one-half of it is $\left[\frac{m}{2(2m + n)}\right]^2$. Adding this to both members, the equation becomes

$$x^2 - \frac{m}{2m + n}x + \left[\frac{m}{2(2m + n)}\right]^2 = \frac{4m^2n^2}{2m + n} + \left[\frac{m}{2(2m + n)}\right]^2.$$

Simplify the right-hand member, thus:

$$\left[\frac{m}{2(2m + n)}\right]^2 = \frac{m^2}{4(4m^2 + 4mn + n^2)} = \frac{m^2}{16m^2 + 16mn + 4n^2}.$$

Therefore,

$$\frac{4m^2n^2}{2m + n} + \frac{m^2}{16m^2 + 16mn + 4n^2} = \frac{4m^2n^2(8m + 4n) + m^2}{16m^2 + 16mn + 4n^2} =$$

$$\frac{32m^3n^2 + 16m^2n^3 + m^2}{16m^2 + 16mn + 4n^2} = \frac{m^2(32mn^2 + 16n^3 + 1)}{4(2m + n)^2}.$$

Extracting the square root of both sides of the completed square, which is now

$$x^2 - \frac{m}{2m + n}x + \left[\frac{m}{2(2m + n)}\right]^2 = \frac{m^2(32mn^2 + 16n^3 + 1)}{4(2m + n)^2},$$

we have $x - \frac{m}{2(2m + n)} = \pm \frac{m}{2(2m + n)}\sqrt{32mn^2 + 16n^3 + 1}$,

or $x = \frac{m}{2(2m + n)} \pm \frac{m}{2(2m + n)}\sqrt{32mn^2 + 16n^3 + 1} =$

$\frac{m}{2(2m + n)}(1 \pm \sqrt{32mn^2 + 16n^3 + 1})$. Ans.

This example is quite as complicated as any the student will probably ever be called upon to solve. It is by no means difficult, if each step is followed by reference to the rule. It should be noted that in all cases the coefficient of the first power of the unknown quantity (i. e., the quantity whose value it is desired to find) consists of all the numbers and letters which are multiplied into the quantity. Thus, in $3a^2bc^3x$ the coefficient of x is $3a^2bc^3$, and one-half of it is $\frac{3a^2bc^3}{2} = \frac{3}{2}a^2bc^3$.

Example.—Solve for x in the equation $80 - 3x^2 - 2x = -5$.

Solution.—Transposing the known term in the left-hand member, $-3x^2 - 2x = -85$. Dividing by the coefficient of x (which is -3 in this case), the equation becomes $x^2 + \frac{2}{3}x = \frac{85}{3}$. One-half of the coefficient of x is $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$, and the square of it is $(\frac{1}{3})^2 = \frac{1}{9}$. Adding $\frac{1}{9}$ to both members, $x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{85}{3} + \frac{1}{9}$. But, $\frac{85}{3} + \frac{1}{9} =$ (reducing to a common denominator) $\frac{255 + 1}{9} = \frac{256}{9}$.

Hence, $x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{256}{9}$. Extracting the square root of both members $\sqrt{x^2 + \frac{2}{3}x + \frac{1}{9}} = \sqrt{\frac{256}{9}}$, or $x + \frac{1}{3} = \pm \frac{16}{3}$; whence, $x = -\frac{1}{3} \pm \frac{16}{3} = 5$, or $-\frac{17}{3}$. Ans.

Example.—Find the value of x in the equation $\frac{x}{x + a} = \frac{b}{x - b}$.

Solution.—Clearing of fractions, $x(x - b) = b(x + a)$ or $x^2 - bx = bx + ab$. The term bx in the right-hand member must be transposed to the other side, so that only the known term shall be on the right. The equation then becomes $x^2 - bx - bx = ab$, or $x^2 - 2bx = ab$. Here the coefficient of x is $2b$ and the square of one-half of it is $\left(\frac{2b}{2}\right)^2 = b^2$. Adding b^2 to both sides, $x^2 - 2bx + b^2 = ab + b^2 = (a + b)b$. Extracting the square root of both members, $x - b = \pm \sqrt{(a + b)b}$, or $x = b \pm \sqrt{(a + b)b}$. Ans.

When extracting the square root of the left-hand member, after the square has been completed, it is easier and quicker to apply the rule given in Art. **457**, than the rule for square root given in Art. **523**.

EXAMPLES FOR PRACTICE.

599. Solve the following equations:

1. $x^2 + 2x = 35$. Ans. $x = 5$ or -7.
2. $9x^2 + 6x = 15$. Ans. $x = 1$ or $-1\frac{2}{3}$.
3. $5x^2 - 24x = 5$. Ans. $x = 5$ or $-\frac{1}{5}$.
4. $x + \frac{24}{x - 1} = 3x - 4$. Ans. $x = 5$ or -2.
5. $-5x^2 + 9x = 2\frac{1}{4}$. Ans. $x = \frac{3}{10}$ or $\frac{3}{2}$.
6. $\frac{x}{x + 1} + \frac{x + 1}{x} = \frac{13}{6}$. Ans. $x = 2$ or -3.

7. $\frac{9x}{12x+6b} = \frac{3b}{4x-2b}$. Ans. $x = \frac{b}{4}(3 \pm \sqrt{17})$.

8. $\frac{2(ax-x^2)}{3a-2x} = \frac{a}{4}$. Ans. $x = \frac{3a}{4}$ or $\frac{a}{2}$.

EQUATIONS IN THE QUADRATIC FORM.

600. An equation is in the *quadratic form* when it contains only two powers of the unknown quantity, *and the exponent of one power is twice as great as the exponent of the other.* Such equations are solved by the rules for quadratics.

Example.—Solve the equation $x^4 + 4x^2 = 12$.

Solution.—Completing the square,

$$x^4 + 4x^2 + 4 = 16.$$

Extracting the square root, $x^2 + 2 = \pm 4$,

or $x^2 = 2$ or -6.

Extracting the square root again,

$$x = \pm\sqrt{2} \text{ or } \pm\sqrt{-6}. \quad \text{Ans.}$$

Example.—Solve the equation $x^6 + 20x^3 - 10 = 59$.

Solution.—Transposing the -10,

$$x^6 + 20x^3 = 69.$$

Completing the square, $x^6 + 20x^3 + 100 = 169$.

Extracting the square root, $x^3 + 10 = \pm 13$,

or $x^3 = 3$ or -23.

Extracting the cube root, $x = \sqrt[3]{3}$ or $-\sqrt[3]{23}$. Ans.

Example.—Solve the equation $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$.

Solution.—Completing the square,

$$x^{\frac{6}{5}} + x^{\frac{3}{5}} + \tfrac{1}{4} = \frac{3{,}025}{4}.$$

$$x^{\frac{3}{5}} + \tfrac{1}{2} = \pm\frac{55}{2}.$$

$$x^{\frac{3}{5}} = 27 \text{ or } -28.$$

Now, to obtain a value for x, we must extract the cube root of both members, and then raise both members to the 5th power. This will clear x of its fractional exponent.

Extracting the cube root, $x^{\frac{1}{5}} = 3$ or $-\sqrt[3]{28}$.

Raising to the 5th power, $x = 243$ or $-\sqrt[3]{28^5}$. Ans.

EXAMPLES FOR PRACTICE.

601. Solve the following equations:

1. $x^4 + 4x^2 = 117$. Ans. $x = \pm 3$ or $\pm\sqrt{-13}$.
2. $x^6 - 2x^3 = 48$ Ans. $x = 2$ or $-\sqrt[3]{6}$.

3. $x^6 - 8x^3 = 513$. Ans. $x = 3$ or $-\sqrt[3]{19}$.

4. $x^3 - x^{\frac{3}{2}} = 56$. Ans. $x = 4$ or $(-7)^{\frac{2}{3}}$.

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

602. In quadratics, where two answers are obtained by solving equations, it is usually the case that only one answer, the positive value, is required. In some instances, however, the negative value is the one sought. In works treating on higher mathematics, the negative value is used as frequently as the positive value.

EXAMPLE.—There are two numbers whose sum is 40, and the sum of their squares is 818. What are the numbers?

SOLUTION.—Let $x =$ one number, and $40 - x =$ the other number.

Then, by the conditions, $x^2 + (40 - x)^2 = 818$;

whence, $x^2 + 1{,}600 - 80x + x^2 = 818$.

Combining, $2x^2 - 80x = -782$,

or $x^2 - 40x = -391$.

Completing the square, $x^2 - 40x + 400 = 9$.

Extracting the square root, $x - 20 = \pm 3$;

$$\left.\begin{aligned} \text{whence,}\quad x &= 23 \text{ or } 17, \\ \text{and}\quad 40 - x &= 17 \text{ or } 23. \end{aligned}\right\} \text{Ans.}$$

Both answers fulfil the conditions.

EXAMPLE.—An iron bar weighs 36 pounds. If it had been 1 foot longer, each foot would have weighed ½ a pound less. Find the length of the bar.

SOLUTION.—Let $x =$ the length of the bar in feet.

Then, $\frac{36}{x} =$ the weight per foot, and

$\frac{36}{x+1} =$ the weight per foot if the bar were 1 foot longer.

By the conditions, $\frac{36}{x} - \frac{36}{x+1} = \frac{1}{2}$.

Clearing of fractions, $72x + 72 - 72x = x^2 + x$,

or $x^2 + x = 72$.

Completing the square, $x^2 + x + \frac{1}{4} = \frac{289}{4}$.

Extracting the square root, $x + \frac{1}{2} = \pm\frac{17}{2}$;

whence, $x = 8$ ft. or -9 ft. Ans.

PROOF.— $\frac{36}{8} = 4\frac{1}{2}$; $\frac{36}{8+1} = 4$; $4\frac{1}{2} - 4 = \frac{1}{2}$.

Or, $\frac{36}{-9} = -4$; $\frac{36}{-9+1} = -4\frac{1}{2}$; $-4 - (-4\frac{1}{2}) = \frac{1}{2}$.

Only the positive value is required, although both values will satisfy the equation.

Example.—A number of men ordered a yacht to be built for $6,300. Each man was to pay the same amount, but two of them withdrew, making it necessary for those remaining to advance $200 more than they otherwise would have done. How many men were there at first?

Solution.—Let $x =$ the number of men at first.

Then, $\frac{6,300}{x} =$ what each was to have paid, and

$\frac{6,300}{x-2} =$ what each finally paid.

By the conditions, $\frac{6,300}{x-2} - \frac{6,300}{x} = 200.$

Clearing of fractions and combining,

$$200x^2 - 400x = 12,600,$$

$$\text{or} \quad x^2 - 2x = 63.$$

Completing the square and solving,

$$x^2 - 2x + 1 = 64.$$

$$x - 1 = \pm 8.$$

$$x = 9 \text{ or } -7. \quad \text{Ans.}$$

Proof.— $\frac{6,300}{9} = 700;\ \frac{6,300}{9-2} = 900;\ 900 - 700 = 200.$

Or, $\frac{6,300}{-7} = -900;\ \frac{6,300}{-7-2} = -700;\ -700-(-900) = 200.$

Only the positive value can be used.

Example.—A and B start at the same time to travel 150 miles. A travels 3 miles an hour faster than B, and finishes his journey $8\frac{1}{3}$ hours before him. How many miles did each travel per hour?

Solution.—Let $x =$ number of miles A traveled per hour, and
$x - 3 =$ number of miles B traveled per hour.

Then, $\frac{150}{x} =$ the time in which A performs the journey, and

$\frac{150}{x-3} =$ the time in which B performs the journey.

By the conditions, $\frac{150}{x-3} - \frac{150}{x} = 8\frac{1}{3}.$

Clearing of fractions and combining,

$$25x^2 - 75x = 1,350,$$

$$\text{or} \quad x^2 - 3x = 54.$$

Completing the square and solving,

$$x^2 - 3x + \tfrac{9}{4} = \tfrac{225}{4}.$$

$$x - \tfrac{3}{2} = \pm \tfrac{15}{2};$$

$$\text{whence,} \quad x = 9 \text{ or } -6,$$

$$\text{and} \quad x - 3 = 6 \text{ or } -9.$$

Using the positive values, A traveled 9 miles per hour and B traveled 6 miles per hour. Ans.

603. As an illustration of the use of the negative values, consider the following explanation, which refers to the preceding example: In Fig. 1 let C be the starting point. Call any advance in the direction of the upper arrow, or from C towards D, positive, and in the opposite direction, negative. Let E and D be each 150 miles from C. Suppose that a train of cars 150 miles long has one end at C and the other end at D, and that the train is moving in the direction from C to E at the rate of 15 miles per hour. Now, if A and B start towards D, running on the train at the rate of 9 and 6 miles per hour, respectively, while the train moves 15 miles per hour towards E, the rate of travel of A towards D is $9 - 15 = -6$ miles per hour, and of B, $6 - 15 = -9$ miles per hour. It is now evident that A is traveling towards D 3 miles per hour faster than B. When A has traveled 150 miles—in other words, when he has reached the end of the train—B has reached the point E; he has traveled negatively farther than A, but if he travels to the end of the train it will take him $8\frac{1}{3}$ hours longer than it did A.

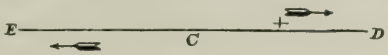

FIG. 1.

The preceding paragraph is also an illustration of the statement in Art. **380,** that of two negative values, the one which has the less value numerically is the greater.

EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

604. In the third problem in Art. **586** it was shown how, under certain conditions, more than one unknown quantity *in an example* may be represented *in an equation*, by expressing the value of each quantity in terms of x, thus producing only the *one unknown quantity x in the equation.*

605. Sometimes, however, each quantity is represented by a different letter, as x, y or z, in which case it is necessary to have as many equations as there are unknown quantities, in order to effect a solution. For example, if it were required to find the value of x in the equation $x + y = 10$, x and y being unknown quantities, we should have $x = 10 - y$, x being still undetermined because its value is in terms of the unknown quantity y. There must be another equation, therefore, expressing some other relation between the unknown quantities x and y, in order to fix their values. The equations which fix the values of the unknown quantities must be *independent and simultaneous*.

606. Independent equations are those which express different relations between the unknown quantities. Thus, $x + y = 4$ and $xy = 6$ express different relations between x and y, and are independent. But $x + y = 4$ and $3x + 3y = 12$ are not independent, because, by dividing both members of the second equation by 3, it reduces to the first equation, and thus expresses the *same* relations between the unknown quantities.

607. Simultaneous equations are such as will be satisfied (Art. **572**) by substituting the same values for the same unknown quantities in each equation.

608. Equations containing more than one unknown quantity are solved by so combining them as to obtain a single equation containing but one unknown quantity. This process is called **elimination.** In what follows, equations containing two unknown quantities will be considered.

609. To Eliminate by Substitution:

Rule.—*From one equation, find the value of one of the unknown quantities in terms of the other. Substitute this value for the same unknown quantity in the other equation.*

EXAMPLE.—Solve the equations:

$$2x + 3y = 18. \qquad (1)$$
$$3x - 2y = 1. \qquad (2)$$

SOLUTION.—It will be more convenient to first find the value of x in (2), since, after transposing $-2y$ to the second member, it will become positive.

Transposing $-2y$ in (2), $3x = 1 + 2y$.

Dividing both members by 3,

$$x = \frac{1 + 2y}{3}. \qquad (3)$$

This gives the value of x in terms of y.

Substituting this value of x for the x in (1),

$$\frac{2(1 + 2y)}{3} + 3y = 18.$$

Removing the parenthesis,

$$\frac{2 + 4y}{3} + 3y = 18.$$

Clearing of fractions, $2 + 4y + 9y = 54.$

Transposing the 2 and uniting the $4y$ and $9y$,

$$13y = 52;$$

whence, $y = 4.$ Ans.

Now, having the value of y, we may substitute it for y in any of the above equations containing both x and y, and thus obtain a value for x.

Substituting this value in equation (3),

$$x = \frac{1 + 2 \times 4}{3};$$

whence, $x = 3.$ Ans.

610. To Eliminate by Comparison:

Rule.—*From each equation find the value of one of the unknown quantities in terms of the other. Form a new equation by placing these two values equal to each other and solve.*

Elimination by comparison depends upon the principle that quantities which are equal to the same or two equal quantities are equal to each other. Thus, if $y = 2$, and $x = 2$, y is evidently equal to x.

Example.—Solve the same equations as before:

$$2x + 3y = 18. \qquad (1)$$

$$3x - 2y = 1. \qquad (2)$$

Solution.—First obtain the value of x in each equation, it being more convenient to obtain in this case than y.

Transposing $3y$ in (1), $2x = 18 - 3y$,

$$\text{or} \qquad x = \frac{18 - 3y}{2}. \qquad (3)$$

Transposing $-2y$ in (2), $3x = 1 + 2y$,

$$\text{or} \qquad x = \frac{1 + 2y}{3}. \qquad (4)$$

Placing the values of x in (3) and (4) equal to each other,

$$\frac{18 - 3y}{2} = \frac{1 + 2y}{3}.$$

Clearing of fractions, $54 - 9y = 2 + 4y.$

Transposing and uniting terms,

$$-13y = -52;$$

whence, $y = 4$. Ans.

Substituting this value in (4),

$$x = \frac{1+8}{3} = 3. \quad \text{Ans.}$$

611. To Eliminate by Addition or Subtraction:

Rule.—*Select the unknown quantity to be eliminated, and multiply the equations by such numbers as will make the coefficients of this quantity equal in the resulting equations. If the signs of the terms having the same coefficient are alike, subtract one equation from the other; if unlike, add the two equations.*

It is evident that this will not destroy the equality, because adding or subtracting two equations is equivalent to adding the same quantity to, or subtracting it from, both members.

EXAMPLE.—Solve the same equations as before:

$$2x + 3y = 18. \qquad (1)$$

$$3x - 2y = 1. \qquad (2)$$

FIRST SOLUTION.—Since the signs of the terms containing x in each equation are alike, x may be eliminated by subtraction. If the first equation be multiplied by 3 and the second by 2, the coefficients of x in each equation will become equal. Hence,

Multiplying (1) by 3, $6x + 9y = 54$. (3)

Multiplying (2) by 2, $6x - 4y = 2$. (4)

Subtracting (4) from (3), $13y = 52$;

whence, $y = 4$. Ans.

Substituting this value of y for the y in (2),

$$3x - 8 = 1.$$

Transposing, $3x = 9$,

or $x = 3$. Ans.

SECOND SOLUTION.— $2x + 3y = 18$. (1)

$$3x - 2y = 1. \qquad (2)$$

Since the signs of the terms containing y in each equation are unlike, y may be eliminated by addition.

Multiplying (1) by 2, $4x + 6y = 36$. (3)

Multiplying (2) by 3, $9x - 6y = 3$. (4)

Adding (3) and (4), $13x = 39$;

whence, $x = 3$. Ans.

Substituting in (1), $6 + 3y = 18$.

$$3y = 12.$$

$y = 4$. Ans.

MISCELLANEOUS EXAMPLES.

612. From the foregoing it will be seen that any one of the three methods of elimination *can* be applied to the solution of equations. The student must use his judgment as to which is the *best* one to apply in any case.

EXAMPLE.—Solve the equations:

$$\frac{3}{x} + \frac{1}{y} = \frac{5}{4}. \qquad (1)$$

$$\frac{2}{x} - \frac{3}{y} = -1. \qquad (2)$$

SOLUTION.—Multiplying (1) by 3,

$$\frac{9}{x} + \frac{3}{y} = \frac{15}{4}. \qquad (3)$$

Adding (2) and (3),

$$\frac{11}{x} = \frac{15}{4} - 1 = \frac{15}{4} - \frac{4}{4} = \frac{11}{4}.$$

Clearing of fractions, $44 = 11x$,

or $x = 4$. Ans.

Substituting this value of x in (1),

$$\frac{3}{4} + \frac{1}{y} = \frac{5}{4}.$$

Clearing of fractions, $3y + 4 = 5y$,

Transposing, $-2y = -4$,

or $y = 2$. Ans.

EXAMPLE.—Solve the equations:

$$x + 36y = 900. \qquad (1)$$

$$36x + y = 1320. \qquad (2)$$

SOLUTION.—Adding (1) and (2),

$$37x + 37y = 2220. \qquad (3)$$

Dividing by 37, $x + y = 60$. (4)

Subtracting (4) from (1), $35y = 840$.

$y = 24$. Ans.

Substituting this value in (4),

$$x + 24 = 60.$$

$x = 36$. Ans.

EXAMPLE.—Solve the equations:

$$\frac{m}{x} + \frac{n}{y} = a. \qquad (1)$$

$$\frac{n}{x} + \frac{m}{y} = b. \qquad (2)$$

SOLUTION.—Multiplying (1) by m,

$$\frac{m^2}{x} + \frac{mn}{y} = am. \qquad (3)$$

Multiplying (2) by n, $\frac{n^2}{x} + \frac{mn}{y} = bn$. (4)

Subtracting (4) from (3),

$$\frac{m^2 - n^2}{x} = am - bn.$$

Clearing of fractions, $m^2 - n^2 = (am - bn)x$;

whence, $x = \dfrac{m^2 - n^2}{am - bn}$. Ans.

Multiplying (1) by n, $\dfrac{mn}{x} + \dfrac{n^2}{y} = an.$ (5)

Multiplying (2) by m, $\dfrac{mn}{x} + \dfrac{m^2}{y} = bm.$ (6)

Subtracting (6) from (5),

$$\frac{n^2 - m^2}{y} = an - bm.$$

Clearing of fractions, $n^2 - m^2 = (an - bm)y$;

whence, $y = \dfrac{n^2 - m^2}{an - bm}$, or $\dfrac{m^2 - n^2}{bm - an}$. Ans.

EXAMPLES FOR PRACTICE.

613. Solve the following equations:

1. $\left.\begin{array}{l} 3x + 7y = 33. \\ 2x + 4y = 20. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = 4. \\ y = 3. \end{array}\right.$

2. $\left.\begin{array}{l} 8y + 12x = 116. \\ 2x - y = 3. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = 5. \\ y = 7. \end{array}\right.$

3. $\left.\begin{array}{l} ax + by = m. \\ cx + dy = n. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = \dfrac{dm - bn}{ad - bc}. \\ y = \dfrac{an - cm}{ad - bc}. \end{array}\right.$

4. $\left.\begin{array}{l} \dfrac{a}{x} + \dfrac{b}{y} = m. \\ \dfrac{c}{x} + \dfrac{d}{y} = n. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = \dfrac{ad - bc}{dm - bn}. \\ y = \dfrac{bc - ad}{cm - an}. \end{array}\right.$

5. $\left.\begin{array}{l} \dfrac{6}{x} - \dfrac{3}{y} = 4. \\ \dfrac{8}{x} + \dfrac{15}{y} = -1. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = 2. \\ y = -3. \end{array}\right.$

QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

614. The methods of solving will be illustrated by the solution of a few examples:

Case I.—*When elimination may be performed by the methods given for simple equations in Arts.* **609–611.**

Example.—Solve the equations:

$$x^2 + y^2 = 13. \qquad (1)$$
$$x + y = 1. \qquad (2)$$

Solution.—Transposing the x in (2),

$$y = 1 - x. \qquad (3)$$

Substituting the value of y in (1),

$$x^2 + (1 - x)^2 = 13,$$
$$\text{or} \qquad x^2 + 1 - 2x + x^2 = 13.$$

Transposing and uniting terms,

$$2x^2 - 2x = 12,$$
$$\text{or} \qquad x^2 - x = 6.$$

Completing the square, and solving,

$$x^2 - x + \tfrac{1}{4} = \tfrac{25}{4}.$$
$$x - \tfrac{1}{2} = \pm \tfrac{5}{2}.$$
$$x = 3 \text{ or } -2.$$

Now, two values must be found for y which will satisfy the equations when $x = 3$ and $x = -2$.

Substituting these values of x in (3), we have,

$$\left.\begin{array}{l}\text{when } x = 3,\ y = -2; \\ \text{when } x = -2,\ y = 3.\end{array}\right\} \text{Ans.}$$

This is the form in which answers to simultaneous quadratic equations should always be written.

Example.—Solve the equations:

$$4x^2 - 3y^2 = -11. \qquad (1)$$
$$11x^2 + 5y^2 = 301. \qquad (2)$$

Solution.—Multiplying (1) by (5),

$$20x^2 - 15y^2 = -55. \qquad (3)$$

Multiplying (2) by (3),

$$33x^2 + 15y^2 = 903. \qquad (4)$$

Adding (3) and (4), $53x^2 = 848,$

$$\text{or} \qquad x^2 = 16.$$

Extracting the square root, $x = \pm 4.$

Substituting $+4$ for x in (2),

$$11 \times 16 + 5y^2 = 301.$$
$$\text{or} \qquad 5y^2 = 125.$$
$$y^2 = 25.$$
$$y = \pm 5.$$

Substituting -4 for x in (2) will evidently give the same result, since $(-4)^2 = 16$, the same as 4^2. Hence,

$$\left.\begin{array}{l}\text{when } x = 4,\ y = \pm 5; \\ \text{when } x = -4,\ y = \pm 5.\end{array}\right\} \text{Ans.}$$

615. Case II.—*When the equations may be so combined or reduced as to produce an equation having for the first* *member an expression of the form* $x^2 + 2xy + y^2$, *or* $x^2 - 2xy + y^2$.

No rule can be given for solving examples under this case. The student must depend upon his own ingenuity.

EXAMPLE.—Solve the equations:

$$x^2 + y^2 = 25. \qquad (1)$$

$$xy = 12. \qquad (2)$$

SOLUTION.—Multiplying (2) by 2,

$$2xy = 24. \qquad (3)$$

Adding (1) and (3), $x^2 + 2xy + y^2 = 49. \qquad (4)$

Subtracting (3) from (1),

$$x^2 - 2xy + y^2 = 1. \qquad (5)$$

Extracting the square root of (4),

$$x + y = \pm 7. \qquad (6)$$

Extracting the square root of (5),

$$x - y = \pm 1. \qquad (7)$$

This gives two simple equations, from which either x or y may be eliminated by subtraction or addition. Adding (6) and (7), the first member of the new equation will be $2x$, and the second member may have four values as follows:

$$7 + 1,\ 7 - 1,\ -7 + 1 \text{ or } -7 - 1,$$

$$\text{or} \qquad 2x = 8,\ 6,\ -6 \text{ or } -8;$$

$$\text{whence,} \qquad x = 4,\ 3,\ -3 \text{ or } -4.$$

By substituting these values in (2), we have for the corresponding values of y, $y = 3, 4, -4$, or -3.

These values may also be obtained by subtracting (7) from (6). The answers would be written,

$$\left.\begin{array}{l} \text{when} \quad x = 4,\ y = 3;\ x = 3,\ y = 4; \\ x = -3,\ y = -4;\ x = -4,\ y = -3. \end{array}\right\} \text{Ans.}$$

NOTE.—In solving examples under this case, the object is always to produce two equations, one with $x + y$ and one with $x - y$ for the first member, from which the value of x or y can easily be found.

EXAMPLE.—Solve the equations:

$$x^3 + y^3 = 133. \qquad (1)$$

$$x^2 - xy + y^2 = 19. \qquad (2)$$

SOLUTION.—Since both members of (1) may be divided by the same quantity without destroying the equality, they may be divided by equal quantities. Hence, dividing (1) by (2), member by member,

$$x + y = 7. \qquad (3)$$

This gives at once an equation with $x + y$ for the first member. To obtain a value for $x - y$, it will be noticed that the first member of (2) lacks only one $-xy$ of being $x^2 - 2xy + y^2$, from which $x - y$ may be obtained; hence, we should proceed to obtain a value for $-xy$, to add to (2).

Squaring (3), $x^2 + 2xy + y^2 = 49.$ (4)

Writing (2) under (4), $\underline{x^2 \quad - xy + y^2 = 19,}$

and subtracting, $3xy \quad = 30,$

of $xy \quad = 10.$ (5)

Subtracting (5) from (2),

$$x^2 - 2xy + y^2 = 9.$$

Extracting the square root,

$$x - y = \pm 3. \qquad (6)$$

Adding (6) and (3), $2x = 10$ or $4.$

$x = 5$ or $2.$

Subtracting (6) from (3), $2y = 4$ or $10.$

$y = 2$ or $5.$

Or, solving (5) for x, $x = \frac{10}{y}.$

Substituting the value of x in (3),

$$\frac{10}{y} + y = 7.$$

Clearing of fractions and changing signs,

$$y^2 - 7y = -10.$$

Solving for y, $y = 5$ or $2.$

Substituting their values in (3),

$$x = 2 \text{ or } 5.$$

Hence, when $\left.\begin{array}{l} x = 5,\ y = 2; \\ x = 2,\ y = 5. \end{array}\right\}$ Ans.

EXAMPLES FOR PRACTICE.

616. Solve the following equations:

1. $\left.\begin{array}{l} x^2 + y^2 = 29. \\ x + y = 3. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = 5,\ y = -2. \\ x = -2,\ y = 5. \end{array}\right.$

2. $\left.\begin{array}{l} 2x^2 + y^2 = 9. \\ 5x^2 + 6y^2 = 26. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = 2,\ y = \pm 1. \\ x = -2,\ y = \pm 1. \end{array}\right.$

3. $\left.\begin{array}{l} x + y = -1. \\ xy = -56. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = 7,\ y = -8. \\ x = -8,\ y = 7. \end{array}\right.$

PROBLEMS LEADING TO EQUATIONS WITH TWO UNKNOWN QUANTITIES.

617. Most problems requiring the use of quadratics involving two unknown quantities, pertain to mensuration. As this subject is not treated before the paper on geometry is reached, no examples of this character will be given.

EXAMPLE.—A certain fraction becomes equal to $\frac{1}{2}$ if 3 is added to its numerator, and equal to $\frac{2}{7}$ if 3 is added to its denominator. What is the fraction?

SOLUTION.—Let $\frac{x}{y}$ = the required fraction.

By the conditions $\frac{x+3}{y}=\frac{1}{2}$, and

$$\frac{x}{y+3}=\frac{2}{7}.$$

Solving these equations, $x=6$ and $y=18$.

That is, the fraction is $\frac{6}{18}$. Ans.

EXAMPLE.—A crew can row 20 miles in 2 hours down stream, and 12 miles in 3 hours up stream. Required, the rate per hour of the current, and the rate per hour at which the crew would row in still water.

SOLUTION.—Let x = rate per hour of crew in still water, and
y = rate per hour of current.

Then, $x+y$ = rate per hour rowing down stream, and
$x-y$ = rate per hour rowing up stream.

Since they row 20 miles in two hours down stream, in one hour, they would row $\frac{20}{2}=10$ miles, or at the rate of 10 miles per hour. Also, in rowing up stream, they would row at the rate of $\frac{12}{3}=4$ miles per hour.

Consequently, $x+y=10$. (1)

$x-y=4$. (2)

Adding, $2x=14$,

or $x=7$.

Subtracting, $2y=6$,

or $y=3$.

Hence, the rate of the crew is 7 miles per hour, and of the current, 3 miles per hour. Ans.

EXAMPLE.—A wine merchant has two kinds of wine, which cost 72 cents and 40 cents a quart, respectively. How much of each must he take to make a mixture of 50 quarts worth 60 cents a quart?

SOLUTION.—Let x = required number of quarts at 72 cents, and
y = required number of quarts at 40 cents.

Then, $72x$ = cost in cents of the first kind;
$40y$ = cost in cents of the second kind, and
$60\times 50=3{,}000$ = cost in cents of the mixture.

By the conditions, $x+y=50$, and

$72x+40y=3{,}000$.

Solving, $x=31\frac{1}{4}$ qts. and $y=18\frac{3}{4}$ qts. Ans.

LOGARITHMS.

EXPONENTS.

618. By the use of logarithms, the processes of multiplication, division, involution, and evolution, are greatly shortened, and some operations may be performed that would be impossible without them. Ordinary logarithms cannot be applied to addition and subtraction.

619. The **logarithm** of a number is that *exponent* by which some fixed number, called the **base,** must be affected in order to equal the number. Any number may be taken as the base. Suppose we choose 4. Then, the logarithm of 16 is 2, because 2 is the exponent by which 4 (the base) must be affected in order to equal 16, since $4^2 = 16$. In this case, instead of reading 4^2 as 4 square, read it 4 exponent 2. With the same base, the logarithms of 64 and 8 would be 3 and 1.5, respectively, since $4^3 = 64$, and $4^{1.5} = 4^{\frac{3}{2}} = 8$. In these cases, as in the preceding, read 4^3 and $4^{1.5}$ as 4 exponent 3, and 4 exponent 1.5, respectively.

620. Although any number *can* be used as a base, and a table of logarithms calculated, but two numbers have ever been employed. For all arithmetical operations (except addition and subtraction), the logarithms used are called the **Briggs** or **common** logarithms, and the base used is 10. In abstract mathematical analysis, the logarithms used are variously called **hyperbolic, Napierian,** or **natural** logarithms, and the base is 2.718281828+. The common logarithm of any number may be converted into a Napierian logarithm by multiplying the common logarithm by 2.30258508+, which is usually abbreviated to 2.3026, and sometimes to 2.3. Only the common system of logarithms will be considered in this Course.

621. Since, in the common system, the base is 10, it follows that, since $10^1 = 10$; $10^2 = 100$; $10^3 = 1{,}000$, etc., the logarithm (exponent) of 10 is 1; of 100 is 2; of 1,000 is 3, etc. For the sake of brevity in writing, the words "logarithm of," are abbreviated to "log." Thus, instead of writing logarithm of 100 = 2, write log 100 = 2. When speaking, however, the words for which "log" stands should always be pronounced in full.

622. From the above it will be seen (see Arts. **439** and **529**) that, when the base is 10,

Since $10^0 = 1$, the exponent $0 = \log 1$;
" $10^1 = 10$, " " $1 =$ " 10;
" $10^2 = 100$, " " $2 =$ " 100;
" $10^3 = 1{,}000$, " " $3 =$ " $1{,}000$, etc.

Also,

Since $10^{-1} = \frac{1}{10} = .1$ the exponent $-1 = \log .1$;
" $10^{-2} = \frac{1}{100} = .01$, " " $-2 =$ " $.01$;
" $10^{-3} = \frac{1}{1000} = .001$, " " $-3 =$ " $.001$, etc.

From this it will be seen that the logarithms of exact powers of 10 and of decimals like .1, .01, and .001 are the whole numbers 1, 2, 3, etc., and − 1, − 2, − 3, etc., respectively. *Only numbers consisting of 1 and one or more ciphers have whole numbers for logarithms.*

623. Now, it is evident that to produce a number between 1 and 10, the exponent of 10 must be a fraction; to produce a number between 10 and 100, it must be 1 plus a fraction; to produce a number between 100 and 1,000, it must be 2 plus a fraction, etc. Hence, the logarithm of any number between 1 and 10 is a fraction; of any number between 10 and 100, 1 plus a fraction; of any number between 100 and 1,000, 2 plus a fraction, etc. A logarithm, therefore, usually consists of two parts, a whole number, called the **characteristic,** and a fraction, called the **mantissa.** The mantissa is always expressed as a decimal. For example, to produce 20, 10 must have an exponent of *approximately* 1.30103, or $10^{1.30103} = 20$, very nearly, the degree of exactness depending upon the number of decimal places used. Hence, log 20 = 1.30103, 1 being the characteristic and .30103 the mantissa.

624. Referring to the second part of the table, Art. **622,** it is clear that the logarithms of all numbers less than 1 are negative, the logarithms of those between 1 and .1 being -1 plus a fraction. For, since log .1 = -1, the logarithms of .2, .3, etc. (which are all greater than .1, but less than 1), must be greater than -1; i. e., they must equal -1 *plus* a fraction. For the same reason, to produce a number between .1 and .01, the logarithm (exponent of 10) would be equal to -2 plus a fraction, and for a number between .01 and .001, it would be equal to -3 plus a fraction. Hence, the logarithm of any number between 1 and .1 has a negative characteristic of 1, and a positive mantissa; of a number between .1 and .01, a negative characteristic of 2, and a positive mantissa; of a number between .01 and .001, a negative characteristic of 3, and a positive mantissa; of a number between .001 and .0001, a negative characteristic of 4, and a positive mantissa, etc. *The negative characteristics are distinguished from the positive by the − sign written over the characteristic.* Thus, $\bar{3}$ indicates that 3 is negative.

It must be remembered that in all cases the mantissa is positive. Thus, the logarithm 1.30103 means $+1+.30103$, and the logarithm $\bar{1}.30103$ means $-1+.30103$. Were the minus sign written in front of the characteristic, it would indicate that the entire logarithm was negative. Thus, $-1.30103=-1-.30103$.

625. Rules for Characteristic.—From Art. **624,** it follows that:

I. *For a number greater than 1 the characteristic is one less than the number of integral places in the number.*

By "integral places" is meant the figures (including ciphers) to the left of the decimal point.

II. *For a number wholly decimal, the characteristic is negative, and is numerically one greater than the number of* *ciphers between the decimal point and the first digit of the decimal.*

For example, the characteristics of the logarithms of 256, 31.24, 7.53, and 1,728.0036, are 2, 1, 0, and 3, respectively, or *one less* than the number of integral places in each case; the characteristics of the logarithms of .0005, .0674, and .50072, are $\overline{4}$, $\overline{2}$, and $\overline{1}$, respectively, or numerically *one greater* than the number of ciphers immediately following the decimal point. It will be noticed that in the last number there are no ciphers, and the characteristic is $0 + 1 = 1$.

THE LOGARITHMIC TABLE.

TO FIND THE LOGARITHM OF A NUMBER.

626. To aid in obtaining the mantissas of logarithms, *tables of logarithms* have been calculated, some of which are very elaborate and convenient. In the Table of Logarithms, the mantissas of the logarithms of numbers, from 1 to 9,999, are given to five places of decimals, and the mantissas of logarithms of larger numbers can be found by interpolation. The table contains the *mantissas only*, and the characteristics may be easily found by the rules of Art. **625.**

The table depends upon the principle, which will be explained later, that all numbers having the same figures in the same order, have their mantissas alike, without regard to the position of the decimal point, which affects the characteristic only. To illustrate, if log 206 = 2.31387, then,

$$\log 20.6 = 1.31387. \qquad \log .206 = \overline{1}.31387.$$
$$\log 2.06 = .31387. \qquad \log .0206 = \overline{2}.31387, \text{ etc.}$$

627. To find the logarithm of a number not having more than four figures:

Rule.—*Find the first three significant figures of the number whose logarithm is desired, in the left-hand column; find the fourth figure in the column at the top (or bottom) of the page, and in the column under (or above) this figure, and* *opposite the first three figures previously found will be the mantissa or decimal part of the logarithm. The characteristic being found as described in Art.* ***625,*** *write it at the left of the mantissa, and the resulting expression will be the logarithm of the required number.*

628. Example.—Find the logarithm (*a*) of 476; (*b*) of 25.47; (*c*) of 1.073, and (*d*) of .06313.

Solution.—(*a*) In order to economize space, and make the labor of finding the logarithms easier, the first two figures of the mantissa are given only in the column headed 0. The last three figures of the mantissa, opposite 476 in the column headed N (N stands for number), page 9 of the tables, are 761, found in the column headed 0; glancing upwards, we find the first two figures of the mantissa, viz., 67. The characteristic is 2; hence, log 476 = 2.67761. Ans.

Note.—Since all numbers in the table are decimal fractions, the decimal point is omitted throughout; this is customary in all tables of logarithms.

(*b*) To find the logarithm of 25.47, we find the first three figures 254, in the column headed N on page 5, and on the same horizontal line, under the column headed 7 (the fourth figure of the given number), will be found the last three figures of the mantissa, viz., 603. The first two figures are evidently 40, and the characteristic is 1; hence, log 25.47 = 1.40603. Ans.

(*c*) For 1.073, the last three figures of the mantissa are found in the usual manner, in the column headed 3, opposite 107 in the column headed N on page 2, to be 060. It will be noticed that these figures are printed *060, the star meaning that instead of glancing *upwards* in the column headed 0, and taking 02 for the first two figures, we must glance *down* and take the two figures opposite the number 108, in the left-hand column, i. e., 03. The characteristic being 0, log 1.073 = 0.03060, or, more simply, .03060.

(*d*) For .06313, the last three figures of the mantissa are found opposite 631, in column headed 3 on page 12, to be 024. In this case, the first two figures occur in the same row, and are 80. Since the characteristic is $\overline{2}$, log .06313 = $\overline{2}$.80024. Ans.

629. If the original number contains but one digit (a cipher is not a digit), annex mentally two ciphers to the right of the digit; if the number contains but two digits (with no ciphers between, as in 4008), annex mentally one cipher on the right, before seeking the mantissa. Thus, if the logarithm of 7 is wanted, seek the mantissa for 700, which is .84510; or, if the logarithm of 48 is wanted, seek the mantissa for

480, which is .68124. Or, find the mantissas of logarithms of numbers between 0 and 100, on the first page of the tables.

The process of finding the logarithm of a number from the table is technically called **taking out the logarithm.**

630. To take out the logarithm of a number consisting of more than four figures, it is inexpedient to use more than five figures of the number, when using five-place logarithms (the logarithms given in the accompanying table are five-place). Hence, if the number consists of more than five figures, and the sixth figure is less than 5, replace all figures after the fifth with ciphers; if the sixth figure is 5 or more, increase the fifth figure by one, and replace the remaining figures with ciphers. Thus, if the number is 31,415,926, find the logarithm of 31,416,000; if 31,415,426, find the logarithm of 31,415,000.

631. EXAMPLE.—Find log 31,416.

SOLUTION.—Find the mantissa of the logarithm of the first four figures, as explained above. This is, in the present case, .49707 (see page 6). Now, subtract the number in the column headed 1, opposite 314 (the first three figures of the given number), from the next greater consecutive number, in this case 721, in the column headed 2. 721 − 707 = 14; this number is called the **difference.** At the extreme right of the page will be found a secondary table headed P. P., and at the top of one of these columns, in this table, in bold-face type, will be found the difference. It will be noticed that each column is divided into two parts by a vertical line, and that the figures on the left of this line run in sequence from 1 to 9. Considering the difference column headed 14, we see opposite the number 6 (6 is the last or fifth figure of the number whose logarithm we are taking out) the number 8.4 and we add this number to the mantissa found above, disregarding the decimal point in the mantissa, obtaining 49707 + 8.4 = 49715.4. Now, since 4 is less than 5, we reject it, and obtain for our complete mantissa .49715. Since the characteristic of the logarithm of 31,416 is 4, log 31,416 = 4.49715. Ans.

632. EXAMPLE.—Find log 380.93.

SOLUTION.—Proceeding in exactly the same manner as above, the mantissa for 3,809 is 58081 (the star directs us to take 58 instead of 57 for the first two figures), the next greater mantissa is 58092, found in the column headed 0, opposite 381 in column headed N. The difference is 092 − 081 = 11. Looking in the section headed P. P., for column headed 11, we find opposite 3, 3.3; neglecting the .3, since it is less than 5, 3 is the amount to be added to the mantissa of the logarithm of 3809 to form the logarithm of 38093. Hence, 58081 + 3 = 58084, and since the characteristic is 2, log 380.93 = 2.58084. Ans.

633. EXAMPLE.—Find log 1,296,728.

SOLUTION.—Since this number consists of more than five figures and the sixth figure is less than 5, we find the logarithm of 1,296,700, and call it the logarithm of 1,296,728. The mantissa of log 1,296 is found on page 2 to be 11261. The difference is 294 — 261 = 33. Looking in the P. P. section for column headed 33, we find opposite 7 on the extreme left, 23.1; neglecting the .1, the amount to be added to the above mantissa is 23. Hence, the mantissa of log 1,296,728 = 11261 + 23 = 11284; since the characteristic is 6, log 1,296,728 = 6.11284. Ans.

634. EXAMPLE.—Find log 89.126.

SOLUTION.—Log 89.12 = 1.94998. Difference between this and log 89.13 = 1.95002 − 1.94998 = 4. The P. P. (proportional part) for the fifth figure of the number, 6, is 2.4, or 2.

Hence, log 89.126 = 1.94998 + .00002 = 1.95000. Ans.

635. EXAMPLE.—Find log .096725.

SOLUTION.— Log .09672 = $\overline{2}$.98552. Difference = 4.
P. P. for 5 = 2

Hence, log .096725 = $\overline{2}$.98554. Ans.

636. To find the logarithm of a number consisting of five or more figures:

Rule.—I. *If the number consists of more than five figures and the sixth figure is 5 or greater, increase the fifth figure by 1, and write ciphers in place of the sixth and remaining figures.*

II. *Find the mantissa corresponding to the logarithm of the first four figures, and subtract this mantissa from the next greater mantissa in the table; the remainder is the difference.*

III. *Find in the secondary table headed P. P. a column headed by the same number as that just found for the difference, and in this column opposite the number corresponding to the fifth figure (or fifth figure increased by 1) of the given number (this figure is always situated at the left of the dividing line of the column) will be found the P. P. (propor-*

tional part) for that number. The P. P. thus found is to be added to the mantissa found in II, as in the preceding examples, and the result is the mantissa of the logarithm of the given number, as nearly as may be found with five-place tables.

EXAMPLES FOR PRACTICE.

637. Find the logarithms of the following numbers:

1. .062. Ans. 2.79239.
2. 620. Ans. 2.79239.
3. 21.4. Ans. 1.33041.
4. .000067. Ans. $\overline{5}.82607$.
5. 89.42. Ans. 1.95143.
6. .785398. Ans. 1.89509.
7. .0010823. Ans. 3.03435.
8. 10,000. Ans. 4.
9. 1,923.208. Ans. 3.28403.
10. 3.00026. Ans. .47717.

TO FIND A NUMBER WHOSE LOGARITHM IS GIVEN.

638. Rule I.—*Consider the mantissa first. Glance along the different columns of the table which are headed 0 until the first two figures of the mantissa are found. Then glance down the same column until the third figure is found (or 1 less than the third figure). Having found the first three figures, glance to the right along the row in which they are situated until the last three figures of the mantissa are found. Then, the number which heads the column in which the last three figures of the mantissa are found is the fourth figure of the required number, and the first three figures lie in the column headed N, and in the same row in which lie the last three figures of the mantissa.*

II. *If the mantissa cannot be found in the table, find the mantissa which is nearest to, but less than, the given mantissa, and which call the* **next less mantissa.** *Subtract the next less mantissa from the next greater mantissa in the table to obtain the difference. Also subtract the next less mantissa from the mantissa of the given logarithm, and call the remainder the P. P. Looking in the secondary table headed P. P. for* *the column headed by the difference just found, find the number opposite the P. P. just found (or the P. P. corresponding most nearly to that just found); this number is the fifth figure of the required number; the fourth figure will be found at the top of the column containing the next less mantissa, and the first three figures in the column headed N and in the same row which contains the next less mantissa.*

III. *Having found the figures of the number as above directed, locate the decimal point by the rules for the characteristic, annexing ciphers to bring the number up to the required number of figures if the characteristic is greater than 4.*

639. EXAMPLE.—Find the number whose logarithm is 3.56867.

SOLUTION.—The first two figures of the mantissa, 56, are found on page 7; glancing down the column, we find the third figure, 8 (in connection with 820), opposite 370 in the N column. Glancing to the right along the row containing 820, the last three figures of the mantissa, 867, are found in the column headed 4; hence, the fourth figure of the required number is 4, and the first three figures are 370, making the figures of the required number 3704. Since the characteristic is 3, there are four figures to the left of the decimal point, and the number whose logarithm is 3.56867 is 3,704. Ans.

640. EXAMPLE.—Find the number whose logarithm is 3.56871.

SOLUTION.—The mantissa is not found in the table. The next less mantissa is 56867; the difference between this and the next greater mantissa is $879 - 867 = 12$, and the P. P. is $56871 - 56867 = 4$. Looking in the P. P. section for the column headed 12, we do not find 4, but we do find 3.6 and 4.8. Since 3.6 is nearer 4 than 4.8, we take the number opposite 3.6 for the fifth figure of the required number; this is 3. Hence, the fourth figure is 4; the first three figures 370, and the figures of the number are 37043. The characteristic being 3, the number is 3,704.3. Ans.

641. EXAMPLE.—Find the number whose logarithm is 5.95424.

SOLUTION.—The mantissa is found in the column headed 0 on page 18, opposite 900 in the column headed N. Hence, the fourth figure is 0, and the number is 900,000, the characteristic being 5. Ans. Had the logarithm been $\overline{5}.95424$, the number would have been .00009.

642. EXAMPLE.—Find the number whose logarithm is .93036.

SOLUTION.—The first three figures of the mantissa, 930, are found in the 0 column opposite 852 in the N column, but since the last two figures of all the mantissas in this row are greater than 36, we must seek the next less mantissa in the preceding row. We find it to be 93034 (the star directing us to use 93 instead of 92 for the first two figures) in the column headed 8. The difference for this case is 039 − 034 = 5, and the P. P. is 036 − 034 = 2. Looking in the P. P. section for the column headed 5, we find the P. P., 2, opposite 4. Hence, the fifth figure is 4; the fourth figure is 8; the first three figures 851, and the number is 8.5184, the characteristic being 0. Ans.

643. Example.—Find the number whose logarithm is $\overline{2}.05753$.

Solution.—The next less mantissa is found in column headed 1 opposite 114 in the N column, page 2; hence, the first four figures are 1141. The difference for this case is 767 − 729 = 38, and the P. P. is 753 − 729 = 24. Looking in the P. P. section for the column headed 38, we find that 24 falls between 22.8 and 26.6. The difference between 24 and 22.8 is 1.2, and between 24 and 26.6 is 2.6; hence, 24 is nearer 22.8 than it is to 26.6, and 6, opposite 22.8, is the fifth figure of the number. Hence, number whose logarithm is $\overline{2}.05753$ = .011416. Ans.

EXAMPLES FOR PRACTICE.

644. Find the numbers corresponding to the following logarithms:

1. .74429. Ans. 5.55.
2. 4.38202. Ans. 24,100.
3. $\overline{1}.84510$. Ans. .7.
4. 1.84510. Ans. 70.
5. $\overline{4}.96047$. Ans. .000913.
6. 3.78942. Ans. 6,157.7.
7. .50210. Ans. 3.1776.
8. $\overline{3}.63491$. Ans. .0043143.
9. $\overline{1}.07619$. Ans. .11918.
10. $\overline{3}.23417$. Ans. .0017146.

645. In order to calculate by means of logarithms, a table is absolutely necessary. Hence, for this reason, we do not explain the method of calculating a logarithm. The work involved in calculating even a single logarithm is very great, and no method has yet been demonstrated, of which we are aware, by which the logarithm of a number like 121 can be calculated directly. Moreover, even if the logarithm could be readily obtained, it would be useless without a complete table, such as that which forms a part of this paper, for the reason that after having used it, say to extract a root, the number corresponding to the logarithm of the result could not be found.

MULTIPLICATION BY LOGARITHMS.

646. The principle upon which the process is based may be illustrated as follows: Let X and Y represent two numbers whose logarithms are x and y. To find the logarithm of their product, we have, from the definition of a logarithm,

$$10^x = X, \qquad (1)$$
$$\text{and } 10^y = Y. \qquad (2)$$

Since both members of (1) may be multiplied by the same quantity without destroying the equality, they evidently may be multiplied by equal quantities like 10^y and Y. Hence, multiplying (1) by (2), member by member,

$$10^x \times 10^y = 10^{x+y} = X\ Y \text{ (Art. } \mathbf{418}\text{);}$$

or, by the definition of a logarithm, $x + y = \log X\ Y$. But $X\ Y$ is the product of X and Y, and $x + y$ is the sum of their logarithms; from which it follows that the sum of the logarithms of two members is equal to the logarithm of their product. Hence,

647. To multiply two or more numbers by using logarithms:

Rule.—*Add the logarithms of the several numbers, and the sum will be the logarithm of the product. Find the number corresponding to this logarithm, and the result will be the number sought.*

Example.—Multiply 4.38, 5.217, and 83 together.

Solution.—
$$\begin{aligned} \text{Log } 4.38 &= .64147 \\ \text{Log } 5.217 &= .71742 \\ \text{Log } 83 &= 1.91908 \\ \hline \text{Adding,} \quad & 3.27797 = \log (4.38 \times 5.217 \times 83). \end{aligned}$$

Number corresponding to 3.27797 = 1,896.6. Hence, 4.38 × 5.217 × 83 = 1,896.6, nearly. Ans. By actual multiplication, the product is 1,896.58818, showing that the result obtained by using logarithms was correct to five figures.

648. When adding logarithms, their *algebraic* sum is always to be found. Hence, if some of their numbers multiplied together are wholly decimal, the algebraic sum of the characteristics will be the characteristic of the product. It must be remembered that the mantissas are always positive.

EXAMPLE.—Multiply 49.82, .00243, 17, and .97 together.

SOLUTION.—Log 49.82 = 1.69740
Log .00243 = $\overline{3}$.38561
Log 17 = 1.23045
Log .97 = $\overline{1}$.98677

Adding, 0.30023 = log (49.82 × .00243 × 17 × .97).

Number corresponding to 0.30023 = 1.9963. Hence, 49.82 × .00243 × 17 × .97 = 1.9963. Ans.

In this case the sum of the mantissas was 2.30023. The integral 2 added to the positive characteristics makes their sum = 2 + 1 + 1 = 4; sum of negative characteristics = 3 + $\overline{1}$ = 4, whence 4 + (− 4) = 0. If, instead of 17, the number had been .17 in the above example, the logarithm of .17 would have been $\overline{1}$.23045, and the sum of the logarithms would have been $\overline{2}$.30023; the product would then have been .019963.

649. It can now be shown why, as stated in Art. **626,** all numbers with figures in the same order have the same mantissa without regard to the decimal point. Thus, suppose it were known that log 2.06 = .31387. Then, log 20.6 = log (2.06 × 10) = log 2.06 + log 10 = .31387 + 1 = 1.31387. And so it might be proved with the decimal point in any other position.

EXAMPLES FOR PRACTICE.

650. Find the products of the following by the use of logarithms:

1. 100, 32, and 31.64. Ans. 101,250.
2. 23.1, 59.64, and 7.863. Ans. 10,833.
3. .00354, .275, and .0198. Ans. .000019275.
4. 2.763, 59.87, .264, and .001702. Ans. .074328.

DIVISION BY LOGARITHMS.

651. As before, let X and Y represent two numbers, whose logarithms are x and y. To find the logarithm of their quotient we have, from the definition of a logarithm,

$$10^x = X, \qquad (1)$$

and

$$10^y = Y. \qquad (2)$$

Dividing (1) by (2), $10^{x-y} = \frac{X}{Y}$ (Art. **438**), or, by the definition of a logarithm, $x - y = \log \frac{X}{Y}$. But $\frac{X}{Y}$ is the quotient of $X \div Y$, and $x - y$ is the difference of their logarithms, from which it follows that *the difference between the logarithms of two numbers is equal to the logarithm of their quotient.* Hence,

652. To divide one number by another by means of logarithms:

Rule.—*Subtract the logarithm of the divisor from the logarithm of the dividend, and the result will be the logarithm of the quotient.*

Example.—Divide 6,784.2 by 27.42.

Solution.—

$$\begin{aligned} \text{Log } 6{,}784.2 &= 3.83150 \\ \text{Log } 27.42 &= 1.43807 \\ \hline \textit{difference} &= 2.39343 = \log (6{,}784.2 \div 27.42). \end{aligned}$$

Number corresponding to 2.39343 = 247.42. Hence, 6,784.2 ÷ 27.42 = 247.42. Ans.

653. When subtracting logarithms, their *algebraic* difference is to be found. The operation may sometimes be confusing, because the mantissa is always positive, and the characteristic may be either positive or negative. *When the logarithm to be subtracted is greater than the logarithm from which it is to be taken, or when negative characteristics appear, subtract the mantissa first, and then the characteristic, by changing its sign and adding.* (Art. **399.**)

Example.—Divide 274.2 by 6,784.2.

Solution.—

$$\begin{aligned} \text{Log } 274.2 &= 2.43807 \\ \text{Log } 6{,}784.2 &= 3.83150 \\ \hline &\ \overline{2}.60657. \end{aligned}$$

First subtracting the mantissa .83150 gives .60657 for the mantissa of the quotient. In subtracting, 1 had to be taken from the characteristic of the minuend leaving a characteristic of 1. Subtract the characteristic 3 from this, by changing its sign and adding $1 - 3 = \overline{2}$, the characteristic of the quotient. Number corresponding to $\overline{2}.60657$ = .040418. Hence, 274.2 ÷ 6,784.2 = .040418. Ans.

Example.—Divide .067842 by .002742.

Solution.—

$$\begin{aligned} \text{Log } .067842 &= \overline{2}.83150 \\ \text{Log } .002742 &= \overline{3}.43807 \\ \hline \textit{difference} &= 1.39343. \end{aligned}$$

Subtracting, .83150 − .43807 = .39343 and − 2 + 3 = 1. Number corresponding to 1.39343 = 24.742. Hence, .067842 ÷ .002742 = 24.742. Ans

654. The only case that need cause trouble in subtracting is where the logarithm of the minuend has a negative characteristic, or none at all, and a mantissa less than the mantissa of the subtrahend. For example, let it be required to subtract the logarithm 3.74036 from the logarithm $\overline{3}.55145$. The logarithm $\overline{3}.55145$ is equivalent to $-3 + .55145$. Now, if we add both + 1 and − 1 to this logarithm, it will not change its value. Hence, $\overline{3}.55145 = -3 - 1 + 1 + .55145 = \overline{4} + 1.55145$. Therefore, $\overline{3}.55145 - 3.74036 =$

$$\begin{array}{r} \overline{4} + 1.55145 \\ 3 + .74036 \\ \hline \end{array}$$

$$\textit{difference} = \overline{7} + .81109 = \overline{7}.81109.$$

Had the characteristic of the above logarithm been 0 instead of $\overline{3}$, the process would have been exactly the same. Thus, $.55145 = \overline{1} + 1.55145$; hence,

$$\begin{array}{r} \overline{1} + 1.55145 \\ 3 + .74036 \\ \hline \end{array}$$

$$\textit{difference} = \overline{4} + .81109 = \overline{4}.81109.$$

Example.—Divide .02742 by 67.842.

Solution.—Log .02742 = $\overline{2}.43807 = \overline{3} + 1.43807$
Log 67.842 = $1.83150 = 1 + .83150$

$$\textit{difference} = \overline{4} + .60657 = \overline{4}.60657.$$

Number corresponding to $\overline{4}.60657 = .00040417$. Hence, .02742 ÷ 67.842 = .00040417. Ans.

Example.—What is the reciprocal of 3.1416?

Solution.—Reciprocal of $3.1416 = \frac{1}{3.1416}$, and $\log \frac{1}{3.1416} = \log 1 - \log 3.1416 = 0 - .49715$. Since $0 = -1 + 1$,

$$\begin{array}{r} \overline{1} + 1.00000 \\ .49715 \\ \hline \end{array}$$

$$\textit{difference} = \overline{1} + .50285 = \overline{1}.50285.$$

Number whose logarithm is $\overline{1}.50285 = .31831$. Ans.

EXAMPLES FOR PRACTICE.

655. Find the quotients of the following by the use of logarithms:

1. $564.35 \div 34.96$.	Ans. 16.143.
2. $9.643 \div 200.04$.	Ans. .048204.
3. $.16071 \div 76.8$.	Ans. .0020926.
4. $.00624 \div 3.096$.	Ans. .0020155.
5. $.000119 \div .0719$.	Ans. .0016551.
6. $1.19 \div 719$.	Ans. .0016551.
7. $1 \div 1{,}728$.	Ans. .00057870.

INVOLUTION BY LOGARITHMS.

656. If X represents a number whose logarithm is x, we have, from the definition of a logarithm,

$$10^x = X.$$

Raising both numbers to some power, as the nth, the equation becomes, by Art. **511**,

$$10^{xn} = X^n.$$

But X^n is the required power of X, and xn is its logarithm, from which it follows that the logarithm of a number multiplied by the exponent of the power to which it is raised is equal to the logarithm of the power. Hence,

657. To raise a number to any power by the use of logarithms:

Rule.—*Multiply the logarithm of the number by the exponent which denotes the power to which the number is to be raised, and the result will be the logarithm of the required power.*

Example.—What is the square of (*a*) 7.92? (*b*) the cube of 94.7? (*c*) the 1.6 power of 512, that is, $512^{1.6}$?

Solution.—(*a*) Log $7.92 = .89873$; the exponent of the power is 2. Hence, $.89873 \times 2 = 1.79746 = \log 7.92^2$. Number corresponding to $1.79746 = 62.727$. Hence, $7.92^2 = 62.727$, nearly. Ans.

(*b*) Log $94.7 = 1.97635$; $1.97635 \times 3 = 5.92905 = \log 94.7^3$. Number corresponding to $5.92905 = 849{,}280$. Hence, $94.7^3 = 849{,}280$, nearly. Ans.

(*c*) Log $512^{1.6} = 1.6 \times \log 512 = 1.6 \times 2.70927 = 4.334832$, or 4.33483 (when using five-place logarithms) $= \log 21{,}619$. Hence, $512^{1.6} = 21{,}619$, nearly. Ans.

658. If the number is wholly decimal, so that the characteristic is negative, *multiply the two parts of the logarithm separately by the exponent of the number. If, after multiplying the mantissa, the product has a characteristic, add it, algebraically, to the negative characteristic, multiplied by the exponent, and the result will be the negative characteristic of the required power.*

EXAMPLE.—Raise .0751 to the fourth power.

SOLUTION.—Log $.0751^4 = 4 \times \log .0751 = 4 \times \bar{2}.87564$. Multiplying the parts separately, $4 \times 2 = 8$ and $4 \times .87564 = 3.50256$. Adding the 3 and 8, $3 + (-8) = -5$; therefore, $\log .0751^4 = \bar{5}.50256$. Number corresponding to this $= .00003181$. Hence, $.0751^4 = .00003181$. Ans.

659. A decimal may be raised to a power whose exponent contains a decimal as follows:

EXAMPLE.—Raise .8 to the 1.21 power.

SOLUTION.—Log $.8^{1.21} = 1.21 \times \bar{1}.90309$. There are several ways of performing the multiplication.

First Method.—Adding the characteristic and mantissa algebraically, the result is $-.09691$. Multiplying this by 1.21 gives $-.1172611$, or $-.11726$, when using 5-place logarithms. To obtain a positive mantissa, add $+1$ and -1; whence, $\log .8^{1.21} = -1 + 1 - .11726 = \bar{1}.88274$.

Second Method.—Multiplying the characteristic and mantissa separately gives $-1.21 + 1.09274$. Adding characteristic and mantissa algebraically gives $-.11726$; then, adding $+1$ and -1, $\log .8^{1.21} = \bar{1}.88274$.

Third Method.—Multiplying the characteristic and mantissa separately gives $-1.21 + 1.09274$. Adding the decimal part of the characteristic to the mantissa gives $-1 + (-.21 + 1.09274) = \bar{1}.88274 = \log .8^{1.21}$. The number corresponding to the logarithm $\bar{1}.88274 = .76338$. Ans.

Any one of the above three methods may be used, but we recommend the first or the third. The third is the most elegant, and saves figures, but requires the exercise of more caution than the first method does. Below will be found the entire work of multiplication for both $.8^{1.21}$ and $.8^{.21}$.

$$
\begin{array}{r}
\bar{1}.90309 \\
1.21 \\
\hline
90309 \\
180618 \\
90309 \\
\hline
1.0927389 \\
-1.21 \\
\hline
\bar{1}.8827389, \text{ or } \bar{1}.88274.
\end{array}
\qquad
\begin{array}{r}
\bar{1}.90309 \\
.21 \\
\hline
90309 \\
180618 \\
\hline
+1.1896489 \\
-1-.21 \\
\hline
\bar{1}.9796489, \text{ or } \bar{1}.97965.
\end{array}
$$

In the second case, the negative decimal obtained by multiplying -1 and $.21$ was greater than the positive decimal obtained by multiplying $.90309$ and $.21$; hence, $+1$ and -1 were added as shown.

EXAMPLES FOR PRACTICE.

660. Find the values of the following by logarithms:

1. $1,728^2$. Ans. 2,985,900.
2. $2.49^{1.24}$. Ans. 3.0995.
3. $32.16^{.42}$. Ans. 4.2961.
4. $.64^4$. Ans. .16777.
5. $.64^{.4}$. Ans. .8365.
6. $.0241^{2.8}$. Ans. .000029489.

EVOLUTION BY LOGARITHMS.

661. If X represents a number whose logarithm is x, we have, from the definition of a logarithm,

$$10^x = X.$$

Extracting some root of both members, as the nth, the equation becomes, by Art. **521**,

$$10^{\frac{x}{n}} = \sqrt[n]{X}.$$

But $\sqrt[n]{X}$ is the required root of X, and $\frac{x}{n}$ is its logarithm, from which it follows that the logarithm of a number, divided by the index of the root to be extracted, is equal to the logarithm of the root. Hence,

662. To extract any root of a number by means of logarithms:

Rule.—*Divide the logarithm of the number by the index of the root; the result will be the logarithm of the root.*

Example.—Extract (*a*) the square root of 77,851; (*b*) the cube root of 698,970; (*c*) the 2.4 root of 8,964,300.

Solution.—(*a*) Log $77,851 = 4.89127$; the index of the root is 2; hence, log $\sqrt{77,851} = 4.89127 \div 2 = 2.44564$; number corresponding to this $= 279.02$. Hence, $\sqrt{77,851} = 279.02$, nearly. Ans.

(*b*) Log $\sqrt[3]{698,970} = 5.84446 \div 3 = 1.94815 =$ log 88.746; or, $\sqrt[3]{698,970} = 88.747$, nearly. Ans.

(c) Log $\sqrt[2.4]{8,964,300} = 6.95251 \div 2.4 = 2.89688 =$ log 788.64; or $\sqrt[2.4]{8,964,300} = 788.64$, nearly. Ans.

663. If it is required to extract a root of a number wholly decimal, and the negative characteristic will not exactly contain the index of the root, without a remainder, proceed as follows:

Separate the two parts of the logarithm; add as many units (or parts of a unit) to the negative characteristic as will make it exactly contain the index of the root. Add the same number to the mantissa, and divide both parts by the index. The result will be the characteristic and mantissa of the root.

EXAMPLE.—Extract the cube root of .0003181.

SOLUTION.—Log $\sqrt[3]{.0003181} = \frac{\log .0003181}{3} = \frac{\bar{4}.50256}{3}$.

$$(\bar{4} + \bar{2} = \bar{6}) + (2 + .50256 = 2.50256).$$

$$(\bar{6} \div 3 = \bar{2}) + (2.50256 \div 3 = .83419; \text{ or,}$$

$$\log \sqrt[3]{.0003181} = \bar{2}.83419 = \log .068263.$$

Hence, $\sqrt[3]{.0003181} = .068263$. Ans.

EXAMPLE.—Find the value of $\sqrt[1.41]{.0003181}$.

SOLUTION.—Log $\sqrt[1.41]{.0003181} = \frac{\log .0003181}{1.41} = \frac{\bar{4}.50256}{1.41}$.

If $-.23$ be added to the characteristic, it will contain 1.41 exactly 3 times. Hence,

$$[-4 + (-.23) = -4.23] + [.23 + .50256 = .73256].$$

$$(-4.23 \div 1.41 = 3) + (.73256 \div 1.41 = .51955); \text{ or,}$$

$$\log \sqrt[1.41]{.0003181} = \bar{3}.51955 = \log .0033079.$$

Hence, $\sqrt[1.41]{.0003181} = .0033079$. Ans.

EXAMPLES FOR PRACTICE.

664. Find the values of the following by logarithms:

1. $\sqrt[4]{906.8}$. Ans. 5.4876.
2. $\sqrt[5]{11}$. Ans. 1.6154.
3. $.0497^{\frac{1}{3}}$. Ans. .36766.
4. $.128^{\frac{1}{6}}$. Ans. .7099.
5. $\sqrt[2.5]{.0227}$. Ans. .21999.
6. $\sqrt[.6]{.756}$. Ans. .62738.

665. Example.—Solve this expression by logarithms:

$$\frac{497 \times .0181 \times 762}{3{,}300 \times .6517} = ?$$

Solution.—

$$\begin{aligned}
\text{Log } 497 &= 2.69636\\
\text{Log } .0181 &= \overline{2}.25768\\
\text{Log } 762 &= 2.88195\\
\text{Log product} &= 3.83599
\end{aligned}$$

$$\begin{aligned}
\text{Log } 3{,}300 &= 3.51851\\
\text{Log } .6517 &= \overline{1}.81405\\
\text{Log product} &= 3.33256
\end{aligned}$$

$$3.83599 - 3.33256 = .50343 = \log 3.1874$$

Hence, $\frac{497 \times .0181 \times 762}{3{,}300 \times .6517} = 3.1874.$ Ans.

Example.—Solve, $\sqrt[3]{\frac{504{,}203 \times 507}{1.75 \times 71.4 \times 87}}$ by logarithms

Solution.—

$$\begin{aligned}
\text{Log } 504{,}203 &= 5.70260\\
\text{Log } 507 &= 2.70501\\
\text{Log product} &= 8.40761
\end{aligned}$$

$$\begin{aligned}
\text{Log } 1.75 &= .24304\\
\text{Log } 71.4 &= 1.85370\\
\text{Log } 87 &= 1.93952\\
\text{Log product} &= 4.03626
\end{aligned}$$

$$\frac{8.40761 - 4.03626}{3} = 1.45712 = \log 28.65.$$

Hence, $\sqrt[3]{\frac{504{,}203 \times 507}{1.75 \times 71.4 \times 87}} = 28.65.$ Ans.

666. Logarithms can often be applied to the solution of equations.

Example.—Solve the equation $2.43x^5 = \sqrt[6]{.0648}$.

Solution.— $2.43x^5 = \sqrt[6]{.0648}.$

Dividing by 2.43, $x^5 = \frac{\sqrt[6]{.0648}}{2.43}.$

Taking the logarithm of both numbers,

$$5 \times \log x = \frac{\log .0648}{6} - \log\ 2.43;$$

or, $$5 \log x = \frac{\overline{2}.81158}{6} - .38561.$$

$$= \overline{1}.80193 - .38561.$$

$$= \overline{1}.41632.$$

Dividing by 5, $\log x = \overline{1}.88326;$

whence, $x = .7643.$ Ans.

EXAMPLES FOR PRACTICE.

667. Find the values of the following:

1. $\frac{89 \times 753 \times .0097}{36,709 \times .08497}$. Ans. .20840.

2. $\sqrt[3]{\frac{7,932 \times .00657 \times .80464}{.03274 \times .6428}}$. Ans. 12.583.

3. $\sqrt[7]{\frac{.03271^2 \times 53.429 \times .77542^3}{32.769 \times .000371^4}}$. Ans. 33.035.

Find the value of x in the following:

4. $5x^7 = \frac{129.4 \times .71}{30}$. Ans. $x = .93237$.

5. $38x^{.42} = \frac{129.4 \times .71^2}{\sqrt{30}}$. Ans. $x = .063133$.

GEOMETRY AND TRIGONOMETRY.

GEOMETRY.

668. **Geometry** is that branch of mathematics which treats of the properties of lines, angles, surfaces, and volumes.

LINES AND ANGLES.

669. A **point** indicates position only. It has neither length, breadth, nor thickness.

670. A **line** has only one dimension: length.

671. A **straight line,** Fig. 2, is one that does not change its direction throughout its whole length. A straight line is also frequently called a **right line.**

FIG. 2.

672. A **curved line,** Fig. 3, changes its direction at every point.

FIG. 3.

673. A **broken line,** Fig. 4, is one made up wholly of straight lines lying in different directions.

FIG. 4.

674. **Parallel lines,** Fig. 5, are equally distant from each other at all points.

FIG. 5.

675. A line is **perpendicular** to another when it meets that line so as not to incline towards it on either side, Fig. 6.

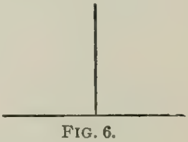

FIG. 6.

676. A **horizontal line** is a line parallel to the horizon, or water level, Fig. 7.

677. A **vertical line,** Fig. 7, is a line perpendicular to a horizontal line; consequently, it has the direction of a plumb line.

FIG. 7.

678. When two lines cross or cut each other, as in Fig. 8, they are said to **intersect,** and the point at which they intersect is called the **point of intersection,** as *A*.

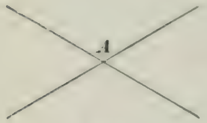

FIG. 8.

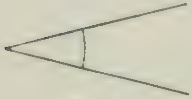

FIG. 9

679. An **angle,** Fig. 9, is the opening between two lines which intersect or meet; the point of meeting is called the **vertex** of the angle.

680. In order to distinguish one line from another, two of its points are given if it is a straight line, and as many more as are considered necessary if it is a broken or curved line. Thus, in Fig. 10, the line *A B* would mean the straight line included between the points *A* and *B*. Similarly, the straight line between *C* and *B*, or between *B* and *D*, would be called the line *C B*, or the line *B D*. The broken line made up of the lines *A B* and *C B*, or *A B* and *B D*, would be called the broken line *C B A* or *A B C*, and *A B D* or *D B A*, according to the point started from.

681. To distinguish angles, name a point on each line, and the point of their intersection, or vertex of the angle. Thus, in Fig. 10, the angle formed by the lines *A B* and *C B* is called the angle *A B C* or the angle *C B A*: the letter at the vertex is always placed in the middle. The angle formed by the lines *A B* and *B D* is called the angle *A B D* or the angle *D B A*.

When an angle stands alone so that it cannot be mistaken for any other angle, only the vertex letter need be given; thus, the angle *O*, or the angle *P*, etc.

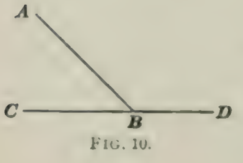

FIG. 10.

682. If one straight line meets another straight line at a point between its ends (see Figs. 10 and 11), two angles, *A B C* and *A B D*, are formed, which are called **adjacent angles.**

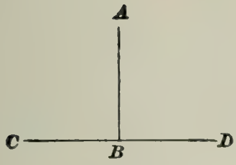

FIG. 11.

683. When these adjacent angles, *A B C* and *A B D*, are equal, they are called **right angles;** see Fig. 11.

684. An **acute angle** is less than a right angle. *A B C*, Fig. 12, is an acute angle.

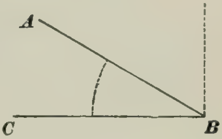

FIG. 12.

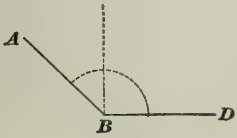

FIG. 13.

685. An **obtuse angle** is greater than a right angle. *A B D*, Fig. 13, is an obtuse angle.

686. When two straight lines intersect, they form four angles about the point of intersection. Thus, in Fig. 14, the lines *A B* and *C D*, intersecting at the point *O*, form four angles, *B O D*, *D O A*, *A O C*, and *C O B*, about the point *O*. The angles which lie on the *same* side of one straight line, as *D O B* and *D O A*, are **adjacent angles.** The angles which lie *opposite* each other are called **opposite angles.** Thus, *A O C* and *D O B*, also *D O A* and *B O C*, are opposite angles.

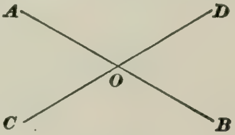

FIG. 14.

When one straight line intersects another straight line, as in Fig. 14, the opposite angles are equal. Thus, $D\,O\,B = A\,O\,C$, and $D\,O\,A = B\,O\,C$.

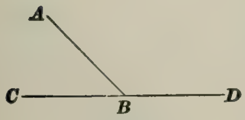

FIG. 15.

When one straight line meets another straight line at a point between its ends, the sum of the two adjacent angles *A B D* and *A B C*, Fig. 15, equals two right angles.

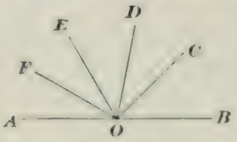

FIG. 16.

687. If a number of straight lines on the same side of a given straight line meet at the same point, the sum of all the angles formed is equal to two right angles. Thus, in Fig. 16, $C O B + D O C + E O D + F O E + A O F =$ two right angles.

688. If a straight line intersects another straight line, so that the adjacent angles are equal, the lines are *said to be* **perpendicular** *to each other.* In such a case, four right angles are formed about the point of intersection. Thus, in Fig. 17, $B O C = C O A$; hence, $B O C$, $C O A$, $A O D$, and $D O B$ are right angles. From this it is seen that *four right angles* are all that can be formed about a given point.

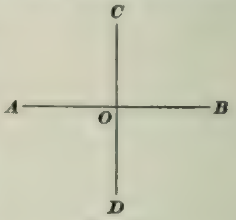

FIG. 17.

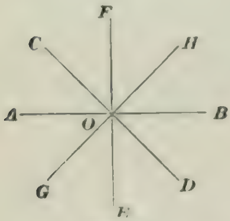

FIG. 18.

689. Through a given point any number of straight lines may be drawn; and the sum of all the angles formed about the point of intersection equals four right angles. Thus, in Fig. 18, $H O F + F O C + C O A + A O G + G O E + E O D + D O B + B O H =$ four right angles.

EXAMPLE.—In a fly-wheel with 12 arms, what part of a right angle is included between the center lines of any two arms, the arms being spaced equally?

SOLUTION.—Since there are 12 arms, there are 12 angles. The sum of all the angles equals four right angles. Hence, one angle equals $\frac{1}{12}$ of four right angles, or $\frac{4}{12} = \frac{1}{3}$ of one right angle.

690. A perpendicular drawn from a point over or under a given straight line is the shortest distance from the point to the line, or to the line extended. Thus, if A, Fig. 19, is the given point, and $C D$ the given line, then the perpendicular $A B$ is the shortest distance from A to $C D$.

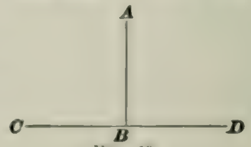

FIG. 19

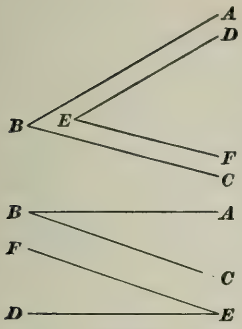

FIG. 20.

691. If two angles have their sides parallel and lie in the same or in opposite directions, they are equal. Thus, if the side $A\ B$, Fig. 20, is parallel to the side $D\ E$, and if the side $B\ C$ is parallel to the side $E\ F$, then the angle $E =$ the angle B.

692. If two sides of an angle are perpendicular to two sides of another angle, the two angles are equal. Thus, if $D\,E$ and $G\,H$, Fig. 21, are perpendicular to $B\,A$, and $E\,F$ and $H\,K$ are perpendicular to $B\ C$, then will angle $E =$ angle $B =$ angle H.

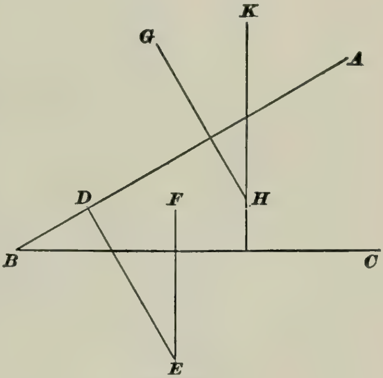

FIG. 21.

EXAMPLES FOR PRACTICE.

1. In a pulley with five arms, what part of a right angle is included between the center lines of any two arms? Ans. $\frac{4}{5}$ of a right angle.

2. If one straight line meets another straight line so as to form an angle equal to $1\frac{2}{5}$ right angles, what part of a right angle does its adjacent angle equal? Ans. $\frac{3}{5}$ of a right angle.

3. If a number of straight lines meet a given straight line at a given point, all being on the same side of the given line, so as to form six equal angles, what part of a right angle is contained in each angle? Ans. $\frac{1}{3}$ of a right angle.

PLANE FIGURES.

693. A **surface** has only two dimensions: *length* and *breadth*.

694. A **plane surface** is a flat surface. If a straight edge be laid on a plane surface, every point along the edge of the straight edge will touch the surface, no matter in what direction it is laid.

695. A **plane figure** is any part of a plane surface bounded by straight or curved lines.

696. When a plane figure is bounded by **straight** lines only, it is called a **polygon.** The bounding lines are called the **sides,** and the broken line that bounds it (or the whole distance around it) is called the **perimeter** of the polygon.

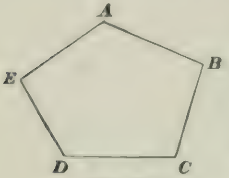

FIG. 22.

697. The angles formed by the sides are called the **angles** of the polygon. Thus, $A\ B\ C\ D\ E$, Fig. 22, is a polygon. $A\ B$, $B\ C$, etc., are the **sides;** $E\ A\ B$, $B\ C\ D$, etc., are the **angles,** and the broken line $A\ B\ C\ D\ E\ A$ is the **perimeter.**

698. Polygons are classified according to the number of their sides: One of three sides is called a **triangle;** one of four sides, a **quadrilateral;** one of five sides, a **pentagon;** one of six sides, a **hexagon;** one of seven sides, a **heptagon;** one of eight sides, an **octagon;** one of ten sides, a **decagon;** one of twelve sides, a **dodecagon,** etc.

699. Equilateral polygons are those in which the sides are all equal. Thus, in Fig. 23, $A\ B = B\ C = C\ D = D\ A$; hence, $A\ B\ C\ D$ is an equilateral polygon.

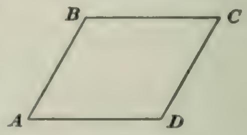

FIG. 23.

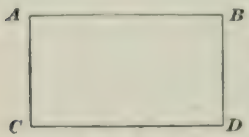

FIG. 24.

700. An **equiangular polygon** is one in which all of the angles are equal. Thus, in Fig. 24, angle $A =$ angle $B =$ angle $D =$ angle C; hence, $A\ B\ D\ C$ is an equiangular polygon.

701. A **regular polygon** is one in which all of the sides and all of the angles are equal. Thus, in Fig. 25, $A\ B = B\ D = D\ C = C\ A$, and angle $A =$ angle $B =$ angle $D =$ angle C· hence, $A\ B\ D\ C$ is a regular polygon.

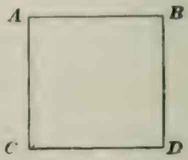

FIG. 25

702. Other regular polygons are shown in Fig. 26.

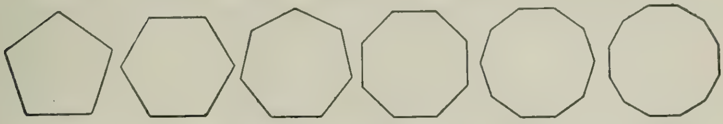
Pentagon. Hexagon. Heptagon. Octagon. Decagon. Dodecagon.

FIG. 26.

703. The sum of all the interior angles of any polygon equals two right angles, multiplied by a number which is two less than the number of sides in the polygon. Thus, $A\,B\,C\,D\,E\,F$, Fig. 27, is a polygon of six sides (hexagon), and the sum of all the interior angles, $A + B + C + D + E + F =$ two right angles $\times$ 4 ($= 6 - 2$), or 8 right angles.

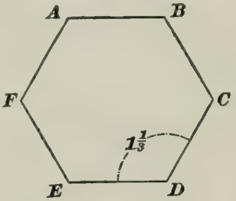

FIG. 27.

EXAMPLE.—If the above figure is a regular hexagon (has equal sides and equal angles), how many right angles are there in each interior angle?

SOLUTION.—$6 - 2 = 4$. Two right angles $\times 4 = 8$ right angles $=$ the total number of right angles in the polygon; and as there are six equal angles, we have $8 \div 6 = 1\frac{1}{3}$ right angles $=$ the number of right angles in each interior angle. Ans.

THE TRIANGLE.

704. Triangles are divided into four classes: **Isosceles** triangles, **scalene** triangles, **right-angled** triangles, and **oblique-angled** triangles.

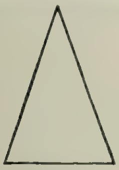

FIG. 28.

705. An **isosceles** triangle, Fig. 28, is one having two of its sides equal.

706. When the three sides are equal, as in Fig. 29, it is called an **equilateral** triangle.

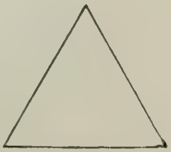

FIG. 29.

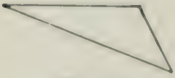

FIG. 30.

707. A **scalene** triangle, Fig. 30, is one having no two of its sides equal.

708. A **right-angled** triangle, Fig. 31, is any triangle having one right angle. The side opposite the right angle is called the **hypotenuse.** Among mathematicians, a right-angled triangle is now termed a **right triangle.**

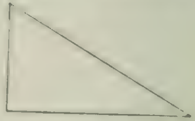

FIG. 31.

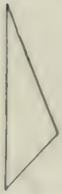

FIG. 32.

709. An **oblique** triangle, Fig. 32, is one which has no right angles.

710. The **base** of any triangle is the side upon which the triangle is supposed to stand.

711. The **altitude** of any triangle is a line drawn from the vertex of the angle opposite the base perpendicular to the base, or to the base extended. Thus, in Figs. 33 and 34, $B\ D$ is the *altitude* of the triangles $A\ B\ C$.

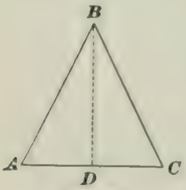

FIG. 33.

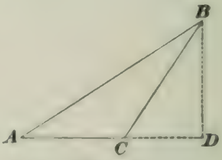

FIG. 34.

In an isosceles triangle, the angles opposite the equal sides are equal. Thus, in Fig. 35, $A\ B = B\ C$; hence, angle $C =$ angle A.

In any isosceles triangle, if a perpendicular be drawn from the vertex opposite the unequal side to that side, it bisects (cuts in halves) the side. Thus, $A\ C$, Fig. 35, is the unequal side in the isosceles triangle $A\ B\ C$; hence, the perpendicular $B\ D$ bisects $A\ C$, or $A\ D = D\ C$.

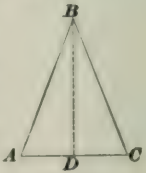

FIG. 35.

If two angles of any triangle are equal, the triangle is isosceles.

712. In any **triangle,** the sum of the three angles equals two right angles. Thus, in Fig. 36, the sum of the angles at A, B, and C = two right angles; that is, $A + B + C$ = two right angles. Hence, if any two angles of a triangle are given, the third may be found by subtracting the sum of the two from two right angles. Suppose that $A + B = 1\frac{7}{10}$ right angles; then, C must equal $2 - 1\frac{7}{10} = \frac{3}{10}$ of a right angle.

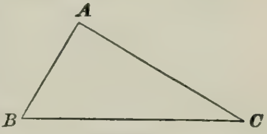

FIG. 36.

713. In any **right-angled** triangle there can be but one right angle, and since the sum of all the angles equals two right angles, it is evident that the sum of the two acute angles must equal a right angle. Therefore, if in any right-angled triangle one acute angle is known, the other can be found by subtracting the known angle from a right angle. Thus, in Fig. 37, $A\ B\ C$ is a right-angled triangle, right-angled at C. Then, the angles $A + B$ = one right angle. If $A = \frac{3}{7}$ of a right angle, $B = 1 - \frac{3}{7} = \frac{4}{7}$ of a right angle.

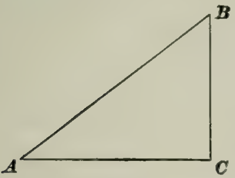

FIG. 37.

714. In any right-angled triangle, the square described on the hypotenuse is equal to the sum of the squares described upon the other two sides. If $A\ B\ C$, Fig. 38, is a right-angled triangle, right-angled at B, then the square described upon the hypotenuse $A\ C$ is equal to the sum of the squares described upon the sides $A\ B$ and $B\ C$; consequently, if the lengths of the sides $A\ B$ and $B\ C$ are known, we can find the length of the hypotenuse by adding the squares of the lengths of the sides $A\ B$ and $B\ C$, and then extracting the square root of the sum.

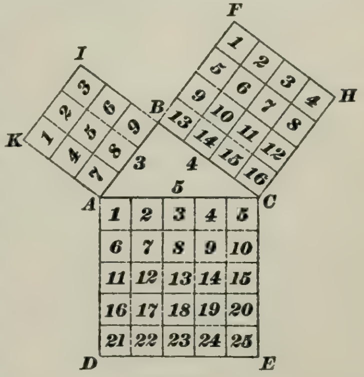

FIG. 38

EXAMPLE.—If $A B = 3$ inches, and $B C = 4$ inches, what is the length of the hypotenuse $A C$?

SOLUTION.— $3^2 = 9$; $4^2 = 16$; adding,
$9 + 16 = 25$. $\sqrt{25} = 5$.
Therefore, $A C = 5$ inches. Ans.

If the hypotenuse and one side are given, the other side can be found by subtracting the square of the given side from the square of the hypotenuse, and then extracting the square root of the remainder.

EXAMPLE.—The side given is 3 inches, the hypotenuse is 5 inches; what is the length of the other side?

SOLUTION.—$3^2 = 9$; $5^2 = 25$. $25 - 9 = 16$, and $\sqrt{16} = 4$ inches. Ans.

EXAMPLE.—If, from a church steeple which is 150 feet high, a rope is to be attached to the top, and to a stake in the ground 85 feet from its foot (the ground being supposed to be level), what must be the length of the rope?

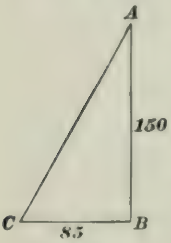

FIG. 39.

SOLUTION.—In Fig. 39, $A B$ represents the steeple 150 feet high; C, a stake 85 feet from the foot of the steeple, and $A C$, the rope. Here we have a right-angled triangle, right-angled at B, and $A C$ is the hypotenuse.

The square of $A C = 85^2 + 150^2 = 7{,}225 + 22{,}500 = 29{,}725$.

Therefore, $A C = \sqrt{29{,}725} = 172.4$ feet, nearly. Ans.

715. Two triangles are **equal** when the *sides* of one are equal to the sides of the other.

716. Two triangles are **similar** when the *angles* of one are equal to the angles of the other. *The corresponding sides of similar triangles are proportional.*

For example, in the triangles $A B C$ and $a b c$, Fig. 40, side $a c$ is perpendicular to $A C$; the side $a b$ to $A B$, and side $c b$ to $B C$; hence, angle A = angle a, since the sides of one are perpendicular to the sides of the other. In like manner, angle B = angle b, and angle C = angle c. The two triangles are, therefore, similar.

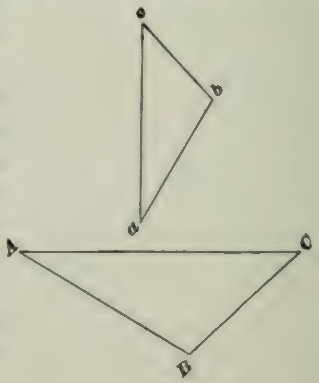

FIG. 40.

and their corresponding sides are proportional. That is, any two sides of one triangle are to each other as the two corresponding sides of the other triangle; or, one side of one triangle is to the corresponding side of the other as another side of the first triangle is to the corresponding side of the second. The following are examples of the many proportions that may be written. In this case, the corresponding sides of the two triangles are the ones that are perpendicular to each other:

$$\begin{aligned} AB : BC &= ab : bc, \\ AB : AC &= ab : ac, \\ BC : bc &= AB : ab, \\ AC : ac &= BC : bc, \text{ etc.} \end{aligned}$$

Example.—The sides of a triangle are 18 inches and 21 inches, and the base is 24 inches long; what are the lengths of the sides of a similar triangle whose base is 8?

Solution.—Since the sides are proportional, we have the proportions $24 : 8 = 21 : x$, and $24 : 8 = 18 : x$. From the first, $x = 7$, and from the second, $x = 6$. Ans.

717. If a straight line is drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally. Thus, in Fig. 41, let the line DE be drawn parallel to the side BC in the triangle ABC. Then,

$$AD : DE = AB : BC.$$

It is to be noticed, also, that the triangles ADE and ABC are similar, and their sides are proportional. The proportion $AD : DB = AE : EC$ is a useful one.

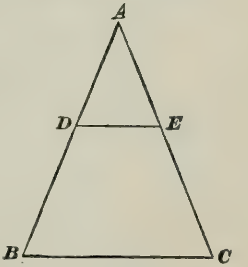

Fig. 41.

Example.—In the last figure, if $AE = 14$, $AD = 12$ and $EC = 9$, what does DB equal?

Solution.—From the proportion $AD : DB = AE : EC$, $12 : DB = 14 : 9$, whence $DB = 7\frac{5}{7}$. Ans.

Example.—The base of a right-angled triangle is 12 inches, and its altitude 40 inches. How wide is the triangle at 24 inches from the base?

SOLUTION.—Since the triangle is right-angled, the length of the perpendicular side equals the altitude, or 40 inches. By drawing a line parallel to the base, and 24 inches above it, a second and similar triangle will be found whose corresponding side = 40 − 24, or 16 inches, and the length of whose base is the required width. Hence, $40 : 12 = 16 : x$, or $x = 4.8$ inches. Ans.

EXAMPLES FOR PRACTICE.

1. How many right angles are there in one of the interior angles of a regular heptagon? Ans. $1\frac{3}{7}$ right angles.

2. The angle at the vertex of an isosceles triangle equals one-half a right angle. What do the other angles equal? Ans. $\frac{3}{4}$ of a right angle.

3. One of the acute angles of a right-angled triangle equals $\frac{5}{9}$ of a right angle. What is the size of the other acute angle?
Ans. $\frac{4}{9}$ of a right angle.

4. If the two sides about the right angle in a right-angled triangle are 52 and 39 feet long, how long is the hypotenuse? Ans. 65 feet.

5. A ladder 65 feet long reaches to the top of a house when its foot is 25 feet from the house. How high is the house, supposing the ground to be level? Ans. 60 feet.

6. In a triangle $A\ B\ C$, side $A\ B = 32$ feet, $B\ C = 34$ feet and $A\ C = 48$ feet. If side $A\ B$ of a similar triangle is 72 feet long, what are the lengths of the other two sides?
Ans. $A\ C = 108$ feet; $B\ C = 76.5$ feet.

7. The base of a right-angled triangle is 24 inches, and its altitude, 72 inches. At what distance from the top is the triangle 16 inches wide?
Ans. 48 inches.

THE CIRCLE.

FIG. 42.

718. A **circle,** Fig. 42, is a plane figure bounded by a curved line, called the **circumference,** every point of which is equally distant from a point within, called the **center.**

719. The **diameter** of a circle, $A\ B$, Fig. 43, is a straight line passing through the center and terminated at both ends by the circumference.

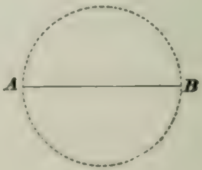

FIG. 43.

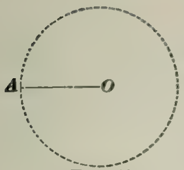

FIG. 44.

720. The **radius** of a circle, *O A*, Fig. 44, is a straight line drawn from the center to the circumference. It is equal in length to one-half the diameter. The plural of radius is **radii.** All radii of any circle are equal in length.

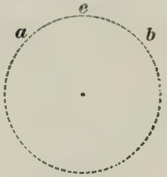

FIG. 45.

721. An **arc** of a circle, *a e b*, Fig. 45, is any part of its circumference.

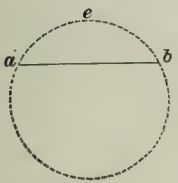

FIG. 46.

722. A **chord** is a straight line joining any two points in a circumference; or, it is a straight line joining the extremities of an arc.

Thus, in Fig. 46, *a b* is the *chord* of the arc *a e b*.

723. A **segment** of a circle is the space included between the arc and its chord.

724. A **sector** of a circle is the space included between an arc and two radii drawn to the extremities of the arc. Thus, in Fig. 47, the space included between the arc *A B* and the radii *O A* and *O B* is a sector of the circle.

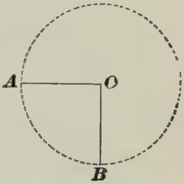

FIG. 47.

725. Two circles are equal when the *radius or diameter* of one equals the *radius or diameter* of the other.

Two arcs are equal when the *radius and chord* of one equals the *radius and chord* of the other.

726. If *A D B C*, Fig. 48, is a circle in which two diameters *A B* and *C D* are drawn at right angles to each other, then, *A O D*, *D O B*, *B O C* and *C O A* are right angles. The circumference is thus divided into four equal parts; each of these parts is called a **quadrant.**

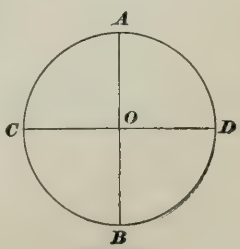

FIG. 48.

727. In Geometry, **angles** are measured by the number of right angles, or parts of a right angle, which they contain; since, in the circle, a right angle intercepts a quadrant, an angle is also measured by the number of quadrants, or parts of a quadrant, that it intercepts.

728. An angle at the center is measured by its intercepted arc.

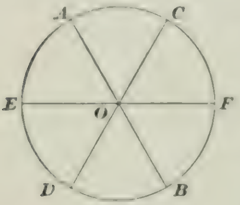

FIG. 49.

EXAMPLE.—If a circle is divided into six equal sectors, how many quadrants, or parts of a quadrant, are contained in the angle of each sector?

SOLUTION.—In Fig. 49, *A C F B D E* is a circle divided into six equal sectors. The sum of all the quadrants in the circle is 4. Hence, $4 \div 6 = \frac{2}{3}$ of a quadrant in each sector. Ans.

729. An **inscribed** angle is one whose vertex lies on the circumference of a circle, and whose sides are chords. It is measured by *one-half* the intercepted arc. Thus, in Fig. 50, *A B C* is an inscribed angle, and it is measured by one-half the arc *A D C*.

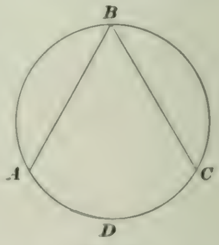

FIG. 50.

EXAMPLE.—If in the figure the arc *A D C* = $\frac{2}{3}$ of the circumference, what is the measurement of the inscribed angle *A B C*?

SOLUTION.—Since the angle is an inscribed angle, it is measured by one-half the intercepted arc, or $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$ of the circumference. The whole circumference contains four quadrants; hence, $4 \times \frac{1}{3} = \frac{4}{3}$ of a quadrant, or $\frac{4}{3}$ of a right angle; therefore, the measurement of the angle *A B C* is $\frac{4}{3}$ of a quadrant.

730. If a circle is divided into halves, each half is called a **semicircle**, and each half circumference is called a **semicircumference.**

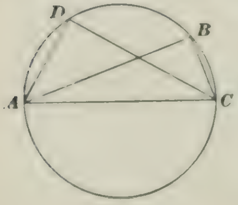

FIG. 51.

731. Any angle that is inscribed in a semicircle and intercepts a semicircumference, as *A B C*, or *A D C*, Fig. 51, is a right angle, since it is measured by one-half a semicircumference, or by a quadrant.

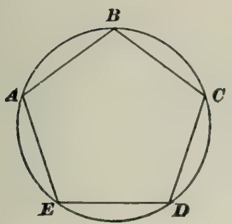

FIG. 52.

732. An **inscribed** polygon is one whose vertexes lie on the circumference of a circle, and whose sides are chords, as *A B C D E*, Fig. 52.

733. If, in any circle, a radius be drawn perpendicular to any chord, it bisects (cuts in halves) the chord. Thus, if the radius *O C*, Fig. 53, is perpendicular to the chord *A B*, *A D* = *D B*.

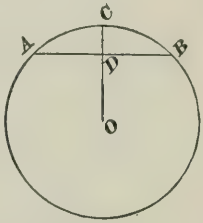

FIG. 53.

EXAMPLE.—If a regular pentagon is inscribed in a circle, and a radius is drawn perpendicular to one of the sides, what are the lengths of the two parts of the side, the perimeter of the pentagon being 27 inches?

SOLUTION.—A pentagon has five sides, and since it is a regular pentagon, all the sides are of equal lengths: the perimeter of the pentagon, which is the distance around it, and equals the sum of all the sides, or 27 inches. Therefore, the length of one side $= 27 \div 5 = 5\frac{2}{5}$ inches. Since the pentagon is an inscribed pentagon, its sides are chords, and as a radius perpendicular to a chord bisects it, we have $5\frac{2}{5} \div 2 = 2\frac{7}{10}$ inches, for the length of each of the parts of the side, cut by a radius perpendicular to it. Ans.

734. If a straight line be drawn perpendicular to any chord at its middle point, it must pass through the center of the circle.

Through any three points not in the same straight line, a circumference can be drawn. Let *A*, *B* and *C*, Fig. 54, be any three points. Join *A* and *B*, and *B* and *C*, by straight lines. At the middle point of *A B*, draw *H K* perpendicular to *A B*; at the middle point of *B C* draw *E F* perpendicular to *B C*. These two perpendiculars intersect at *O*. With *O* as a center, and *O B* as a radius, describe a circle; it will pass through *A*, *B*, and *C*.

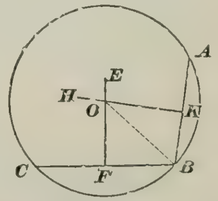

FIG. 54.

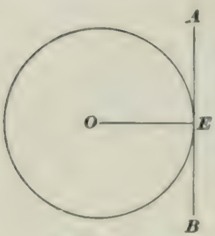

FIG. 55.

735. A **tangent** to a circle is a straight line which touches the circle at one point only; it is always perpendicular to a radius drawn to that point. Thus, in Fig. 55, $A\ B$ drawn perpendicular to the radius $O\ E$ at its extremity E, is a *tangent* to the circle.

If a straight line is perpendicular to a radius at its extremity, it is tangent to the circle.

736. If two circles intersect each other, the line joining their centers bisects at right angles the line joining the two points of intersection. If the two circles, whose centers are O and P, Fig. 56, intersect at A and B, the line $O\ P$ bisects at right angles the line $A\ B$; or $A\ C = B\ C$.

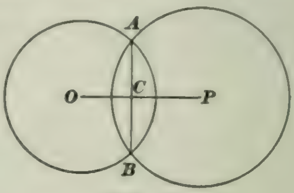

FIG. 56.

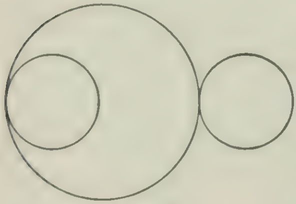

FIG. 57.

737. One circle is said to be **tangent** to another circle when they touch each other at one point only, as in Fig. 57. This point is called the **point of contact,** or the **point of tangency.**

738. When two or more circles are described from the same center, as in Fig. 58, they are called **concentric circles.**

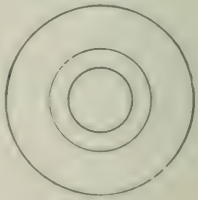

FIG. 58.

739. If, from any point on the circumference of a circle, a perpendicular be let fall upon a given diameter, this perpendicular will be a mean proportional between the two parts into which it divides the diameter.

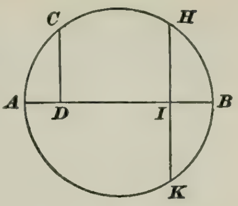

FIG. 59.

If $A\,B$, Fig. 59, is the given diameter, and C any point on the circumference, then is the perpendicular $C\,D$ a mean proportional between $A\,D$ and $D\,B$, or $A\,D : C\,D = C\,D : D\,B$.

Therefore, $\overline{C\,D}^2 = A\,D \times D\,B$,
and $C\,D = \sqrt{A\,D \times D\,B}$.

EXAMPLE.—If $H\,K = 30$ feet, and $I\,B = 8$ feet, what is the diameter of the circle, $H\,K$ being perpendicular to $A\,B$?

SOLUTION.— 30 feet ÷ 2 feet = 15 feet = $I\,H$. And
$B\,I : I\,H = I\,H : I\,A$, or $8 : 15 = 15 : I\,A$.

Therefore, $I\,A = \frac{15^2}{8} = \frac{225}{8} = 28\frac{1}{8}$ feet, and $I\,A + I\,B = 28\frac{1}{8} + 8 = 36\frac{1}{8}$ feet $= A\,B$, the diameter of the circle. Ans.

EXAMPLE.—The diameter of the circle $A\,B$ is $36\frac{1}{8}$ feet, and the distance $B\,I$ is 8 feet; what is the length of the line $H\,K$?

SOLUTION.—As the diameter of the circle is $36\frac{1}{8}$ feet, and as $B\,I$ is 8 feet, $I\,A$ is equal to $36\frac{1}{8} - 8 = 28\frac{1}{8}$ feet. Hence, $B\,I : I\,H = I\,H : I\,A$, or $8 : I\,H = I\,H : 28\frac{1}{8}$.

Therefore, $I\,H = \sqrt{8 \times 28\frac{1}{8}} = 15$ feet; and as $H\,K = I\,H + I\,K$, or $2\,I\,H$, $H\,K = 15 \times 2 = 30$ feet. Ans.

EXAMPLES FOR PRACTICE.

1. If a circle is divided into ten equal sectors, what part of a quadrant is contained in the angle of each sector? Ans. $\frac{2}{5}$ of a quadrant.

2. An angle inscribed in a circle intercepts one-fourth of the circumference. What is the size of the angle? Ans. $\frac{1}{2}$ of a right angle.

3. The perimeter of a regular inscribed octagon is 100 inches long. If a radius is drawn perpendicular to one of the sides, what are the lengths of the two parts of the side? Ans. $6\frac{1}{4}$ inches.

4. If, in Fig. 59, the diameter $A\,B = 32\frac{1}{2}$ feet, and the distance $I\,B = 8$ feet, what is the length of the chord $H\,K$? Ans. 28 feet.

5. In the same figure, if the distance $B\,I$ is 6 inches, and $H\,K$ 18 inches, what is the diameter of the circle? Ans. 19.5 inches.

TRIGONOMETRY.

740. Trigonometry is that branch of mathematics which treats of the solution of triangles.

Every triangle has six parts—three **sides** and three **angles.** If any three parts are given, one of them being a side, the other three can be found. The process of finding the unknown parts from the given parts is called the **solution** of the triangle.

741. In Trigonometry, the circumference of every circle is supposed to be divided into 360 equal parts, called **degrees;** every degree is subdivided into 60 equal parts, called **minutes,** and every minute is again divided into 60 equal parts, called **seconds.** Degrees, minutes and seconds are denoted by the symbols $^\circ$, $'$, $''$. Thus, the expression $37^\circ\ 14'\ 44''$, is read 37 degrees, 14 minutes and 44 seconds.

Since one degree is $\frac{1}{360}$ of any circumference, it follows that the length of an arc of one degree will be different in circles of different diameters, but the proportion of the length of an arc of one degree to the whole circumference will always be the same, viz., $\frac{1}{360}$ of the circumference.

Hence, in two given circles the length of an arc of 1° will be proportional to the two radii. Thus, if $A\ O\ B$, Fig. 60, is an angle of 1° on the larger circle, it is also 1° on the smaller concentric circle, and the length of the arc $A\ B$ is to the length of the arc $C\ D$ as the radius $O\ B$ is to the radius $O\ D$; or, arc $A\ B$: arc $C\ D = O\ B : O\ D$.

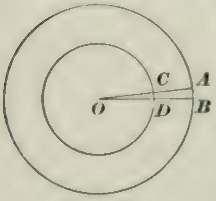

Fig. 60.

Example.—If the arc $C\ D = 2$ inches, radius $O\ D = 5$ inches, and radius $O\ B = 9$ inches, what is the length of the arc $A\ B$?

Solution.— $A\ B : 2 = 9 : 5$, or $A\ B = \frac{9 \times 2}{5} = 3\frac{3}{5}$ inches. Ans.

742. In Trigonometry, the arcs of circles are used to measure angles. All angles are supposed to have their vertexes at the center O of the circle (see Fig. 61), one side of the angle lying to the right of O, and coinciding with the horizontal diameter, as $O\ B$.

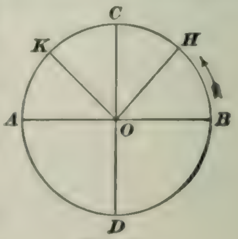

Fig. 61.

The point B on the arc is the starting point in measuring an angle; the angle is supposed to increase by moving around the circumference in the direction indicated by the arrow, until the number of degrees, minutes and seconds in the angle have been measured off on the arc. Suppose that it stops at the point *H*; draw *O H*, and *H O B* will be the angle. If *K* had been the stopping point, *K O B* would have been the angle.

Example.—A given angle equals 51°; to lay it off trigonometrically on a circle: Describe a circle with any convenient radius, and draw the horizontal diameter *A B*. From the point *B*, count off 51° in the direction of the arrow. If the 51° stops at the point *H*, draw *O H*, and *H O B* will equal 51°, and will be the required angle.

743. Since a quadrant is a fourth part of a circle, the number of degrees in a quadrant is one-fourth of 360°, or 90°. Hence, a right angle always contains 90°.

Example.—The earth turns completely around on its axis once in one day; through how many degrees does it turn in one hour?

Solution.—In one day there are 24 hours, and since the earth turns through 360° in 24 hours, in one hour it will turn through $360° \div 24 = 15°$. Ans.

744. In adding two angles together, seconds are added to seconds, minutes to minutes, and degrees to degrees; so, also, in subtracting two angles, seconds are subtracted from seconds, minutes from minutes, and degrees from degrees.

Example.—Add 75° 46′ 17″ and 14° 27′ 34″.

Solution.—

$$\begin{array}{r} 75° \; 46' \; 17'' \\ 14° \; 27' \; 34'' \\ \hline 89° \; 73' \; 51'' \end{array}$$

Since $73' = 1° \; 13'$, the 1° is added to the 89°, and the sum is then written 90° 13′ 51″. Ans.

Example.—What is the difference between 126° 14′ 20″ and 45° 28′ 13″?

Solution.—

$$\begin{array}{r} 126° \; 14' \; 20'' \\ 45° \; 28' \; 13'' \\ \hline 7'' \end{array}$$

Since 28′ cannot be taken from 14′, 1° ($= 60'$) is taken from 126° and added to the 14′, and the above is written:

$$\begin{array}{r} 125° \; 74' \; 20'' \\ 45° \; 28' \; 13'' \\ \hline 80° \; 46' \;\; 7''. \end{array} \quad \text{Ans.}$$

Example.—Subtract 49° 36′ 14″ from 90°.

Solution.—Since 1° = 60′ and 1′ = 60″, we can write, 90° = 89° 59′ 60″, and

$$\begin{array}{r} 89^\circ\ 59'\ 60'' \\ 49^\circ\ 36'\ 14'' \\ \hline 40^\circ\ 23'\ 46''. \end{array}\quad \text{Ans.}$$

Example.—Add 83° 15′ 39″ and 96° 44′ 21″.

Solution.—

$$\begin{array}{r} 83^\circ\ 15'\ 39'' \\ 96^\circ\ 44'\ 21'' \\ \hline 179^\circ\ 59'\ 60'' \end{array}$$

Since 60″ = 1′, add 1′ to 59′ making it 60′; since 60′ = 1°, add 1 to 179°, making it 180°.

Therefore, 83° 15′ 39″ + 96° 44′ 21″ = 180°. Ans.

EXAMPLES FOR PRACTICE.

1. Add 43° 0′ 59″ and 10° 59′ 40″. Ans. 54° 0′ 39″.
2. From 180° 12′ 20″ subtract 3° 12′ 56″. Ans. 176° 59′ 24″.
3. From 84° take 83° 14′ 10″, and to the result add 14′ 10″. Ans. 1°.

THE TRIGONOMETRIC FUNCTIONS.

745. A **function** of a quantity is another quantity depending upon the first one for its value. The circumference of a circle, for example, is a function of the diameter, because the length of the circumference depends upon the length of the diameter.

746. In Trigonometry, the number of degrees contained in an angle is often denoted by certain lines, called the **trigonometric functions,** whose lengths depend upon the arcs which measure the angles. These lines are the characteristic quantities of trigonometry.

The principal trigonometric functions are the *sine*, *cosine*, and *tangent*.

747. The **sine** of an angle is the line drawn from the point where one side of the angle cuts the circumference perpendicular to the other side. Thus, $A\,C$, Fig. 62, is the sine of the angle $B\,O\,A$. In making calculations, the word *sine* is abbreviated to *sin*, and instead of writing sine of $B\,O\,A$, or sine of 34° 15′, write **sin B O A,** and **sin 34° 15′.**

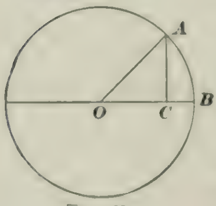

Fig. 62.

748. The **cosine** of an angle is the distance from the foot of the sine to the center of the circle. Thus, in the above figure, *O C* is the cosine of the angle *B O A*. The word *cosine* is abbreviated to *cos.* Thus, the cosine of *B O A* is written ***cos B O A.***

749. If a tangent is drawn at the right extremity of the horizontal diameter of a circle, which forms one side of an angle, and the other side of the angle is prolonged to meet it, the distance intercepted by the two sides of the angle is called the **tangent** of that angle. Thus, in Fig. 63, *D B* is the tangent of *B O A*. The word *tangent* is abbreviated to *tan.* Thus, the tangent of *B O A* is written ***tan B O A.***

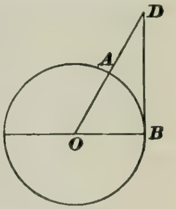

FIG. 63.

750. If a tangent is drawn from the upper extremity of a vertical diameter of a circle, whose horizontal diameter forms one side of an angle, and the other side of the angle is prolonged until it meets this tangent, the distance intercepted on this tangent between the extremity of the vertical diameter and the prolonged line is called the **cotangent** of that angle. Thus, *E F*, Fig. 64, is the cotangent of the angle *B O A*. The word *cotangent* is abbreviated to *cotan.* Thus, the cotangent of *B O A* is written ***cotan B O A.***

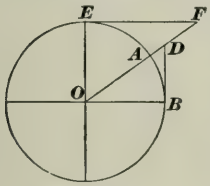

FIG. 64.

These abbreviations must always be pronounced in full. Thus, cos 14° 22′ 46″, is pronounced *cosine of fourteen degrees, twenty-two minutes and forty-six seconds;* tan 45° is pronounced *tangent of forty-five degrees.*

EXAMPLE.—Given, an angle = 60°, to draw its sine, cosine, tangent, and cotangent.

SOLUTION.—Describe any circle, *H E B G*, Fig. 65, and draw the horizontal diameter *H B*, and the vertical diameter *G E*. From *B*, mark off *B A* = 60°, and draw *O A*; then, *B O A* = 60°. At *A*, draw *A C* perpendicular to *O B*. *A C* = sin 60°, and *O C* = cos 60°.

Draw a tangent at *B*, and prolong *O A* until *O A* intersects the tangent at *D*; then *D B* = tan 60°.

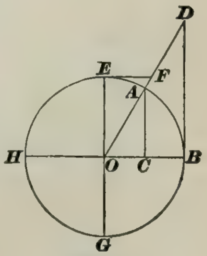

FIG. 65.

Draw a tangent at E, and prolong it until it meets $O\,A$ prolonged at F; then, $E\,F$ = cotan 60°. Ans.

751. The sine, cosine, tangent, and cotangent of the *same* angle in circles of different radii are proportional to the radii. In a circle whose radius is 3, for example, they are three times as great as the sine, cosine, tangent, and cotangent of the *same* angle in a circle whose radius is 1. Thus, in Fig. 66, the angles $B\,O\,A$ and $F\,O\,H$ are the same; but if the radius $O\,B$ is three times as long as the radius $O\,F$, then, by the principle of similar triangles (see Art. **716**), it can easily be shown that the sine $A\,C$ is three times as long as the sine $H\,K$; that the cosine $O\,C$ is three times as long as the cosine $O\,K$; that the tangent $D\,B$ is three times as long as the tangent $E\,F$, and that the cotangent $R\,N$ is three times as long as the cotangent $L\,M$. Hence, we have the two important principles:

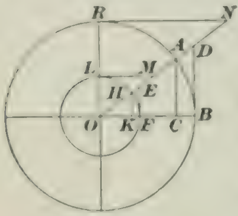

Fig. 66.

If the values of the sine, cosine, tangent, and cotangent are known for any angle in a circle whose radius is 1, their values for the **same** *angle in any other circle can be obtained by multiplying their known values in a circle whose radius is 1, by the radius of the required circle.*

Conversely, if the values of the sine, cosine, tangent, and cotangent are known for an angle in a circle whose radius is other than 1, their values for the **same** *angle in a circle whose radius is 1 can be obtained by dividing the known values by the radius of the given circle.*

EXAMPLE.—Sin 28° 21′ to a radius 1 = .47486; what is the sine of the same angle to a radius 4½?

SOLUTION.— .47486 × 4½ = 2.13687. Ans.

EXAMPLE.—If tan 19° 35′ to a radius 5 = 1.77880, what does the tangent of the same angle to a radius 1 equal?

SOLUTION.— 1.77880 ÷ 5 = .35576. Ans.

752. To facilitate calculations, tables of the trigonometric functions are employed. These tables give the sine, cosine, tangent, cotangent, etc., of the degrees and minutes in a circle whose radius is 1. Hence, when the sine, cosine, tangent, or cotangent is given for an angle in a circle whose radius is greater or less than 1, its value must be found for a circle whose radius is 1, before the table can be used.

For example, suppose we have measured the sine of a certain angle belonging to a circle whose radius is 20 feet, and found that it measures 9.4972 feet, and wanted to find how many degrees the angle contained. In this case the sine of the angle for a circle whose radius is 20 feet = 9.4972 feet. By dividing 9.4972 by 20, we have $\frac{9.4972}{20} = .47486$, which is the sine of the *same* angle in a circle whose radius is 1; and by looking in a table containing the natural sines of angles for different degrees and minutes, we find opposite .47486, 28° 21′, which is the size of the angle. The method of using the table of natural sines, cosines, tangents, and cotangents will be explained later.

753. The *application* of the trigonometric functions is made in right-angled triangles. Thus, in Fig. 67, $A\ B\ C$ is a right-angled triangle, right-angled at C. If A be taken as a center, and $A\ B$ as a radius, and an arc $D\ B\ F$ be described, $B\ C$ will be the *sine* of the angle $B\ A\ C$, and $A\ C$ will be the *cosine* of $B\ A\ C$. If, with the same center A, but with $A\ C$ as a radius, an arc $C\ H\ E$ be described, $B\ C$ will be the *tangent* of $B\ A\ C$.

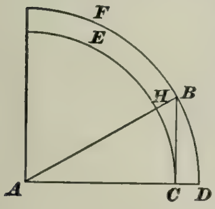

FIG. 67.

754. To show the method of using the sine, cosine and tangent, the following cases will be considered:

CASE I.—Suppose we have a right-angled triangle $A\ B\ C$, Fig. 68, right-angled at C, and that we know the lengths of $A\ B$ and $B\ C$, and wish to find the length of $A\ C$, and the angles A and B.

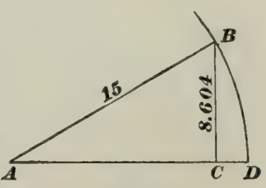

FIG. 68.

As $A\ C\ B$ is a right-angled triangle, we can calculate $A\ C$ by subtracting the square of $B\ C$ from the square of $A\ B$, and extracting the square root of the remainder, or $A\,C = \sqrt{\overline{A\,B}^2 - \overline{B\,C}^2}$. (See Art. **714.**) But we should not know the angles A and B. Hence, we proceed as follows: With A as a center, and $A\,B$ as a radius, describe the arc $B\,D$; then, $B\,C$ is the sine of the angle A, and $A\,B$ is the radius of the circle. Suppose that $A\,B = 15$ feet, and $B\,C = 8.604$ feet. Now, reduce $B\,C$ to a circle whose radius is 1, by dividing 8.604 by 15, or $\frac{8.604}{15} = .5736 =$ sine of angle A, in a circle whose radius is 1.

Looking in a table of natural sines, we find opposite .5736, 35°; hence, angle $A = 35°$. Now, observe from the figure that $A\,B$ is the hypotenuse, and $B\,C$ is the side opposite the angle we wish to find; then, since we divided $B\,C$ by $A\,B$, to reduce the sine to a circle whose radius is 1, we have

Rule 1.—$\frac{\textit{side opposite}}{\textit{hypotenuse}}$ = *sine of angle in a circle whose radius is 1.*

Since angle $A +$ angle $B = 90°$; angle $B = 90° - 35° = 55°$. Having found the angle B, the side $A\,C$ can be easily found by referring to a table of natural sines, in which the sine of 55° will be found to be equal to .81915.

From rule **1**, we have $\frac{\text{side opposite}}{\text{hypotenuse}} =$ sine of angle, or since B is an angle of 55°, and $A\,C$ is the side opposite, $\frac{A\,C}{15} = .81915$, or $A\,C = .81915 \times 15 = 12.29$ feet. We now know all of the sides and angles.

Since $A\,C = .81915 \times 15 =$ sine of the angle $B \times$ hypotenuse, and $A\,C =$ side opposite, we have,

Rule 2.—*Side opposite = hypotenuse × sine.*

CASE II.—Suppose that in a right-angled triangle, $A\,B\,C$, Fig. 69, right-angled at C, the side $A\,C = 14$, and the hypotenuse $A\,B = 18$, had been given, to find $B\,C$, and the angles A and B. $B\,C$ is called the **side opposite** the angle A, and $A\,C$ is called the **side adjacent.** With A as a center,

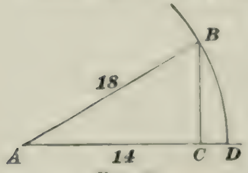

FIG. 69.

and $A\ B$ as a radius, describe the arc $B\ D$. Then $A\ C$ is the cosine of the angle A to a radius $A\ B = 18$; cos A to a radius $1 = 14 \div 18 = .77778$. Referring to a table of natural cosines, .77778 will be found to be the cos of 38° 56′ 32″.

Hence, the angle $A = 38°\ 56'\ 32''$. To find B, subtract A from 90°.

$$\begin{array}{rr} 90° = & 89°\ 59'\ 60'' \\ & 38°\ 56'\ 32'' \\ \hline & 51°\ \ \ 3'\ 28'' \end{array}$$

Therefore, $B = 51°\ 3'\ 28''$.

But since $14 \div 18 = A\ C \div A\ B$, we have

Rule 3.— $\frac{\textit{side adjacent}}{\textit{hypotenuse}} = \textit{cosine.}$

$B\ C$ is side opposite the angle A; therefore, since side opposite = hypotenuse × sine (rule **2**), $B\ C = 18 \times \sin 38°\ 56'\ 32'' = 18 \times .62853 = 11.31$.

Since cosine $= \frac{\text{side adjacent}}{\text{hypotenuse}}$ (rule **3**), we have

Rule 4.—*Side adjacent = hypotenuse × cosine.*

CASE III.—In Fig. 70, $A\ C\ B$ is a right-angled triangle, right-angled at C. Let $B\ C = 9$, and $A\ C = 16$. To find the hypotenuse $A\ B$, and angles A and B. In the preceding cases, the hypotenuse was given; A was taken as a center, and the hypotenuse $A\ B$ as a radius. But since, in the present case, the hypotenuse is unknown, we cannot use it as in rules **1**, **2**, **3**, and **4**. So we take the same center A, and $A\ C$ as a radius, and describe the arc $C\ D$. Then, $B\ C$ is the tangent of the angle A to a radius 16, and $9 \div 16 = .5625$, or tangent A to a radius 1. In a table of natural tangents we find .5625 opposite 29° 21′ 28″.

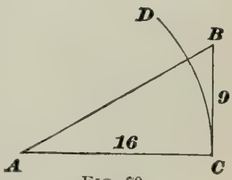

FIG. 70.

Therefore, $A = 29°\ 21'\ 28''$, and $B = 90° - 29°\ 21'\ 28'' = 60°\ 38'\ 32''$.

To find $A\ B$, we use rule **2**, side opposite = hypotenuse × sine, or $9 = \sin 29°\ 21'\ 28'' \times$ hypotenuse. In the table of natural sines, we find that $\sin 29° \, 21' \, 28'' = .49026$. $9 = .49026 \times \text{hypotenuse}$, or $\text{hypotenuse} = \frac{9}{.49026} = 18.36$.

Since $BC = 9$, or side opposite, $AC = 16$, or side adjacent, and $9 \div 16 = BC \div AC$, or tangent, we have

Rule 5.—*Tangent* $= \frac{\textit{side opposite}}{\textit{side adjacent}}$; also,

Rule 6.—*Side opposite = tangent × side adjacent.*

We also have

Rule 7.—*Cotangent* $= \frac{\textit{side adjacent}}{\textit{side opposite}}$; and

Rule 8.—*Side adjacent = cotangent × side opposite.*

By means of these eight rules, we can find the side and angles of any triangle, when three of its parts are given, one part being a side.

They should be *memorized*, and for convenience are again given here.

Rule 1.—*Sine* $= \frac{\textit{side opposite}}{\textit{hypotenuse}}$.

Rule 2.—*Side opposite = hypotenuse × sine.*

Rule 3.—*Cosine* $= \frac{\textit{side adjacent}}{\textit{hypotenuse}}$.

Rule 4.—*Side adjacent = hypotenuse × cosine.*

Rule 5.—*Tangent* $= \frac{\textit{side opposite}}{\textit{side adjacent}}$.

Rule 6.—*Side opposite = side adjacent × tangent.*

Rule 7.—*Cotangent* $= \frac{\textit{side adjacent}}{\textit{side opposite}}$.

Rule 8.—*Side adjacent = cotangent × side opposite.*

To these may be added two more:

Rule 9.—*Hypotenuse* $= \frac{\textit{side opposite}}{\textit{sine}}$.

Rule 10.—*Hypotenuse* $= \frac{\textit{side adjacent}}{\textit{cosine}}$.

When solving triangles by means of these rules, and the hypotenuse is given or required, use rule **1, 2, 3, 4, 9,** or **10;** but, when the two sides about the right angle are given, use rules **5** and **6** or **7** and **8.**

EXAMPLES FOR PRACTICE.

1. Given, an angle = 45°, to draw its sine, cosine and tangent.

2. Given, the tangent of an angle in a circle 3 inches in diameter = 2 inches. Draw a figure, and from it find the sine of the same angle in a circle whose radius is twice as great.

3. The cosine of an angle in a circle whose radius is 1 inch is .9654 inch. (*a*) What is the cosine of the same angle in a circle whose radius measures $3\frac{1}{2}$ inches? (*b*) What in a circle whose radius is .63 inch?

Ans. { (*a*) 3.3789 in.
(*b*) .6082 in.

TRIGONOMETRIC TABLES.

In the foregoing rules, the sines, cosines, tangents, and cotangents are all to be taken from printed tables, which have previously been referred to. There are two kinds of tables in general use; the first is a table of natural sines, cosines, tangents, etc., and gives their actual values for a circle whose radius is 1. In other tables, logarithms are used. The first is the table that we shall employ, and the method of using it will now be explained.

755. Given, an angle, to find its sine, cosine, tangent, and cotangent:

Example.—Let it be required to find the sine, cosine, tangent and cotangent of an angle of 37° 24′.

Solution.—Look in the table of **natural sines** along the tops of the pages and find 37°. The left-hand column is marked (′), meaning that the minutes are to be sought in that column, and begin with 0, 1, 2, 3, etc., up to 60. Glancing *down* this column until 24 is found, find opposite this 24 in the column marked *sine*, and headed 37°, the number .60738; then .60738 = sin 37° 24′ to a radius 1. In exactly the same manner, find in the column marked *cosine*, and headed 37°, the number .79441, which corresponds to cos 37° 24′; or cos 37° 24′ = .79441. So, also, find in the column marked *tangent*, and headed 37°, and opposite 24′, the number .76456; whence, tan 37° 24′ = .76456. Finally, find in the column marked *cotangent* and headed 37°, and opposite 24′, the number 1.30795; whence, cotan 37° 24′ = 1.30795.

756. In most of the tables published, the angles run only from 0° to 45°, at the heads of the columns; to find an angle greater than 45°, *look at the bottom of the page and glance upwards*, ***using the extreme right-hand column to find minutes,*** which begin with 0 at the bottom and run upwards, 1, 2, 3, etc., up to 60.

EXAMPLE.—Find the sine, cosine, tangent, and cotangent of 77° 43′.

SOLUTION.—Since this angle is greater than 45°, look along the bottom of the tables, until the column marked *sine*, and headed 77°, is found. Glancing up the column of minutes on the *right*, until 43 is found, find opposite 43′ in the column marked *sine* at the bottom, and headed 77°, the number .97711; this is the sine of 77° 43′, or sin 77° 43′ =.97711. Similarly, in the column marked *cosine*, and headed 77°, find opposite 43′, in the right-hand column, the number .21275; this is the cosine of 77° 43′, or cos 77° 43′ = .21275. So, also, find that 4.59283 is the tangent of 77° 43′, or tan 77° 43′ = 4.59283. Finally, find in the same manner, that the cotangent of 77° 43′, or cotan 77° 43′ = .21773.

Let it be required to find the sine of 14° 22′ 26″.

EXPLANATION.—The sine of 14° 22′ 26″, lies between the sine 14° 22′ and 14° 23′. Sin 14° 22′ = .24813; sin 14° 23′ = .24841; difference = .00028. Hence, there are 28 parts between sin 14° 22′ and sin 14° 23′, or 28 parts corresponding to a difference of 1′. Now, since 1′ = 60″, the number of parts between sin 14° 22′ and sin 14° 22′ 26″, that is, the number of parts corresponding to a difference of 26″, must be $\frac{26}{60} \times 28 = 12.1$. Since .1 is less than .5, omit it, and we have 12 as the number of parts to be added to the sine of 14° 22′ to produce the sine of 14° 22′ 26″.

Therefore, sin 14° 22′ 26″ = .24813 + .00012 = .24825.

By referring to the table of sines, cosines, tangents, and cotangents, it will be observed that, as the angles increase in size, the sines and tangents increase, while the *cosines and cotangents decrease.* In the above example, therefore, had it been required to find the *cosine* or the *cotangent* of 14° 22′ 26″, the correction for the 26″ would have been *subtracted* from the cosine or the cotangent of 14° 22′, instead of added to it. From the foregoing, we have, to find the sine, cosine, tangent, or cotangent of an angle containing seconds, the following rule:

Rule 11.—*Find in the table the sine, cosine, tangent, or cotangent corresponding to the degrees and minutes of the angle.*

For the seconds, find the difference of the values of the sine, cosine, tangent, or cotangent, taken from the table between which the seconds of the angle fall; multiply this difference *by a fraction whose numerator is the number of seconds in the given angle, and whose denominator is 60.*

If sine or tangent, add this correction to the value first found; if cosine or cotangent, subtract the correction.

EXAMPLE.—Find the sine, cosine, tangent, and cotangent of 56° 43′ 17″.

SOLUTION.—Sin 56° 43′ = .83597. Sin 56° 44′ = .83613. Since 56° 43′ 17″ is greater than 56° 43′ and less than 56° 44′, the value of the sine of the angle lies between .83597 and .83613; the difference = .83613 − .83597 = .00016; multiplying this by the fraction $\frac{17}{60}$, $.00016 \times \frac{17}{60} = .00005$, nearly, which is to be *added* to .83597, the value first found, or .83597 + .00005 = .83602. Hence, sin 56° 43′ 17″ = .83602. Ans.

Cos 56° 43′ = .54878; cos 56° 44′ = .54854; the difference = .54878 − .54854 = .00024, and $.00024 \times \frac{17}{60} = .00007$, nearly. Now, since the *cosine* is desired, we must *subtract* this correction from cos 56° 43′ or .54878; subtracting, .54878 − .00007 = .54871. Hence, cos 56° 43′ 17″ = .54871. Ans.

Tan 56° 43′ = 1.52332; tan 56° 44′ = 1.52429; the difference = .00097, and $.00097 \times \frac{17}{60} = .00027$, nearly. Since the tangent is desired, we must *add*, giving 1.52332 + .00027 = 1.52359. Hence, tan 56° 43′ 17″ = 1.52359. Ans.

Cotan 56° 43′ = .65646; cotan 56° 44′ = .65604; the difference = .00042, and $.00042 \times \frac{17}{60} = .00012$, nearly. Since the cotangent is desired, we must *subtract*, giving .65646 − .00012 = .65634. Hence, cotan 56° 43′ 17″ = .65634. Ans.

757. Given, the sine, cosine, tangent, or cotangent, to find the angle corresponding:

EXAMPLE.—The sine of an angle is .47486; what is the angle*?

SOLUTION.—Consulting the table of natural sines, glance down the columns marked *sine*, until .47486 is found opposite 21′, in the left-hand column, and under the column headed 28°. Therefore, the angle whose sine = .47486 is 28° 21′, or sin 28° 21′ = .47486. Ans.

EXAMPLE.—Find the angle whose cosine is .27032.

SOLUTION.—Looking in the columns marked *cosine*, at the top of the page, it is not found; hence, the angle is greater than 45°; consequently, looking in the columns marked *cosine* at the bottom of the page, it is found opposite 19′, in the *right-hand* column of minutes, and in the column headed 74° at the bottom. Therefore, the angle whose cosine is .27032, is 74° 19′, or cos 74° 19′ = .27032. Ans.

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*NOTE.—Whenever the sine, cosine, tangent, or cotangent of an angle is given, and no radius is specified, the radius is always understood to be 1.

EXAMPLE.—Find the angle whose tangent is 2.15925.

SOLUTION.—On searching a table of natural tangents, it is found to belong to an angle greater than 45°, so it must be looked for in the column marked *tangent* at the bottom. It is found opposite 9′, in the right-hand column of minutes, and in the column headed 65° at the bottom. Therefore, tan 65° 9′ = 2.15925. Ans.

EXAMPLE.—Find the angle whose cotangent is .43412.

SOLUTION.—From the table of natural cotangents, it is found that this value is less than the cotangent of 45°, so it must be found in the column marked *cotangent* at the bottom. Looking there, it is found in the column headed 66° at the bottom, and opposite 32′, in the right-hand column of minutes. Therefore, the angle whose cotangent is .43412, is 66° 32′, or cotan 66° 32′ = .43412. Ans.

Let it be required to find the angle whose sine is .42531.

EXPLANATION.—Referring to the table of sines, this number is found to lie between .42525, the sine of 25° 10′, and .42552, the sine of 25° 11′. The difference between these two numbers = .42552 − .42525 = .00027, or 27 parts; the difference between .42525, the sine of 25° 10′, and .42531, the sine of the given angle, = .42531 − .42525 = .00006, or 6 parts. Therefore, the angle whose sine is .42531, is greater than 25° 10′ by 6 of the 27 parts between sin 25° 10′ and sin 25° 11′. Hence, the angle whose sine = .42531 = 25° $10\frac{6}{27}'$. Since 1′ = 60″, $\frac{6}{27}$ minute = $\frac{6}{27} \times 60 = 13.3''$. Therefore, the angle whose sine is .42531 = 25° 10′ 13.3″.

In this case, the correction is added for the cosine and cotangent, as well as for the sine and tangent, because for any difference between the given sine, cosine, tangent, or cotangent, and the one belonging to the angle next lower, the size of the *angle* always *increases* a corresponding amount.

758. To find the angle corresponding to a given sine, cosine, tangent, or cotangent, whose exact value is not contained in the table:

Rule 12.—*Find the difference of the two numbers in the table between which the given sine, cosine, or tangent falls, and use the number of parts in this difference as the denominator of a fraction.*

Find the difference between the number belonging to the **smaller angle,** *and the given sine, cosine, tangent, or cotangent, and use the number of parts in the difference just found as the numerator of the fraction mentioned above. Multiply this fraction by 60, and the result will be the number of seconds to be added to the* **smaller angle.**

EXAMPLE.—Find the angle whose sine = .57698.

SOLUTION.—Looking in the table of natural sines, in the columns marked *sine*, it is found between .57691 = sin 35° 14′ and .57715 = 35° 15′. The difference between them is .57715 − .57691 = .00024, or 24 parts. The difference between the sine of the smaller angle, or sin 35° 14′ = .57691, and the given sine, or .57698, is .57698 − .57691 = .00007, or 7 parts.

Then, $\frac{7}{24} \times 60 = 17.5''$, and the angle = 35° 14′ 17.5″, or sin 35° 14′ 17.5″ = .57698. Ans.

EXAMPLE.—Find the angle whose cosine is .27052.

SOLUTION.—Looking in the table of cosines, it is found to belong to a greater angle than 45° and, hence, must be sought for in the columns marked *cosine*, at the bottom of the page. It is found between the numbers .27060 = cos 74° 18′ and .27032 = cos 74° 19′. The difference between the two is .27060 − .27032 = .00028, or 28 parts. The cosine of the *smaller angle*, or 74° 18′, is .27060, and the difference between this and the given cosine is .27060 − .27052 = .00008, or 8 parts.

Hence, $\frac{8}{28} \times 60 = 17.14''$, nearly, and the angle whose cosine is .27052 = 74° 18′ 17.14″, or cos 74° 18′ 17.14″ = .27052. Ans.

EXAMPLE.—Find the angle whose tangent is 2.15841.

SOLUTION.—2.15841 falls between 2.15760 = tan 65° 8′, and 2.15925 = tan 65° 9′. The difference = 2.15925 − 2.15760 = .00165, or 165 parts. 2.15841 − 2.15760 = .00081, or 81 parts. $\frac{81}{165} \times 60 = 29.5''$, nearly, and the angle whose tangent is 2.15841 = 65° 8′ 29.5″, or tan 65° 8′ 29.5″ = 2.15841.

EXAMPLE.—Find the angle whose cotangent is 1.26342.

SOLUTION.— 1.26342 falls between 1.26395 = cotan 38° 21′, and 1.26319 = cotan 38° 22′. The difference = 1.26395 − 1.26319 = .00076. 1.26395 − 1.26342 = .00053. $\frac{53}{76} \times 60 = 41.9$, nearly, and the angle whose cotangent is 1.26342 = 38° 21′ 41.9″, or cotan 38° 21′ 41.9″ = 1.26342.

EXAMPLES FOR PRACTICE.

1. Find (*a*) the sine, (*b*) cosine and (*c*) tangent of 48° 17′.

Ans. $\begin{cases} (a)\ .74644. \\ (b)\ .66545. \\ (c)\ 1.12172. \end{cases}$

2. Find the (*a*) sine, (*b*) cosine, and (*c*) tangent of 13° 11′ 6″.

Ans. (*a*) .22810.
(*b*) .97364.
(*c*) .23427.

3. Find the (*a*) sine, (*b*) cosine, and (*c*) tangent of 72° 0′ 1.8″.

Ans. (*a*) .95106.
(*b*) .30901.
(*c*) 3.07777.

4. (*a*) Of what angle is .26489 the sine? (*b*) Of what is it the cosine?

Ans. (*a*) 15° 21′ 37.2″.
(*b*) 74° 38′ 22.8″.

5. (*a*) Of what angle is .68800 the sine? (*b*) Of what the cosine? (*c*) Of what the tangent?

Ans. (*a*) 43° 28′ 20″.
(*b*) 46° 31′ 40″.
(*c*) 34° 31′ 40.5″.

THE SOLUTION OF TRIANGLES.

759. As previously stated, every triangle has six parts, three sides and three angles, and if any three parts are given, one of them being a side, the other three may be found.

In right-angled triangles, it is only necessary to know *two* parts in addition to the right angle, one of which must be a side.

Rules **1** to **12** are sufficient for solving all cases.

RIGHT-ANGLED TRIANGLES.

The method of solving right-angled triangles has already been explained, but a few additional examples will be considered. There are two cases:

Case I.—*When the two given parts are a side and an angle:*

EXAMPLE.—In Fig. 71, the hypotenuse *A B* of the right-angled triangle *A C B* is 24 feet, and the angle *A* is 29° 31′, to find the sides *A C* and *B C*, and the angle *B*.

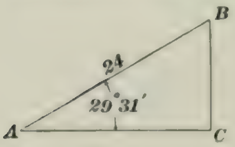

FIG. 71.

NOTE.—When working examples of this kind, construct the figure, and mark the known parts. This is a great help in solving the example. Hence, in the figure draw the angle *A* to represent an angle of 29° 31′, and complete the right-angled triangle *A C B*, right-angled at *C*, as shown. Mark the angle *A* and the hypotenuse, as is done in the figure.

SOLUTION.—Angle *B* = 90° − 29° 31′ = 60° 29′. To find *A C*, use rule **4**; viz., *A C*, or side adjacent = hypotenuse × cosine = 24 × cos 29° 31′ = 24 × .87021 = 20.89 feet, nearly.

To find BC, use the same rule; thus, $BC = 24 \times \cos 60° 29' = 24 \times .49268 = 11.82$ feet, nearly. To find BC, rule **2** could also have been used, viz., side opposite = hypotenuse × sine, or $BC = 24 \times \sin 29° 31' = 24 \times .49268 = 11.82$ feet, nearly.

$$\text{Ans.} \left\{ \begin{array}{l} \text{Angle } B = 60° 29'. \\ \text{Side } AC = 20.89 \text{ ft.} \\ \text{Side } BC = 11.82 \text{ ft.} \end{array} \right.$$

Example.—One side of a right-angled triangle, ABC, Fig. 72, is 37 feet 7 inches long; the angle opposite is 25° 33′ 7″. How long is the hypotenuse, the side adjacent, and what is the other angle?

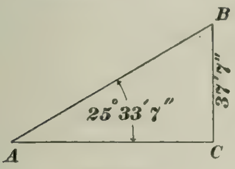

Fig. 72.

Solution.—Angle $B = 90° - 25° 33' 7'' = 64° 26' 53''$.

To find the hypotenuse, use rule **9**,

$$\text{hypotenuse} = \frac{\text{side opposite}}{\text{sine}}.$$

Since the side opposite is given in feet and inches, both must be reduced to feet, or both to inches.

7 in. $= \frac{7}{12}$ of a foot $= .583 +$ of a foot, and $BC = 37.583$ ft.

Therefore, the hypotenuse $= \dfrac{37.583}{\sin 25° 33' 7''} = \dfrac{37.583}{.43133} = 87.133$ feet = 87 ft. 2 in., nearly.

To find the side AC, use rule **4**, side adjacent = hypotenuse × cosine $= 87.133 \times \cos 25° 33' 7'' = 87.133 \times .90219 = 78.61$ feet = 78 ft. $7\frac{1}{4}$ in., nearly.

$$\text{Ans.} \left\{ \begin{array}{l} \text{Angle } B = 64° 26' 53''. \\ AC = 78 \text{ ft. } 7\tfrac{1}{4} \text{ in.} \\ AB = 87 \text{ ft. } 2 \text{ in.} \end{array} \right.$$

The work involved in finding the sine and cosine of 25° 33′ 7″, in the above example, is as follows: Sin 25° 33′ = .43130; sin 25° 34′ = .43156; difference = .00026; $.00026 \times \frac{7}{60} = .00003$. Hence, $\sin 25° 33' 7'' = .43130 + .00003 = .43133$.

Cos 25° 33′ = .90221; cos 25° 34′ = .90208; difference = .00013; $.00013 \times \frac{7}{60} = .00002$, nearly. Hence, $\cos 25° 33' 7'' = .90221 - .00002 = .90219$.

Case II.—*When two sides are given:*

Example.—In the right-angled triangle, ABC, Fig. 73, right-angled at C, $AC = 18$, and $BC = 15$, to find AB and the angles A and B.

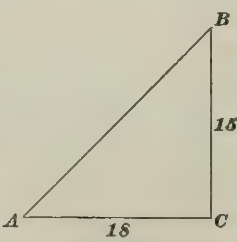

Fig. 73.

Solution.—Since the hypotenuse is not given, use rule **5**, viz.:

$$\text{Tangent } A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{15}{18} = .83333.$$

To find the angle whose tangent is .83333, we have: Tangent of next less angle = .83317 = tan 39° 48′; of the next greater angle = .83366; difference = .00049. The difference between .83317, the tangent of the smaller angle, and .83333, the given tangent, = .83333 − .83317 = .00016. Hence, $\frac{16}{49} \times 60 = 19.6''$, and the angle whose tangent is $.83333 = 39° 48' 19.6'' =$ angle A.

Angle $B = 90° - 39° 48' 19.6'' = 50° 11' 40.4''$.

To find the hypotenuse $A\,B$, use rule **9.**

$$\text{Hypotenuse} = \frac{\text{side opposite}}{\text{sine}} = \frac{15}{\sin 39° 48' 19.6''} = \frac{15}{.64018} = 23.43.$$

$$\text{Ans.} \begin{cases} \text{Angle } A = 39° 48' 19.6''. \\ \text{Angle } B = 50° 11' 40.4''. \\ A\,B = 23.43. \end{cases}$$

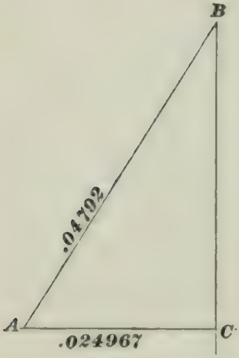

FIG. 74.

EXAMPLE.—In the right-angled triangle, $A\,B\,C$, Fig. 74, right-angled at C, $A\,C = .024967$ mile, and $A\,B = .04792$ mile; to find the other parts.

SOLUTION.—To find angle A, use rule **3.**

$$\text{Cosine } A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{.024967}{.04792} = .52101.$$

The angle whose cosine is $.52101 = 58° 36' =$ angle A. Angle $B = 90° - 58° 36' = 31° 24'$.

To find side $B\,C$, use rule **6.**

Side opposite $A =$ side adjacent × tan A, or $B\,C = .024967 \times 1.63826 = .0409$ mile.

$$\text{Ans.} \begin{cases} \text{Angle } A = 58° 36'. \\ \text{Angle } B = 31° 24'. \\ B\,C = .0409 \text{ mile}. \end{cases}$$

EXAMPLE.—In the right-angled triangle, $A\,B\,C$, Fig. 75, right-angled at C, $A\,B = 308$ feet, and $B\,C = 234$ feet; to find the other parts.

SOLUTION.—To find angle A, use rule **1.**

$$\text{Sine } A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{234}{308} = .75974.$$

The angle whose sine is $.75974 = 49° 26' 28''$, nearly, = angle A. Angle $B = 90° - 49° 26' 28'' = 40° 33' 32''$.

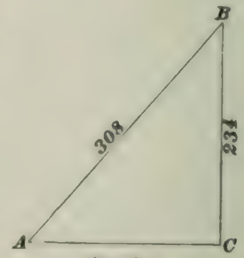

FIG. 75.

To find $A\,C$, use rule **8.**

Side adjacent $A =$ cotan A × side opposite, or $A\,C = .85586 \times 234 = 200.27$ feet.

$$\text{Ans.} \begin{cases} \text{Angle } A = 49° 26' 28''. \\ \text{Angle } B = 40° 33' 32''. \\ A\,C = 200.27 \text{ feet}. \end{cases}$$

EXAMPLES FOR PRACTICE.

1. In a right-angled triangle $A B C$, right-angled at C, the hypotenuse $A B = 40$ inches, and angle $A = 28° 14' 14''$. Solve the triangle.

$$\text{Ans.} \begin{cases} \text{Angle } B = 61° 45' 46''. \\ A C = 35.24 \text{ in.} \\ B C = 18.92 \text{ in.} \end{cases}$$

2. In a right-angled triangle $A B C$, right-angled at C, the side $B C =$ 10 feet 4 inches. If angle $A = 26° 59' 6''$, what do the other parts equal?

$$\text{Ans.} \begin{cases} \text{Angle } B = 63° 0' 54''. \\ A B = 22 \text{ ft. } 9\tfrac{1}{4} \text{ in., nearly.} \\ A C = 20 \text{ ft. } 3\tfrac{1}{2} \text{ in., nearly.} \end{cases}$$

3. In a right-angled triangle, $A B C$, the hypotenuse $A B = 60$ feet, and the side $A C = 22$ feet. Solve the triangle.

$$\text{Ans.} \begin{cases} \text{Angle } A = 68° 29' 22.2''. \\ \text{Angle } B = 21° 30' 37.8''. \\ B C = 55.82 \text{ ft.} \end{cases}$$

4. In a right-angled triangle $A B C$, right-angled at C, side $A C =$ .364 foot and side $B C = .216$ foot. Solve the triangle.

$$\text{Ans.} \begin{cases} \text{Angle } A = 30° 41' 7.5''. \\ \text{Angle } B = 59° 18' 52.5''. \\ A B = .423 \text{ ft.} \end{cases}$$

OBLIQUE TRIANGLES.

760. When three parts of *any* triangle are given, one being a side, the remaining parts can be found by means of right-angled triangles, by drawing a perpendicular from one angle to the opposite side. The parts of these triangles can then be computed, and from them the parts of the required triangle can be found.

761. CAUTION.—*When dividing the triangle into two right-angled triangles, care must be taken that the perpendicular be so drawn that one of the right-angled triangles will have two known parts besides the right angle; otherwise the triangle cannot be solved.*

Case I.—*When the three known parts are a side and two angles, or an angle and two sides:*

EXAMPLE.—In Fig. 76, the angle $A = 46° 14'$, the angle $B = 88° 24' 11''$, and the side $A B = 21$ in.; to find $A C$, $B C$, and the angle C.

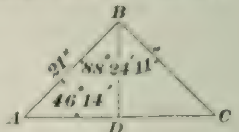

FIG. 76.

SOLUTION.—Since the sum of all the angles of any triangle = 2 right angles, or 180°, we can find the angle C by adding the two known angles, and subtracting their sum from 180°.

$$88° 24' 11'' + 46° 14' = 134° 38' 11''.$$

$$180° - 134° 38' 11'' = 45° 21' 49'' = C.$$

From the vertex B, draw $B D$ perpendicular to $A C$. The triangle $A B C$ is now divided into two right-angled triangles, $A D B$ and $B D C$.

In the right-angled triangle $A D B$, the angle A, the right angle D, and the hypotenuse $A B$ are known; to find $B D$ and $A D$. Use rule **2.** Side opposite or $B D = 21 \times \sin 46° 14' = 21 \times .72216 = 15.17$ in., nearly.

Use rule **4.** Side adjacent, or $A D = 21 \times \cos 46° 14' = 21 \times .69172$, or $A D = 14.53$ in., nearly.

In the right-angled triangle $B D C$, the angle C and the side opposite, or $B D$, are known; to find $B C$ and $D C$.

Use rule **9.** Hypotenuse, or $B C = \frac{B D}{\sin 45° 21' 49''} = \frac{15.17}{.71158} =$ 21.32 in., nearly.

Use rule **4.** $C D$, or side adjacent $= 21.32 \times \cos 45° 21' 49'' = 21.32 \times .70261 = 14.98$ in.

Since $A D + D C = A C$, we have $14.53 + 14.98 = 29.51$ in. $= A C$.

$$\text{Ans.} \begin{cases} A C = 29.51 \text{ in.} \\ B C = 21.32 \text{ in.} \\ \text{Angle } C = 45° 21' 49''. \end{cases}$$

If, in the above example, the angle C had been given instead of the angle A, the dividing line should have been drawn from the angle A to the side $B C$, as in the following example:

EXAMPLE.—In the triangle $A B C$, Fig. 77, given, $A B = 18''$, angle $B = 60°$, and angle $C = 38° 42'$; to find the other three parts.

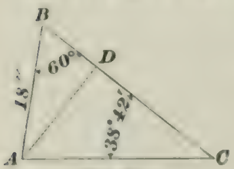

FIG. 77.

SOLUTION.—In the triangle $A B C$, we have angle $A = 180° - (60° + 38° 42') = 81° 18'$.

From the vertex A, draw the line $A D$ perpendicular to $B C$, thus forming the right-angled triangle $A B D$, in which two parts (the side $A B$ and angle B) are known besides the right angle.

Begin with the right-angled triangle $A B D$. To find $B D$, use rule **4.** $B D = 18$ in. $\times \cos 60° = 18 \times .5 = 9$ in. To find $A D$, use rule **2.** $A D = 18$ in. $\times \sin 60° = 18 \times .86603 = 15.59$ in.

In the right-angled triangle $A D C$, $A D$ and the angle C are known.

To find $A\,C$, use rule **9.** $A\,C = \frac{A\,D}{\sin C} = \frac{15.59}{.62524} = 24.93$ in.

To obtain $D\,C$, use rule **4.** $D\,C = A\,C \times \cos C = 24.93 \times .78043 = 19.46$ in.

Since $B\,C = B\,D + D\,C$, $B\,C = 9 + 19.46 = 28.46$ in.

$$\text{Ans.} \begin{cases} \text{Angle } A = 81^\circ\ 18'. \\ A\,C = 24.93 \text{ in.} \\ B\,C = 28.46 \text{ in.} \end{cases}$$

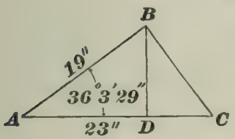

FIG. 78.

EXAMPLE.—In Fig. 78, $A\,B = 19$ in., $A\,C = 23$ in., and the included angle $A = 36^\circ\ 3'\ 29''$; to find the other two angles and the side $B\,C$.

SOLUTION.—From the vertex B, draw $B\,D$ perpendicular to $A\,C$, forming the two right-angled triangles $A\,D\,B$ and $B\,D\,C$. In the right-angled triangle $A\,D\,B$, $A\,B$ is known and also the angle A. Hence, by rule **2,**

$B\,D = 19 \times \sin 36^\circ\ 3'\ 29'' = 19 \times .58861 = 11.18$ in., nearly.

By rule **4,** $A\,D = 19 \times \cos 36^\circ\ 3'\ 29'' = 19 \times .80842 = 15.36$ in.

$A\,C - A\,D = 23 - 15.36 = 7.64$ inches $= D\,C$.

In the right-angled triangle $B\,D\,C$, the two sides $B\,D$ and $D\,C$, about the right angle, are known; hence (rule **5**), $\tan C = \frac{B\,D}{D\,C} = \frac{11.18}{7.64} = 1.46335$, and angle $C = 55^\circ\ 39'\ 10''$.

Applying rule **9,** $B\,C = \frac{B\,D}{\sin 55^\circ\ 39'\ 10''} = \frac{11.18}{.82564} = 13.54$ in.

Angle $B = 180^\circ - (36^\circ\ 3'\ 29'' + 55^\circ\ 39'\ 10'') = 180^\circ - 91^\circ\ 42'\ 39'' = 88^\circ\ 17'\ 21''$.

$$\text{Ans.} \begin{cases} \text{Angle } C = 55^\circ\ 39'\ 10''. \\ \text{Angle } B = 88^\circ\ 17'\ 21''. \\ \text{Side } B\,C = 13.54 \text{ in.} \end{cases}$$

The following example presents a case where the perpendicular must be drawn outside of the triangle in order to form a right-angled triangle two of whose parts are known, besides the right angle:

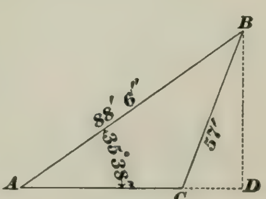

FIG. 79.

EXAMPLE.—Given, the triangle $A\,B\,C$, Fig. 79, in which $A\,B = 88$ ft. 6 in., $B\,C = 57$ ft., and angle $A = 35^\circ\ 0'\ 38''$, to find the other parts.

SOLUTION.—From the vertex B, draw the line $B\,D$ perpendicular to the base $A\,C$ extended, forming the right-angled triangles $A\,D\,B$ and $C\,D\,B$.

In the right-angled triangle $A\,D\,B$, $A\,B$ and angle A are known, to find $A\,D$ and $B\,D$.

By rule **4,** $A\,D = 88$ ft. 6 in. $\times \cos 35^\circ\ 0'\ 38'' = 88.5$ ft. $\times .81905 = 72.49$ ft.

By rule **2,** $BD = 88$ ft. 6 in. $\times \sin 35° 0' 38'' = 88.5 \times .57373 =$ 50.78 ft.

Now, in the right-angled triangle CDB, BC and BD are known, to find angle BCD and side CD.

By rule **1,** $\sin BCD = \frac{BD}{BC} = \frac{50.78}{57} = .89088$; whence, angle $BCD = 62° 59' 4.3''$.

By rule **4,** $CD = 57 \times \cos 62° 59' 4.3'' = 57 \times .45423 = 25.89$ ft.

We now have the data necessary for obtaining the required parts of the triangle ABC. Since the angle $BCD = 62° 59' 4.3''$, the adjacent angle $ACB = 180° - 62° 59' 4.3'' = 117° 0' 55.7''$. Also, angle $ABC = 180° - (35° 0' 38'' + 117° 0' 55.7'') = 180° - 152° 1' 33.7'' = 27° 58' 26.3''$. Since $AD = 72.49$ ft. and $CD = 25.89$ ft., $AC = 72.49 - 25.89 = 46.6$ ft. $= 46$ ft. $7\frac{3}{16}$ in., nearly.

$$\text{Ans.} \begin{cases} \text{Angle } C = 117° 0' 55.7''. \\ \text{Angle } B = 27° 58' 26.3''. \\ \text{Side } AC = 46 \text{ ft. } 7\frac{3}{16} \text{ in.} \end{cases}$$

Case II.—*When the three sides are given, to find the angles.*

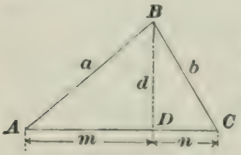

FIG. 80.

This case is solved by drawing a line from the vertex of the angle opposite the longest side, perpendicular to that side, as BD in Fig. 80. The parts m and n of the side AC are then determined from the following proportion:

$$m + n \text{ (or } AC) : a + b = a - b : m - n.$$

This gives the value of $m - n$. The value of $m + n = AC$ is already known, and from the two, m and n may be determined by the principles of algebra. Having m and n, therefore, the right-angled triangles ABD and CBD may be solved.

The above proportion is obtained as follows:

Let a, b, and d represent the sides indicated. Then, in the right-angled triangles ABD and CBD,

$$m^2 + d^2 = a^2 \qquad (1)$$

$$\text{and } n^2 + d^2 = b^2 \qquad (2)$$

Subtracting (2) from (1),

$$m^2 - n^2 = a^2 - b^2.$$

Factoring,

$$(m + n)(m - n) = (a + b)(a - b).$$

Dividing by $m - n$ and $a + b$, $\frac{m + n}{a + b} = \frac{a - b}{m - n}$.

This equation expresses the equality of two ratios, and may be stated as a proportion, or

$$m+n : a+b = a-b : m-n.$$

EXAMPLE.—Given, a triangle whose sides are 17 ft. 3 in., 21 ft., and 32 ft. long. Find the angles.

SOLUTION.—$m+n$, the longest side, $= 32$ ft.

$a+b$, the sum of the two shorter sides, $= 17.25 + 21 = 38.25$ ft.

$a-b$, the difference of the two shorter sides, $= 3.75$ ft. Hence,

$$32 : 38.25 = 3.75 : m-n, \text{ or } m-n = \frac{38.25 \times 3.75}{32} = 4.48 \text{ ft.}$$

Adding, $m+n$ and $m-n$,

$$\begin{aligned} m+n &= 32 \\ m-n &= 4.48 \\ \hline 2m &= 36.48 \\ m &= 18.24 \end{aligned}$$

Subtracting,

$$\begin{aligned} 2n &= 27.52 \\ n &= 13.76 \end{aligned}$$

Now, referring to the last figure, we have, in the triangle $A\,D\,B$, side $a = 21$ and $m = 18.24$ ft.; whence, by rule **3**, $\cos A = \frac{18.24}{21} = .86857$, or $A = 29° 42' 25.7''$.

In triangle $C\,B\,D$, side $b = 17.25$ and $n = 13.76$ ft.; whence, by rule **3**, $\cos C = \frac{13.76}{17.25} = .79768$, or $C = 37° 5' 26.7''$.

Angle $A\,B\,C = 180° - (29° 42' 25.7'' + 37° 5' 26.7'') = 113° 12' 7.6''$.

Ans. $\begin{cases} \text{Angle } A = 29° 42' 25.7''. \\ \text{Angle } B = 113° 12' 7.6''. \\ \text{Angle } C = 37° 5' 26.7''. \end{cases}$

EXAMPLES FOR PRACTICE.

1. Given, an oblique triangle $A\,B\,C$, in which side $A\,B = 21$ feet, angle $A = 22° 10' 16''$, and angle $B = 78° 24' 24''$. Find the other parts.

Ans. $\begin{cases} \text{Angle } C = 79° 25' 20''. \\ A\,C = 20.93 \text{ ft.} \\ B\,C = 8.06 \text{ ft.} \end{cases}$

2. Given, a triangle $A\,B\,C$, in which $A\,B = 32$ inches, angle $B = 54° 16'$, and angle $C = 58° 18' 9''$. Find the other parts.

Ans. $\begin{cases} \text{Angle } A = 67° 25' 51''. \\ A\,C = 30.53 \text{ in.} \\ B\,C = 34.73 \text{ in.} \end{cases}$

3. In a triangle $A\,B\,C$, $A\,B = 20$ feet 6 inches, $B\,C = 16$ feet, and angle $B = 46° 40' 42''$. Find the values of the other parts.

Ans. $\begin{cases} \text{Angle } A = 50° 42' 52''. \\ \text{Angle } C = 82° 36' 26''. \\ A\,C = 15.04 \text{ ft.} \end{cases}$

4. In the triangle $A\,B\,C$, $A\,C = 100$ feet, $B\,C = 60$ feet, and angle $A = 20°$. Solve the triangle.

$$\text{Ans.}\begin{cases}\text{Angle } B = 34°\ 45'\ 7.5''.\\ \text{Angle } C = 125°\ 14'\ 52.5''.\\ A\,B = 143.26 \text{ ft.}\end{cases}$$

5. In a triangle $A\,B\,C$, $A\,B = 98$ inches, $B\,C = 140$ inches, and $A\,C = 210$ inches. Compute the angles A, B and C.

$$\text{Ans.}\begin{cases}A = 34°\ 2'\ 52.5''.\\ B = 122°\ 52'\ 40.2''.\\ C = 23°\ 4'\ 27.3''.\end{cases}$$

MENSURATION.

762. Mensuration is that part of Geometry which treats of the measurement of lines, surfaces and solids.

MENSURATION OF PLANE SURFACES.

763. The **area** of a surface is expressed by the number of unit squares it will contain.

764. A **unit square** is the square having the unit for its side. For example, if the unit is 1 inch, the unit square is the square whose sides measure 1 inch in length, and the area would be expressed by the number of square inches that the surface contains. If the unit were 1 foot, the unit square would measure 1 foot on each side, and the area would be the number of square feet that the surface contains, etc.

765. The square that measures 1 inch on a side is called a **square inch,** and the one that measures 1 foot on a side is called a **square foot.** Square inch and square foot are abbreviated to sq. in. and sq. ft., or to □″ and □′.

THE TRIANGLE.

766. Rule.—*The area of any triangle equals one-half the product of the base and the altitude.*

EXAMPLE.—What is the area of a triangle whose base is 18 feet, and altitude 7 feet 9 inches?

SOLUTION.—9 inches = $\frac{9}{12}$ of a foot = $\frac{3}{4}$ of a foot. $18 \times 7\frac{3}{4} = 139\frac{1}{2}$, and $\frac{1}{2}$ of $139\frac{1}{2} = 69\frac{3}{4}$ square feet. Ans.

THE QUADRILATERAL.

767. A **parallelogram** is a quadrilateral whose opposite sides are parallel.

768. There are four kinds of parallelograms: the **square,** the **rectangle,** the **rhombus** and the **rhomboid.**

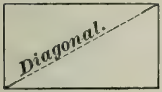

FIG. 81.

769. A **rectangle,** Fig. 81, is a parallelogram whose angles are all right angles.

770. A **square,** Fig. 82, is a rectangle, all of whose sides are equal.

FIG. 82.

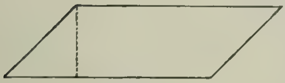

FIG. 83.

771. A **rhomboid,** Fig. 83, is a parallelogram whose opposite sides only are equal, and whose angles are not right angles.

772. A **rhombus,** Fig. 84, is a parallelogram having equal sides, and whose angles are not right right angles.

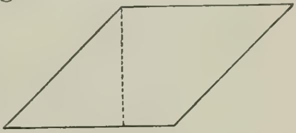

FIG. 84.

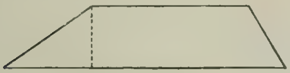

FIG. 85.

773. A **trapezoid,** Fig. 85, is a quadrilateral which has only two of its sides parallel.

774. A **trapezium,** Fig. 86, is a quadrilateral having no two sides parallel.

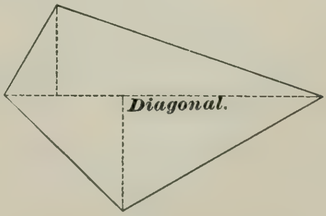

FIG. 86.

775. The **altitude** of a parallelogram, or of a trapezoid, is the perpendicular distance between the parallel sides.

776. A **diagonal** is a straight line drawn from the vertex of any angle of a quadrilateral to the vertex of the angle opposite; a diagonal divides the quadrilateral into two triangles.

A diagonal divides a parallelogram into two *equal* and *similar* triangles.

777. To find the area of a parallelogram:

Rule.—*The area of any parallelogram equals the product of the base and the altitude.*

EXAMPLE.—What is the area of a parallelogram whose base is 12 feet and altitude $7\frac{1}{2}$ feet?

SOLUTION.—Area $= 12 \times 7\frac{1}{2} = 90$ square feet. Ans.

778. To find the area of a trapezoid:

Rule.—*The area of a trapezoid equals one-half the sum of the parallel sides multiplied by the altitude.*

EXAMPLE.—What is the area of a trapezoid whose parallel sides are 9 feet and 15 feet, and whose altitude is 6 feet 7 inches?

SOLUTION.—6 feet 7 inches $= 6\frac{7}{12}$ feet. $\frac{9+15}{2} = 12$. Area $= 12 \times 6\frac{7}{12} = 79$ square feet. Ans.

779. To find the area of an irregular figure bounded by straight lines:

Rule.—*Divide the figure into triangles, and find the area of each triangle separately. The sum of the areas of all the triangles will be the area of the figure.*

EXAMPLE.—The diagonal of a trapezium is 15 feet. The altitudes drawn from the vertexes of the two triangles to this diagonal as a base are 6 feet 8 inches and 4 feet 9 inches, respectively. What is the area of the trapezium?

SOLUTION.—8 inches $= \frac{8}{12}$ of a foot $= \frac{2}{3}$ of a foot. $\frac{15 \times 6\frac{2}{3}}{2} = 50$ square feet = area of one triangle.

9 inches $= \frac{9}{12}$ of a foot $= \frac{3}{4}$ of a foot. $\frac{15 \times 4\frac{3}{4}}{2} = 35.63$ square feet = the area of the other triangle.

$50 + 35.63 = 85.63$ square feet = the area of the trapezium. Ans.

THE CIRCLE.

780. To find the circumference or diameter of a circle:

Rule.—*The circumference of a circle equals the diameter multiplied by 3.1416.*

Rule.—*The diameter of a circle equals the circumference divided by 3.1416.*

Example.—What is the circumference of a circle whose diameter is 15 inches?

Solution.—$15 \times 3.1416 = 47.12$ inches, circumference. Ans.

Example.—What is the diameter of a circle whose circumference is 65.973 inches?

Solution.—$65.973 \div 3.1416 = 21$ inches diameter. Ans.

781. To find the length of an arc of a circle:

Rule.—*The length of an arc of a circle equals the circumference of the circle of which the arc is a part, multiplied by the number of degrees in the arc, and divided by 360.*

Example.—What is the length of an arc of 24°, the radius of the arc being 18 inches?

Solution.—$18 \times 2 = 36$ inches = the diameter of the circle. $36 \times 3.1416 = 113.1$ inches, the circumference of the circle of which the arc is a part.

$$113.1 \times \frac{24}{360} = 7.54 \text{ inches, the length of the arc. Ans.}$$

782. To find the area of a circle:

Rule.—*Square the diameter, and multiply by .7854.*

Example.—What is the area of a circle whose diameter is 15 inches?

Solution.—$15^2 = 225$. $225 \times .7854 = 176.72$ sq. in. Ans.

783. Given, the area of a circle to find its diameter:

Rule.—*Divide the area by .7854 and extract the square root of the quotient.*

Example.—The area of a circle = 17,671.5 square inches. What is its diameter in feet?

SOLUTION.— $\sqrt{\frac{17,671.5}{.7854}} = 150$ inches.

$\frac{150}{12} = 12\frac{1}{2}$ feet, the diameter. Ans.

784. To find the area of a sector:

Rule.—*Divide the number of degrees in the arc of a sector by 360. Multiply the result by the area of the circle of which the sector is a part.*

EXAMPLE.—The number of degrees in the angle formed by drawing radii from the center of a circle to the extremities of the arc of the circle is 75°. The diameter of the circle is 12 inches; what is the area of the sector?

SOLUTION.— $\frac{75}{360} = \frac{5}{24}$. $12^2 \times .7854 = 113.1$ square inches.

$\frac{5}{24} \times 113.1 = 23.56$ square inches, the area. Ans.

785. To find the area of a segment of a circle:

Rule.—*Draw radii from the center of the circle to the extremities of the arc of the segment; find the area of the sector thus formed, subtract from this the area of the triangle formed by the radii and the chord of the arc of the segment, and the result is the area of the segment.*

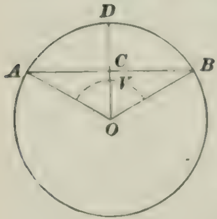

FIG. 87.

In problems requiring the area of the segment, the chord $A B$, Fig. 87, may be given, or the height of the segment $C D$, or the angle V; if any one of these three be given, and the radius of the circle is known, the area can be found.

Also, if any two are given, the radius can be found.

EXAMPLE.—If the diameter of the circle is 10 inches, and the chord of the segment is 7 inches, what is the area of the segment?

SOLUTION.—In the above figure, suppose that the chord $A B = 7$ inches, and the diameter = 10 inches; draw $O A$, $O B$, and a radius perpendicular to the chord, thus dividing $A B$ into two equal parts (see Art. **733**). The triangle $A O B$ is now divided into two equal right-angled triangles, ACO and OCB, in which the hypotenuse = radius, or $\frac{10}{2} = 5$, and one side, $AC = BC = \frac{7}{2}$, or $3\frac{1}{2}$.

$$\text{Sin } COB = \frac{CB}{OB} = \frac{3\frac{1}{2}}{5} = .70000, \text{ and angle } COB = 44° \, 26', \text{ nearly.}$$

Angle $AOB = 44° \; 26' \times 2 = 88° \; 52'$. $CO = OB \times \cos COB = 5 \times .71407 = 3.57''$.

$$\text{Area of sector} = 10^2 \times .7854 \times \frac{88\frac{52}{60}}{360} = 19.39 \text{ sq. in., nearly.}$$

$$\text{Area of triangle} = \frac{7 \times 3.57}{2} = 12.5 \text{ sq. in., nearly.}$$

$19.39 - 12.5 = 6.89$ sq. in., the area of the segment. Ans.

EXAMPLE.—Given, the chord of the arc of a segment = 7 inches, and the height of the segment = 1.43 inches, to find the radius.

SOLUTION.—Suppose that in Fig. 88, $ACBE$ is a circle struck with the required radius, that the chord $AB = 7$ inches, and that the height CD of the segment = 1.43 inches. Join C with A and B, and the right-angled triangle $ADC = BDC$.

$$\text{Tan } CBD = \frac{CD}{BD} = \frac{1.43}{3.5} = .40857.$$

Angle $CBD = 22° \; 13\frac{1}{2}'$, nearly.

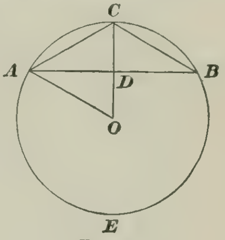

FIG. 88.

Since CBD or its equal CBA is an inscribed angle (see Art. **729**), it is measured by one-half the intercepted arc AC; hence, the number of degrees in arc $AC = 22° \; 13\frac{1}{2}' \times 2 = 44° \; 27'$, or the number of degrees in the angle AOC.

In the right-angled triangle ADO,

$$AO = \frac{\text{side opposite}}{\sin AOD} = \frac{AD}{\sin AOC} = \frac{3.5}{.70029} = 5 \text{ inches, nearly. Ans.}$$

THE ELLIPSE.

786. An **ellipse** is a plane figure bounded by a curved line, to any point of which the sum of the distances from two fixed points within, called the **foci,** is equal to the sum of the distances from the foci to any other point on the curve.

In Fig. 89, let A and B be the foci, and let C and D be any two points on the perimeter. Then, according to the above definition, $A\,C + C\,B = A\,D + D\,B$, and both these sums are also equal to the major axis $F\,E$.

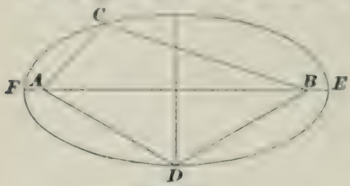

FIG. 89.

787. The long diameter is called the **major axis;** the short diameter, the **minor axis.**

788. To find the perimeter of an ellipse: There is no exact method, but the following is close enough for most cases:

Rule.—*Multiply the major axis by 1.82, and the minor axis by 1.315. The sum of the results will be the perimeter.*

EXAMPLE.—What is the perimeter of an ellipse whose axes are 10 and 4 inches?

SOLUTION.— $10 \times 1.82 = 18.2$ inches. $4 \times 1.315 = 5.26$ inches. $18.2 + 5.26 = 23.46$ inches, or the perimeter. Ans.

789. To find the exact area of an ellipse:

Rule.—*The area of an ellipse is equal to the product of its two diameters multiplied by .7854.*

EXAMPLE.—What is the area of an ellipse whose diameters are 10 and 6 inches?

SOLUTION.— $10 \times 6 \times .7854 = 47.12$ square inches, area. Ans.

EXAMPLES FOR PRACTICE.

1. What is the area in square feet of a rhombus whose base is 84 inches, and whose altitude is 3 feet? Ans. 21 sq. ft.

2. One side of a room is 16 feet long. If the floor contains 240 square feet, what is the length of the other side? Ans. 15 ft.

3. How many square feet in a board 12 feet long, 18 inches wide at one end and 12 inches wide at the other end? Ans. 15 sq. ft.

4. How many square yards of plastering will be required for the ceiling and walls of a room 10×15 feet, and 9 feet high? The room contains one door $3\frac{1}{2} \times 7$ feet, three windows $3\frac{1}{2} \times 6$ feet, and a baseboard 8 inches high. Ans. 53.5 sq. yd.

5. What is the area of a triangle whose base is 10 feet 6 inches long, and whose altitude is 18 feet? Ans. 94.5 sq. ft.

6. The area of a triangle is 16 square inches. If the altitude is 4 inches, what does the base measure? Ans. 8 in.

7. The upper side of a trapezium is 16 inches long, and the lower side 14 inches. If the figure be divided into two triangles by a diagonal whose altitudes, drawn from their vertexes to the two given sides as bases, are 17 and 3 inches, respectively, what is the area of the trapezium? Ans. 157 sq. in.

8. Find the area of a circle 2 feet 3 inches in diameter.
Ans. 3.976 sq. ft.

9. A carriage-wheel was observed to make 71⅔ turns while going 300 yards. What was its diameter? Ans. 4 ft., nearly.

10. Required, the diameter of a circle whose area is 2,004 square inches. Ans. 50.51 in.

11. Required, the area of a regular pentagon inscribed in a circle whose diameter is 20 inches. Ans. 237.77 sq. in.

12. The number of degrees in the angle formed by drawing radii from the center of a circle to the extremities of the arc of the circle is 84 degrees. The diameter of the circle is 17 inches; what is the area of the sector? Ans. 52.96 sq. in.

13. Given, the chord of the arc of a segment = 24 inches, and the height of the segment = 6.5 inches, to find (*a*) the diameter of the circle, and (*b*) the area of the segment.
Ans. $\begin{cases} (a) \ 28.654 \text{ in.} \\ (b) \ 109.87 \text{ sq. in.} \end{cases}$

14. (*a*) What is the perimeter of an ellipse whose axes are 15 and 9 inches, and (*b*) what is the area?
Ans. $\begin{cases} (a) \ 39.14 \text{ in.} \\ (b) \ 106.03 \text{ sq. in.} \end{cases}$

790. Rule.—*The area of any plane figure may be found by dividing it into triangles, quadrilaterals, circles or parts of circles, and ellipses, finding the area of each part separately and adding them together.*

EXAMPLE.—What is the area of a flat circular ring, Fig. 90, whose outside diameter equals 10 inches, and whose inside diameter equals 4 inches?

The area of the large circle = $10^2 \times .7854$ = 78.54 square inches; the area of the small circle = $4^2 \times .7854$ = 12.57 square inches.

78.54 − 12.57 = 65.97 square inches, or the area. Ans.

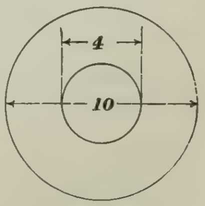

FIG. 90.

EXAMPLE.—What is the exact area in square inches of Fig. 91?

SOLUTION.—Divide the figure into rectangles, triangles, and parts of a circle, as shown by the dotted lines, then the total area equals 8-inch circle − 4-inch circle − segment *A B* + rectangle *A B G F* + 2 times the triangle *C D E* + 2 times the triangle *R S T* + 2 times the rectangle *D E S R* + rectangle *H I K L* + 2 times the rectangle *L M N P* + 2 times the triangle *M O N*.

$$8^2 \times .7854 = 50.27 \,\square''.$$
$$4^2 \times .7854 = 12.57 \,\square''.$$

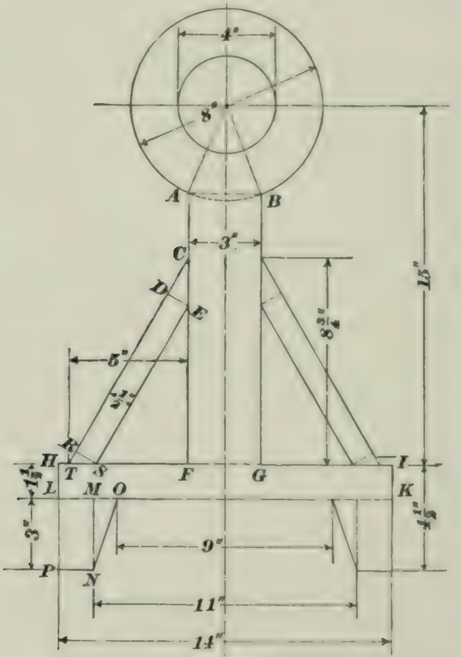

FIG. 91.

The chord *A B* = 3 inches, and the radius of the circle = 4 inches; hence, the sine of one-half the angle at center $= \frac{1.5}{4} = .375$, and one-half the angle at center = 22° 1′ 27″, or angle at center = 44° 2′ 54″ = 44.05°. Area of sector $= 50.27 \times \frac{44.05}{360} = 6.15 \,\square''$.

The altitude of the triangle = 4 × cos 22° 1′ 27″ = 3.71 inches.

The area of the triangle $= \frac{3.71 \times 3}{2} = 5.56 \,\square''$.

The area of the segment $= 6.15 - 5.56 = 0.59 \,\square''$.

The area of the rectangle *A B G F* $= (15 - 3.71) \times 3 = 33.87 \,\square''$

In the triangle CDE, $\tan C = \frac{5}{8\frac{3}{4}} = .57143 = \frac{DE}{CD} = \frac{.5}{CD}$; hence, $CD = \frac{.5}{.57143} = .875$ inch.

The area of the triangle $CDE = \frac{.875 \times .5}{2} = .22 \square''$, nearly

$.22 \times 2 = .44 \square'' =$ twice the area of the triangle CDE. Since in the triangle RST, RS is perpendicular to CR and TS is perpendicular to CF, the angle $S =$ angle C; hence, $\tan S = .57143 = \frac{RT}{SR} = \frac{RT}{.5}$; therefore, $RT = .57143 \times .5 = .29$ inch, nearly.

Area $RST = \frac{.29 \times .5}{2} = .07 \square''$, nearly.

Twice the area of the triangle $RST = .07 \times 2 = .14 \square''$.

Since $\tan C = .57143$, $C = 29° 44' 42''$.

In the rectangle $DESR$, $DR = CT - (CD + RT)$. But $CT = \frac{5}{\sin 29° 44' 42''} = \frac{5}{.49614} = 10.08$ in.

$CD + RT = .875 + .29 = 1.16$. $DR = 10.08 - 1.16 = 8.92$. $8.92 \div 2 = 4.46 \square'' =$ the area of $DESR$.

Twice the area of the rectangle $DESR = 4.46 \times 2 = 8.92 \square''$.

The area of the rectangle $HIKL = 14 \times 1\frac{1}{2} = 21 \square''$.

The area of the rectangle $LMNP = \left(\frac{14-11}{2}\right) \times 3 = 1\frac{1}{2} \times 3 = 4\frac{1}{2} \square''$; and $4\frac{1}{2} \times 2 = 9 \square''$.

The area of the triangle $MON = \left(\frac{11-9}{2} \times 3\right) \div 2 = 1.5 \square''$.

Twice the area of the triangle $MON = 1.5$ inches $\times 2 = 3 \square''$.

Then, $50.27 + 33.87 + 0.44 + 0.14 + 8.92 + 21 + 9 + 3 = 126.64 \square''$. $12.57 + 0.59 = 13.16 \square''$. $126.64 - 13.16 = 113.48 \square''$.

Therefore, the area of the figure $= 113.48 \square''$. Ans.

THE MENSURATION OF SOLIDS.

791. A **solid,** or body, has three dimensions: length, breadth, and thickness. The sides which enclose it are called the **faces,** and their intersections are called **edges.**

792. The **entire surface** of a solid is the area of the whole outside of the solid, including the ends.

793. The **convex surface** of a solid is the same as the entire surface, except that the areas of the ends are not included.

794. The **volume** of a solid is expressed by the number of times it will contain another volume, called the unit of volume. Instead of the word *volume*, the expression **cubical contents** is frequently used.

THE PRISM AND CYLINDER.

795. A **prism** is a solid whose ends are equal polygons and parallel to each other, and whose sides are parallelograms.

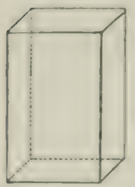

FIG. 92.

796. A **parallelopipedon,** Fig. 92, is a prism whose bases (ends) are parallelograms.

797. A **cube,** Fig. 93, is a parallelopipedon whose faces and ends are squares.

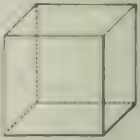

FIG. 93.

798. The cube, whose edges are equal to the unit of length, is taken as the unit of volume when finding the volume of a solid.

Thus, if the unit of length is 1 inch, the unit of volume will be the cube whose edges measure 1 inch, or 1 *cubic inch;* and the number of cubic inches the solid contains will be its volume. If the unit of length is 1 foot, the unit of volume will be one *cubic foot*, etc. Cubic inch, cubic foot, and cubic yard are abbreviated to cu. in., cu. ft., and cu. yd., respectively.

799. Prisms take their names from their bases. Thus, a *triangular prism* is one whose bases are triangles; a *pentagonal prism* is one whose bases are pentagons, etc.

800. A **cylinder,** Fig. 94, is a round body of uniform diameter with circles for its ends.

FIG. 94.

801. A **right prism,** or **right cylinder,** is one whose center line (axis) is perpendicular to its base. In this subject all of the solids will be considered as having their center lines perpendicular to their bases.

802. The **altitude** of a prism or cylinder is the perpendicular distance between its two ends.

803. To find the area of the convex surface of any right prism, or right cylinder:

Rule.—*Multiply the perimeter of the base by the altitude.*

EXAMPLE.—In a right prism whose base is a square, one side of which is 9 inches, and whose altitude is 16 inches, what is its convex area?

SOLUTION.— $9 \times 4 = 36 =$ the perimeter of the base.
$36 \times 16 = 576$ □″, or the convex area. Ans.

To find the entire area, add the areas of the two ends to the convex area:

EXAMPLE.—What is the entire area of the parallelopipedon mentioned in the last question?

SOLUTION.—The area of one end $= 9^2 = 81$ □″. $81 \times 2 = 162$ □″, or the area of both ends. $576 + 162 = 738$ □″, the entire area of the parallelopipedon. Ans.

EXAMPLE.—What is the entire area of a right cylinder whose base is 16 inches in diameter, and whose altitude is 24 inches?

SOLUTION.— $16 \times 3.1416 = 50.27$ inches, or the perimeter (circumference) of the base. $50.27 \times 24 = 1{,}206.48$ □″, the convex area.
$16^2 \times .7854 \times 2 = 402.12$ □″, the area of the ends.
$1{,}206.48 + 402.12 = 1{,}608.6$ □″, the entire area. Ans.

804. To find the volume of a right prism, or cylinder:

Rule.—*The volume of any right prism or cylinder equals the area of the base multiplied by the altitude.*

If the given prism is a cube, the three dimensions are all equal, and the volume equals the cube of one of the edges.

Example.—What is the volume of a rectangular prism whose base is 6×4 inches, and whose altitude is 12 inches?

Solution.—The base of a rectangular prism is a rectangle. Hence, $6 \times 4 = 24$ □″, the area of the base. $24 \times 12 = 288$ cubic inches, or the volume. Ans.

Example.—What is the volume of a cube whose edge is 9 inches?

Solution.— $9^3 = 9 \times 9 \times 9 = 729$ cubic inches, the volume. Ans.

Example.—What is the volume of a cylinder whose base is 7 inches in diameter, and whose altitude is 11 inches?

Solution.— $7^2 \times .7854 = 38.48$ □″, the area of the base. $38.48 \times 11 = 423.28$ cubic inches, the volume. Ans.

THE PYRAMID AND CONE.

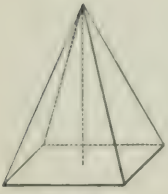

Fig. 95.

805. A **pyramid,** Fig. 95, is a solid whose base is a polygon, and whose sides are triangles uniting at a common point, called the **vertex.**

806. A **cone,** Fig. 96, is a solid whose base is a circle, and whose convex surface tapers uniformly to a point called the **vertex.**

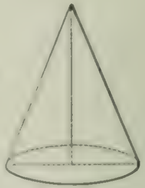

Fig. 96.

807. The **altitude** of a pyramid or cone is the perpendicular distance from the vertex to the base.

808. The **slant height** of a *pyramid* is a line drawn from the vertex perpendicular to one of the sides of the base. The slant height of a *cone* is any straight line drawn from the vertex to the circumference of the base.

809. To find the area of a right pyramid or right cone:

Rule.—*The convex area of a right pyramid or cone equals the perimeter of the base multiplied by one-half the slant height.*

EXAMPLE.—What is the convex area of a pentagonal pyramid, if one side of the base measures 6 inches, and the slant height equals 14 inches?

SOLUTION.—The base of a pentagonal pyramid is a pentagon, and, consequently, it has five sides.

$6 \times 5 = 30$ inches, or the perimeter of the base.

$30 \times \frac{14}{2} = 210 \,\square''$, or the convex area. Ans.

EXAMPLE.—What is the entire area of a right cone whose slant height is 17 inches, and whose base is 8 inches in diameter?

SOLUTION.— $8 \times 3.1416 = 25.1328$ inches, the perimeter.

$25.1328 \times \frac{17}{2} = 213.63 \,\square''$, the convex area.

$8^2 \times .7854 = \underline{\ 50.27} \,\square''$, the area of base.

$sum = 263.90 \,\square''$ = the entire area. Ans.

810. To find the volume of a right pyramid or cone:

Rule.—*The volume of a right pyramid or cone equals the area of the base multiplied by one-third of the altitude.*

EXAMPLE.—What is the volume of a triangular pyramid, one edge of whose base measures 6 inches, and whose altitude is 8 inches?

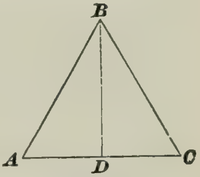

FIG. 97.

SOLUTION.—Draw the base as shown in Fig. 97; it will be an equilateral triangle, all of whose sides are 6 inches long.

Draw a perpendicular, $B\,D$, from the vertex to the base; it will divide the base into two equal parts, and will be the altitude of the triangle.

In the right-angled triangle $B\,D\,A$, the hypotenuse $B\,A = 6$ inches, and side $A\,D = \frac{A\,C}{2} = \frac{6}{2} = 3$ inches, to find the other side:

$B\,D = \sqrt{6^2 - 3^2} = 5.2$ inches, nearly.

Area of $B\,A\,C = \frac{6 \times 5.2}{2} = 15.6 \,\square''$, the area of the base.

$15.6 \times \frac{8}{3} = 41.6$ cubic inches, the volume. Ans.

EXAMPLE.—What is the volume of a cone whose altitude is 18 inches, and whose base is 14 inches in diameter?

SOLUTION.— $14^2 \times .7854 = 153.94 \,\square''$, the area of the base.

$153.94 \times \frac{18}{3} = 923.64$ cubic inches, the volume. Ans.

THE FRUSTUM OF A PYRAMID OR CONE.

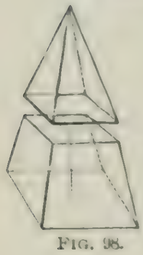

FIG. 98.

811. If a pyramid be cut by a plane parallel to the base, as in Fig. 98, so as to form two parts, the lower part is called the **frustum** of the pyramid.

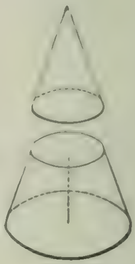

FIG. 99.

812. If a cone be cut in a similar manner, as in Fig. 99, the lower part is called the **frustum** of the cone.

813. The upper end of the frustum of a pyramid or cone is called the **upper base,** and the lower end the **lower base.** The **altitude** of a frustum is the perpendicular distance between the bases.

814. To find the convex area of a frustum of a right pyramid or right cone:

Rule.—*The convex area of a frustum of a right pyramid or right cone equals one-half the sum of the perimeters of its bases multiplied by the slant height of the frustum.*

EXAMPLE.—Given, the frustum of a triangular pyramid in which one side of the lower base measures 10 inches, one side of the upper base measures 6 inches, and whose slant height is 9 inches; find the convex area.

SOLUTION.— 10 inches × 3 = 30 inches, the perimeter of the lower base.

6 inches × 3 = 18 inches, the perimeter of the upper base.

$\frac{30 + 18}{2} = 24$ inches, or one-half the sum of the perimeters of the bases. $24 \times 9 = 216 \square''$, the convex area. Ans.

Example.—If the diameters of the two bases of a frustum of a cone are 12 inches and 8 inches, respectively, and the slant height is 12 inches, what is the entire area of the frustum?

Solution.— $\frac{(12 \times 3.1416) + (8 \times 3.1416)}{2} \times 12 = 376.99\,\square''$, the area of the convex surface.

$$8^2 \times .7854 = 50.27\,\square''.$$
$$12^2 \times .7854 = 113.1\,\square''.$$

$113.1 + 50.27 = 163.37\,\square''$, the area of the two ends.

$376.99 + 163.37 = 540.36\,\square''$, the entire area of the frustum. Ans.

815. To find the volume of the frustum of a pyramid or cone:

Rule.—*Add the areas of the upper base, the lower base, and the square root of the product of the areas of the two bases; multiply this sum by one-third of the altitude.*

Example.—Given, a frustum of a hexagonal pyramid, each edge of the lower base measuring 8 inches, and each edge of the upper base 5 inches, and whose altitude is 14 inches; what is its volume?

Solution.—A hexagonal pyramid is one whose base is a hexagon, or six-sided polygon. Divide the base into 6 equal triangles, as shown in Fig. 100. To find the altitude of one of the triangles, proceed as follows:

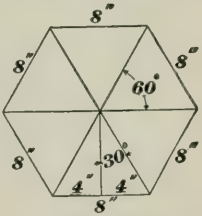

FIG. 100.

The angle at the vertex of one of the triangles will be $\frac{360}{6} = 60°$, and the angle on each side of the perpendicular to the base (or altitude) will be $\frac{60}{2} = 30°$.

The altitude $= \frac{4}{\tan 30°} = \frac{4}{.57735} = 6.93$ inches, nearly.

The area of the triangle $= \frac{8 \times 6.93}{2} = 27.72\,\square''$.

$27.72 \times 6 = 166.32\,\square''$ = the area of the hexagon, or the area of the lower base.

In a similar way, find the area of the upper base to be $64.97\,\square''$. Then, applying the rule, $166.32 + 64.97 + \sqrt{166.32 \times 64.97} = 166.32 + 64.97 + 103.95 = 335.24$.

$335.24 \times \frac{14}{3} = 1{,}564.45$ cubic inches = the volume. Ans.

Example.—What is the volume of a frustum of a cone whose upper base is 8 inches, the lower base is 12 inches in diameter, and whose altitude is 15 inches?

Solution.—The area of the upper base $= 8^2 \times .7854 = 50.27 \square''$. The area of the lower base $= 12^2 \times .7854 = 113.1 \square''$.

The square root of their product $= \sqrt{50.27 \times 113.1} = 75.4$.

$50.27 + 113.1 + 75.4 = 238.77$.

$238.77 \times \frac{15}{3} = 1{,}193.85$ cubic inches, the volume. Ans.

THE SPHERE.

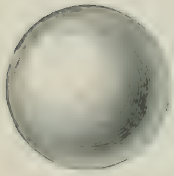

FIG. 101.

816. A **sphere,** Fig. 101, is a solid bounded by a uniformly curved surface, every point of which is equally distant from a point within, called the center.

The word **ball** is commonly used instead of sphere.

817. To find the area of the surface of a sphere:

Rule.—*The area of the surface of a sphere equals the square of the diameter multiplied by 3.1416.*

Example.—What is the area of the surface of a sphere whose diameter is 14 inches?

Solution.—

$14^2 \times 3.1416 = 14 \times 14 \times 3.1416 = 615.75 \square''$, the area. Ans.

818. To find the volume of a sphere:

Rule.—*The volume of a sphere equals the cube of the diameter multiplied by .5236.*

Example.—What is the weight of a lead cannon ball 12 inches in diameter, a cubic inch of lead weighing .41 pound?

Solution.—$12 \times 12 \times 12 \times .5236 = 904.78$ cubic inches, or the volume of the ball. $904.78 \times .41 = 370.96$ pounds. Ans.

819. If any solid be sliced in pieces, whose adjacent surfaces are flat, any piece is called a **plane section** of the solid.

Plane sections are divided into three classes: Longitudinal sections, cross-sections, and right sections. A **longitudinal section** is any plane section taken lengthwise through the solid. Any other plane section is called a **cross-section.** If the surface exposed by taking a plane section of a solid is perpendicular to the center line of the solid, the section is called a **right section.** The surface exposed by any longitudinal section of a cylinder is a rectangle. The surface exposed by a right section of a cube is a square; of a cylinder or cone, a circle; an oblique cross-section of a cylinder is an ellipse. The lower half of a right section of a cone or pyramid is called a frustum of the cone or pyramid.

THE CYLINDRICAL RING.

820. To find the convex area of a cylindrical ring:

A **cylindrical ring** is a cylinder bent to a circle. The **altitude** of the cylinder before bending is the same as the length of the dotted center line *D*. Fig. 102.

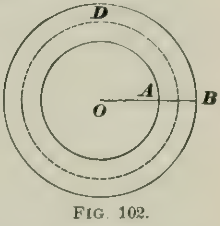

FIG. 102.

821. The **base** will correspond to a cross-section on the line *A B* drawn from the center *O*. Hence, to find the convex area, multiply the circumference of an imaginary cross-section on the line *A B* by the length of the center line *D*.

EXAMPLE.—If the outside diameter of the ring is 12 inches, and the inside diameter is 8 inches, what is its convex area?

SOLUTION.—The diameter of the center circle equals one-half the sum of the inside and outside diameters $= \frac{12 + 8}{2} = 10$, and $10 \times 3.1416 = 31.416$ inches, the length of the center line.

The radius of the inner circle is 4 inches, of the outside circle 6 inches; therefore, the diameter of the cross-section on the line *A B* is 2 inches. Then, $2 \times 3.1416 = 6.2832$ inches, and $6.2832 \times 31.416 = 197.4$ □″, or the convex area. Ans.

822. To find the volume of a cylindrical ring :

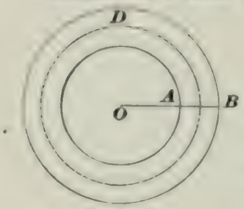

FIG. 103.

The volume will be the same as that of a cylinder whose altitude equals the length of the dotted center line D, Fig. 103, and whose base is the same as a cross-section of the ring on the line A B, drawn from the center O. Hence, to find the volume of a cylindrical ring, multiply the area of an imaginary cross-section on a line A B, by the length of the center line D.

EXAMPLE.—What is the volume of a cylindrical ring whose outside diameter is 12 inches, and whose inside diameter is 8 inches ?

SOLUTION.—The diameter of the center circle equals one-half the sum of the inside and outside diameters $= \frac{12 + 8}{2} = 10$.

$10 \times 3.1416 = 31.416$ inches, the length of the center line.

The radius of the outside circle = 6 inches, of the inside circle = 4 inches; therefore, the diameter of the cross-section on the line $A B$ = 2 inches.

Then, $2^2 \times .7854 = 3.1416$ □′, the area of the imaginary cross-section.

And $3.1416 \times 31.416 = 98.7$ cubic inches, the volume. Ans.

EXAMPLES FOR PRACTICE.

1. Find the weight of an iron bar 16 feet long and 2 inches in diameter, the weight of iron being taken at .28 pound per cubic inch.
Ans. 168.89 lb.

2. What is the area of the entire surface of a hexagonal prism 12 inches long, each edge of the base being 1 inch long ?
Ans. 77.196 sq. in.

3. What is the volume of a triangular pyramid, one edge of whose base measures 3 inches, and whose altitude is 4 inches ? Ans. 5.2 cu. in.

4. Find the volume of a cone whose altitude is 12 inches and the circumference of whose base is 31.416 inches. Ans. 314.16 cu. in.

5. A round tank is 8 feet in diameter at the top (inside) and 10 feet at the bottom. If the tank is 12 feet deep, how many gallons will it hold, there being 231 cubic inches in a gallon ? Ans. 5,734.2 gallons.

6. Required, the area of the convex surface of the frustum of a square pyramid whose altitude is 16 inches, one side of the lower base being 28 inches long, and of the upper base 10 inches.
Ans. 1,395.18 sq. in.

7. What is the volume of a sphere 30 inches in diameter ?
Ans. 14,137.2 cu. in.

8. How many square inches in the surface of the sphere of example 7? Ans. 2,827.44 sq. in.

9. Required, the area of the convex surface of a circular ring, the outside diameter of the ring being 10 inches and the inside diameter $7\frac{1}{2}$ inches. Ans. 107.95 sq. in.

10. Find the cubical contents of the ring in the last example. Ans. 33.734 cu. in.

11. The volume of a sphere is 606.132 cubic inches; required, the area of the convex surface of a cone whose slant height is 10 inches and the diameter of whose base is the same as the diameter of the sphere. Ans. 164.934 sq. in.

12. What is the volume of the frustum of example 6? Ans. 6,208 cu. in.

PROJECTIONS.

823. If perpendiculars be drawn from the extremities of a line, as $A\,B$, Fig. 104 or Fig. 105, to another line, as $H\,K$, as shown in the figures, that portion of $H\,K$ included between the foot of each perpendicular is called the **projection** of $A\,B$ upon $H\,K$. Thus, $C\,D$ is the projection of $A\,B$ upon $H\,K$, the point C is the projection of the point A upon $H\,K$, and the point D is the projection of the point B upon $H\,K$.

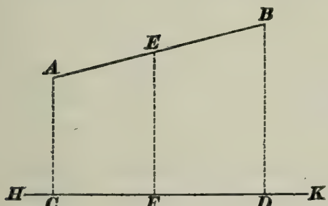

FIG. 104.

The projection of any point of $A\,B$, as E, can be found by drawing a perpendicular from E to $H\,K$, and the point where this perpendicular intersects $H\,K$ is its projection; in this case the point F is the projection of the point E upon $H\,K$.

It makes no difference whether the line is straight or curved, the method of finding the projection is exactly the same.

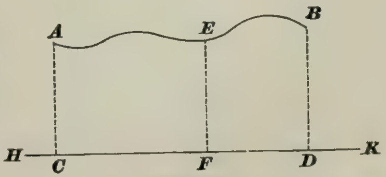

FIG. 105.

From this it is seen that the projection of the hypotenuse of a right-angled triangle upon the base, as *A D* (see Fig. 106), is the cosine of the angle which the hypotenuse makes with the base, and the projection of the hypotenuse upon the side opposite, or *B D*, is the sine of the angle which the hypotenuse makes with the base.

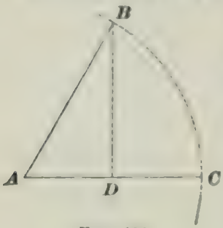

FIG. 106.

In a similar way, a surface is projected upon a flat surface.

Thus, it is desired to project the irregular surface *a b d c*, Fig. 107, upon the flat surface *A B D C*. Draw the lines *a a'*, *b b'* perpendicular to the flat surface; join the points *a'* and *b'* where these perpendiculars intersect the flat surface *A B D C* by a straight line *a' b'*, and *a' b'* is the projection of *a b* upon *A B D C*. The projection of the surface *a b d c* upon the plane *A B D C* is, in this case, the quadrilateral *a' b' d' c'*.

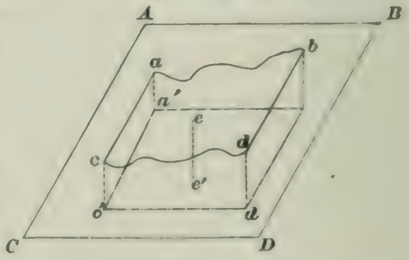

FIG. 107.

SYMMETRICAL AND SIMILAR FIGURES.

824. An **axis of symmetry** is any line so drawn that, if the part of the figure on one side of the line be folded on this line, it will coincide exactly with the other part, point for point and line for line. Thus, in Fig. 108, if the upper half be folded over on the diameter *C D*, it will coincide exactly with the lower half; also, if the part on the right of the diameter *A B* be folded over on *A B*, it will coincide exactly with the part on the left of this line.

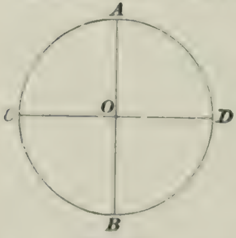

FIG. 108.

It is evident from the above that a circle may have any number of axes of symmetry. In certain cases, however, a figure may be symmetrical with regard to only one axis. Thus, the isosceles triangle *A B C*, Fig. 109, is symmetrical with regard to the axis *B D*, because the part *B C D* would coincide with the part *B A D*, if folded over on the line *B D*; but no other axis of symmetry could be drawn. A rectangle has two axes of symmetry at right angles to each other. A hexagon has six axes of symmetry.

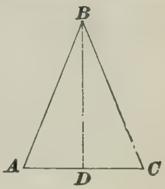

FIG. 109.

825. **Similar figures** are those which are alike in form. As in the case of triangles, which have been considered, two figures, to be similar, must have their corresponding sides in proportion, and the angles of one equal to the corresponding angles of the other. Any two circles are similar.

826. The areas of two similar figures are to each other as the squares of any one dimension. Thus, a parallelogram 10 inches long and 4 inches wide contains 40 square inches. A similar parallelogram 20 inches long would be 8 inches wide, and would contain 160 square inches, while the two areas would be to each other as the squares of the corresponding sides of the parallelograms. That is,

$$40 : 160 = 10^2 : 20^2,$$

$$\text{or } 40 : 160 = 4^2 : 8^2.$$

EXAMPLE.—A circle 10 inches in diameter contains 78.54 square inches; what is the area of one 12 inches in diameter?

SOLUTION.—Let x = the area of the larger circle. Then,

$$78.54 : x = 10^2 : 12^2, \text{ or } x = \frac{78.54 \times 144}{100} = 113.0976 \text{ sq. in. Ans.}$$

827. The cubical contents (and weights) of **similar solids** are to each other as the *cubes* of any one dimension.

EXAMPLE.—If a cast iron ball 9 inches in diameter weighs 100 pounds, what would one 15 inches in diameter weigh?

SOLUTION.—$100 : x = 9^3 : 15^3$, or $x = \frac{100 \times 3{,}375}{729} = 462.96$ pounds, the weight of the larger ball. Ans.

EXAMPLE.—A regular hexagon has sides 5″ long; how much greater will the area of another regular hexagon be whose sides are 30″ long?

SOLUTION.—$30 \div 5 = 6$, or the length of a side of a 30″ hexagon is 6 times as great as the length of a side of a 5″ hexagon; the area will be $6^2 = 36$ times as great. Ans.

This example may also be solved by letting 1 represent the area of the 5″ hexagon. Then, $1 : x = 5^2 : 30^2$, or $x = \frac{900}{25} = 36$.

ELEMENTARY MECHANICS.

MATTER AND ITS PROPERTIES.

DEFINITIONS.

828. Matter is anything that occupies space. It is the substance of which all bodies are composed. Matter is composed of *molecules* and *atoms.*

829. A **molecule** is the smallest portion of matter that can exist without changing its nature.

830. An **atom** is an indivisible portion of matter.

Atoms unite to form molecules, and a collection of molecules form a mass or body.

A drop of water may be divided and subdivided, until each particle is so small that it can only be seen by the most powerful microscope, but each particle will still be water. Now, imagine the division to be carried on still farther until a limit is reached beyond which it is impossible to go without changing the nature of the particle. The particle of water is now so small that, if it be divided again, it will cease to be water, and will be something else; we call this particle a *molecule.*

If a molecule of water be divided, it will yield two atoms of hydrogen gas, and one of oxygen gas. If a molecule of sulphuric acid be divided, it will yield two atoms of hydrogen, one of sulphur, and four of oxygen.

It has been calculated that the diameter of a molecule is larger than $\frac{1}{1250000000}$ of an inch, and smaller than $\frac{1}{5000000000}$ of an inch.

831. Bodies are composed of collections of molecules. Matter exists in three conditions or forms: *solid*, *liquid*, and *gaseous.*

832. A **solid body** is one whose molecules change their relative positions with great difficulty; as iron, wood, stone, etc.

833. A **liquid body** is one whose molecules tend to change their relative positions easily. Liquids readily adapt themselves to the vessel which contains them, and their upper surface always tends to become perfectly level. Water, mercury, molasses, etc., are liquids.

834. A **gaseous body,** or gas, is one whose molecules tend to separate from one another; as air, oxygen, hydrogen, etc.

Gaseous bodies are sometimes called **aeriform** (air-like) **bodies.** They are divided into two classes—the so-called "*permanent*" *gases*, and *vapors*.

835. A **permanent gas** is one which remains a gas at ordinary temperatures and pressures.

836. A **vapor** is a body which at ordinary temperatures is a liquid or solid, but, when heat is applied, becomes a gas, as steam.

One body may be in all three states; as, for example, mercury, which at ordinary temperatures is a liquid, becomes a solid (freezes) at 40° below zero, and a vapor (gas) at 600° above zero. By means of great cold, all gases, even hydrogen, have been liquefied, and some solidified.

By means of heat, all solids have been liquefied and a great many vaporized. It is probable that if we had the means of producing sufficiently great extremes of heat and cold, all solids might be converted into gases, and all gases into solids.

837. Every portion of matter possesses certain qualities called *properties*. Properties of matter are divided into two classes: *general* and *special*.

838. **General properties of matter** are those which are common to all bodies. They are as follows: *Extension*, *impenetrability*, *weight*, *indestructibility*, *inertia*, *mobility*, *divisibility*, *porosity*, *compressibility*, *expansibility*, and *elasticity*.

839. Special properties are those which are not possessed by all bodies. Some of the most important are as follows: *Hardness*, *tenacity*, *brittleness*, *malleability*, and *ductility*.

840. Extension is the property of occupying space. Since all bodies must occupy space, it follows that extension is a general property.

841. By **impenetrability** we mean that no two bodies can occupy exactly the same space at the same time.

842. Weight is the measure of the earth's attraction upon a body. All bodies have weight. In former times it was supposed that gases had no weight, since, if unconfined, they tend to move away from the earth; but, nevertheless, they will finally reach a point beyond which they cannot go, being held in suspension by the earth's attraction. Weight is measured by comparing it with a standard. The standard is a bar of platinum owned and kept by the government; it weighs one pound.

843. Inertia means that a body cannot put itself in motion nor bring itself to rest. To do so, it must be acted upon by some force.

844. Mobility means that a body can be changed in position by some force acting upon it.

845. Divisibility is that property of matter which indicates that a body may be separated into parts.

846. Porosity is that property of matter which indicates that there is space between the molecules of a body. Molecules of bodies are supposed to be spherical, and, hence, there is space between them, as there would be between peaches in a basket. The molecules of water are larger than those of salt; so that when salt is dissolved in water, its molecules wedge themselves between the molecules of the water, and, unless too much salt is added, the water will occupy no more space than it did before. This does not prove that water is penetrable, for the molecules of salt occupy the space that the molecules of water did not.

Water has been forced through iron by pressure, thus proving that iron is porous.

847. Compressibility is that property of matter which indicates that the molecules of a body may be crowded nearer together, so as to occupy a smaller space.

848. Expansibility is that property of matter which indicates that the molecules of a body may be forced apart, so as to occupy a greater space.

849. Elasticity is that property of matter which indicates that if a body be distorted within certain limits, it will resume its original form when the distorting force is removed. Glass, ivory, and steel are very elastic.

850. Indestructibility indicates that matter can never be destroyed. A body may undergo thousands of changes; be resolved into its molecules, and its molecules into atoms, which may unite with other atoms to form other molecules and bodies, which may be entirely different from the original body, but the same number of atoms remains. The whole number of atoms in the universe is exactly the same now as it was millions of years ago, and will always be the same. *Matter is indestructible.*

851. Hardness is that property of matter which indicates that some bodies may scratch other bodies. Fluids and gases do not possess hardness. The diamond is the hardest of all substances.

852. Tenacity is that property of matter which indicates that some bodies resist a force tending to pull them apart. Steel is very tenacious.

853. Brittleness is that property of matter which indicates that some bodies are easily broken; as glass, crockery, etc.

854. Malleability is that property of matter which indicates that some bodies may be hammered or rolled into sheets. Gold is the most malleable of all substances.

855. **Ductility** is that property of matter which indicates that some bodies may be drawn into wire. Platinum is the most ductile of substances.

856. **Mechanics** is that science which treats of the action of forces upon bodies, and the effects which they produce; it treats of the laws which govern the movement and equilibrium of bodies, and shows how they may be utilized.

MOTION AND REST.

VELOCITY.

857. **Motion** is the opposite of rest, and indicates a changing of position in relation to some object. If a large stone is rolled down hill, it is in motion in relation to the hill.

If a person is on a railway train, and walks in the opposite direction from that in which the train is moving, and with the same speed, he will be in motion as regards the train, but at rest with respect to the earth, since, until he gets to the end of the train, he will be directly over the spot at which he was when he started to walk.

858. The **path** of a body in motion is the line described by its *central point.* No matter how irregular the shape of the body may be, nor how many turns and twists it may make; the line which indicates the direction of the center of the body for every instant that it was in motion, is the path of the body.

859. **Velocity** is rate of motion. It is measured by a unit of space passed over in a unit of time. When equal spaces are passed over in equal times, the velocity is said to be **uniform.** In all other cases it is **variable.**

If the fly-wheel of an engine keeps up a constant speed of a certain number of revolutions per minute, the velocity of any point on the wheel is uniform. A railway train having a constant speed of 40 miles per hour, moves 40 miles every hour, or $\frac{40}{60} = \frac{2}{3}$ of a mile every minute; and, since equal spaces are passed over in equal times, the velocity is uniform

Let S = the length of space passed over uniformly;
t = the time occupied in passing over the space S;
V = the velocity.

Then, the velocity V must equal the space S, divided by the time t, or

$$V = \frac{S}{t}. \qquad (7.)$$

Also, the space S must equal the velocity V, multiplied by the time, or

$$S = Vt. \qquad (8.)$$

The time t must equal the space S, divided by the velocity, or

$$t = \frac{S}{V}. \qquad (9.)$$

860. Unless stated otherwise, the *space passed over* will be the *length of the path of the body*, and will be measured in *feet* and decimals of a foot, and, unless otherwise stated, the *time* will be measured in *seconds*.

When these *units* are used, the *velocity will be in feet per second*, which means that the center of the body passed over a certain *number of feet every second*, and the *unit* will be *one foot in one second*.

Example.—The velocity of sound in still air is 1,092 feet per second. If I see the flash of a cannon when it is fired, but do not hear the report until 5 seconds afterwards, how far away is the cannon?

Solution.—$S = Vt = 1,092 \times 5 = 5,460$ feet. Ans.

Example.—The velocity of light is 186,000 miles a second. If the average distance from the earth to the sun is 93,000,000 miles, how long does it take for a beam of light to reach the earth from the sun?

Solution.—$t = \frac{S}{V} = \frac{93,000,000}{186,000} = 500$ seconds, or 8 minutes 20 seconds. Ans.

Example.—If a body passes over a space of 4,800 feet uniformly in 8 minutes, what is its velocity in feet per second?

Solution.—8 minutes = 480 seconds. $V = \frac{S}{t} = \frac{4,800}{480} = 10$. Hence, the velocity is 10 feet per second. Ans.

In examples concerning *work* the *unit of velocity* is usually taken as *one foot in one minute*.

The *unit of time* may be a *second*, *minute*, *hour*, *day*, or *year*. The *unit of space* may be *feet*, *miles*, *the earth's radius*, or the *distance from the earth to the sun*, according to the conditions of the example. The larger units are used only in astronomy.

EXAMPLE.—The distance from the earth to the moon is about 60 times the radius of the earth; how many miles is it from here to the moon?

SOLUTION.—The radius of the earth is nearly 4,000 miles; hence, $4,000 \times 60 = 240,000$ miles, the distance to the moon, nearly. Ans.

EXAMPLES FOR PRACTICE.

1. The piston speed of a steam engine is 10 feet per second; how many miles will the piston travel in one hour? Ans. $6\frac{9}{11}$ mi.

2. If a railroad train travels 70 miles in one hour, what is its velocity in feet per second? Ans. $102\frac{2}{3}$ ft. per sec.

3. A man runs 100 yards in 12 seconds; how long will it take him to run a mile at the same rate? Ans. 3 min. 31.2 sec.

4. The outside diameter of an engine fly-wheel is 13 feet 9 inches. A point on the rim travels 45,000 feet in 5 minutes; what is the velocity in feet per second? Ans. 150 ft. per sec.

FORCE.

THE THREE LAWS OF MOTION.

861. A **force** is that which produces, or tends to produce or destroy, motion. Forces are called by various names, according to the effects which they produce upon a body, as *attraction*, *repulsion*, *cohesion*, *adhesion*, *accelerating* force, *retarding* force, *resisting* force, etc., but all are equivalent to a push or pull, according to the direction in which they act upon a body. That the effect of a force upon a body may be compared with another force, it is necessary that three conditions be fulfilled in regard to both forces; they are as follows:

(1.) *The point of application, or point at which the force acts upon the body, must be known.*

(2.) *The direction of the force, or, what is the same thing, the straight line along which the force tends to move the point of application, must be known.*

(3.) *The magnitude or value of the force, when compared with a given standard, must be known.*

862. The unit of magnitude of forces will *always* be taken as *one pound*, in this section on Elementary Mechanics, and all forces will be spoken of as a certain number of pounds.

863. According to the effects which forces produce upon a body, the science of Mechanics is subdivided as follows:

(1.) *Mechanics of Solid Bodies.*
(2.) *Mechanics of Fluids.*
(3.) *Mechanics of Heat, or Thermodynamics.*

Mechanics of Solids is further divided into *Statics* and *Kinetics*, or *Dynamics*, as it is commonly called.

Mechanics of Fluids is further divided into *Mechanics of Air and Gases*, or *Pneumatics*, and *Mechanics of Liquids.* The Mechanics of Liquids is divided into *Hydrostatics* and *Hydrokinetics;* the latter is also called *Hydraulics* and *Hydrodynamics.*

864. Statics treats of the conditions of the equilibrium of bodies. A body is in **equilibrium** under the action of forces, when the forces acting upon the body balance each other.

865. Kinetics, or **Dynamics,** treats of bodies in motion, and the effects which they may produce.

866. Pneumatics treats of the laws of the pressure and of the movement of air and other gaseous bodies.

867. Hydrostatics treats of the equilibrium of liquids.

868. Hydrokinetics (also called Hydraulics and Hydrodynamics) treats of liquids in motion, and the effects which they may produce.

869. Thermodynamics treats of the mechanical effects of heat upon bodies.

870. The fundamental principles of the relations between force and motion were first stated by Sir Isaac Newton. They are called "Newton's Three Laws of Motion," and are as follows:

(I.) *All bodies continue in a state of rest, or of uniform motion, in a straight line, unless acted upon by some external force that compels a change.*

(II.) *Every motion, or change of motion, is proportional to the acting force, and takes place in the direction of the straight line along which the force acts.*

(III.) *To every action there is always opposed an equal and contrary reaction.*

From the *first law of motion*, it is inferred that a body once set in motion by any force, no matter how small, will move forever in a straight line, and always with the same velocity, unless acted upon by some other force which compels a change. It is not possible to actually verify this law, on account of the earth's attraction for all bodies, but, from astronomical observations, we are certain that the law is true. This law is often called *the law of inertia.*

871. The word **inertia** is so abused that a full understanding of its meaning is necessary. Inertia is not a force, although it is often so called. If a force acts upon a body and puts it in motion, the effect of the force is stored in the body; and a second body, in stopping the first, will receive a blow equal in every respect to the original force, assuming that there has been no resistance of any kind to the motion of the first body.

It is dangerous for a person to jump from a fast-moving train, for the reason that, since his body has the same velocity as the train, it has the same force stored in it that would cause a body of the same weight to take the same velocity as the train, and the effect of a sudden stoppage is the same as the effect of a blow necessary to give the person that velocity. But, by "bracing" himself and jumping in the same direction that the train is moving, and running, he brings himself gradually to rest, and thus reduces the danger. If a body is at rest, it must be acted upon by a force in order to be put in motion, and, no matter how great the force may be, the body cannot be *instantly* put in motion.

The resistance thus offered to being put in motion is commonly, but erroneously, called, the *Resistance of Inertia*. It should be called the *Resistance due to Inertia*.

From the *second law*, it is seen that, if two or more forces act upon a body, their final effect upon the body will be in proportion to their magnitude and to the directions in which they act. Thus, if the wind is blowing due west, with a velocity of 50 miles per hour, and a ball is thrown due north with the same velocity, or 50 miles per hour, the wind will carry the ball just as far west as the force of the throw carried it north, and the combined effect will be to cause it to move north-west. The amount of departure from due north will be proportional to the force of the wind, and independent of the velocity due to the force of the throw.

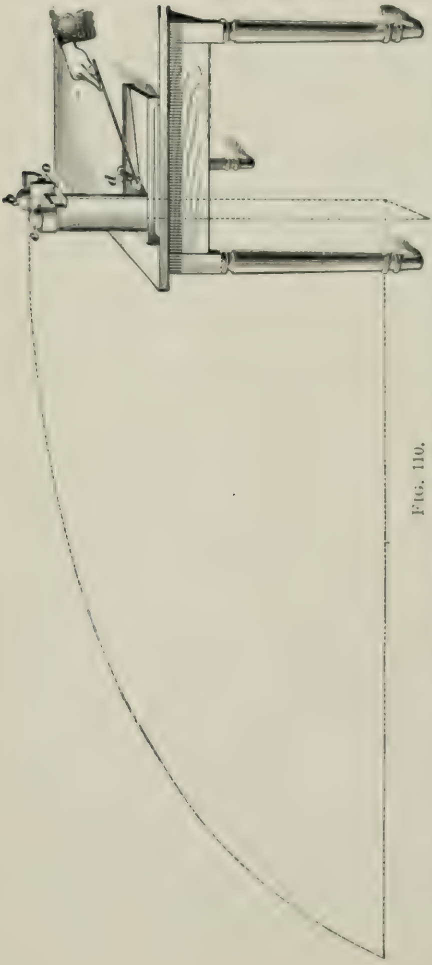

Fig. 110.

In Fig. 110, a ball *e* is supported in a cup, the bottom of which is attached to the lever *o* in such a manner that a movement of *o* will swing the bottom horizontally and allow the ball to drop. Another ball *b* rests in a horizontal groove that is provided with a slit in the bottom. A swinging arm is actuated by the spring *d* in such a manner that, when drawn back as shown and then released, it will strike the lever *o* and the ball *b* at the same time. This gives *b* an impulse in a horizontal direction and swings *o* so as to allow *e* to fall.

On trying the experiment, it is found that *b* follows a path shown by the curved dotted line, and reaches the floor at the same instant as *e*, which drops vertically. This shows that the force which gave the first ball its horizontal movement, had no effect on the vertical force which compelled both balls to fall to the floor, the vertical force producing the same effect as if the horizontal force had not acted. The second law may also be stated as follows: *A force has the same effect in producing motion, whether it acts upon a body at rest, or in motion, and whether it acts alone or with other forces.*

The *third law* states that action and reaction are equal and opposite. A man cannot lift himself by his boot-straps, for the reason that he presses downwards with the same force that he pulls upwards; the downward reaction equals the upward action, and is opposite to it.

In springing from a boat we must exercise caution, or the reaction will drive the boat from the shore. When we jump from the ground, we tend to push the earth from us, while the earth reacts and pushes us from it.

872. A **force** may be represented by a line; thus, in Fig. 111, let *A* be the *point of application* of the force; let the length of the line *A B* represent its *magnitude*, and let the arrow-head indicate the *direction* in which the force acts, then the line *A B* fulfils the three conditions (see Art. **861**), and the force is fully represented.

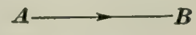

FIG. 111.

THE COMPOSITION OF FORCES.

873. When two forces act upon a body at the same time, but at different angles, their final result may be obtained as follows:

In Fig. 112, let A be the common *point of application* of the two forces, and let $A\ B$ and $A\ C$ represent the *magnitude* and *direction* of the forces. According to the second law of motion, the final effect of the movement due to these two forces would be the same whether they acted singly or together. Suppose that the line $A\ B$ represents the distance that the force $A\ B$ would cause the body to move; similarly, that $A\ C$ represents the distance which the force $A\ C$ would cause the body to move when both forces were acting separately. The force $A\ B$, acting alone, would carry the body to B; if the force $A\ C$ were now to act upon the body, it would carry it along the line $B\ D$, parallel to $A\ C$, to a point D, at a distance from B equal to $A\ C$. Join C and D, then $C\ D$ is parallel to $A\ B$, and $A\ B\ D\ C$ is a parallelogram. Draw the diagonal $A\ D$. According to the second law of motion, the body will stop at D, whether the forces act separately or together, but if they act together, the path of the body will be along $A\ D$, the diagonal of the parallelogram. Moreover, the length of the line $A\ D$ represents the *magnitude* of a force which, acting at A in the *direction* $A\ D$, would cause the body to move from A to D; in other words, $A\ D$, measured to the same scale as $A\ B$ and $A\ C$, represents, in *magnitude* and *direction*, the combined effect of the two forces $A\ B$ and $A\ C$.

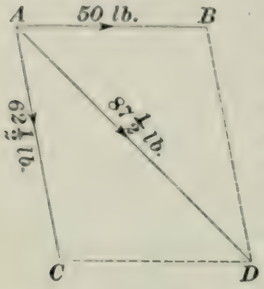

FIG. 112.

874. This line $A\ D$ is called the **resultant.** Suppose that the scale used was 50 pounds to the inch; then, if $A\ B = 50$ pounds, and $A\ C = 62\frac{1}{2}$ pounds, the length of $A\ B$ would be $\frac{50}{50} = 1$ inch, and the length of $A\ C$ would be $\frac{62.5}{50} = 1\frac{1}{4}$ inches. If $A\ D$, or the *resultant*, measures $1\frac{3}{4}$ inches, its *magnitude* would be $1\frac{3}{4} \times 50 = 87\frac{1}{2}$ pounds.

Therefore, a force of $87\frac{1}{2}$ pounds acting upon a body at A in the direction $A\ D$, will produce the same result as the combined effects of a force of 50 pounds acting in the direction $A\ B$, and a force of $62\frac{1}{2}$ pounds acting in the direction $A\ C$.

875. The above method of finding the resulting action of two forces acting upon a body at a common point, is correct, whatever may be their direction and magnitudes. Hence, to find the **resultant** of two forces when their common point of application, their direction and magnitudes are known:

Rule I.—*Assume a point, and draw two lines parallel to the directions of the lines of action of the two forces. With any convenient scale, measure off from the point of intersection (common point of application), distances corresponding to the magnitudes of the respective forces, and complete the parallelogram. From the common point of application, draw the diagonal of the parallelogram; this diagonal will be the resultant, and its direction will be away from the point of application. Its magnitude should be measured with the same scale that was used to measure the two forces.*

This method is called the **graphical method of the parallelogram of forces.**

876. EXPERIMENTAL PROOF.—The principle of the parallelogram of forces is clearly shown in Fig. 113. $A\ B\ D\ C$ is a wooden frame, jointed to allow motion at its four corners. The length $A\ B$ equals $C\ D$; that of $A\ C$ equals $B\ D$, and the corresponding adjacent sides are in the ratio of two to three. Cords pass over the pulleys M and N, carrying weights W and w, of 90 and 60 pounds. The ratio between the weights equals the ratio of the corresponding adjacent sides. A weight V of 120 pounds is hung from the corner A.

When the frame comes to rest, the sides $A\ B$ and $A\ C$ lie in the direction of the cords. These sides $A\ B$ and $A\ C$ are accurate graphic representations of the two forces acting upon the point A. It will be found that the diagonal $A\ D$ is vertical, and twice as long as $A\ C$; hence, since $A\ C$ represents a force of 60 pounds, $A\ D$ will represent a force of 2×60, or 120 pounds.

Thus, we see that the line $A\ D$ represents the *resultant* of the two forces $A\ B$ and $A\ C$; in other words, it represents the resultant of the two weights W and w. This resultant is equal and opposite to the vertical force, which is due to the weight of V.

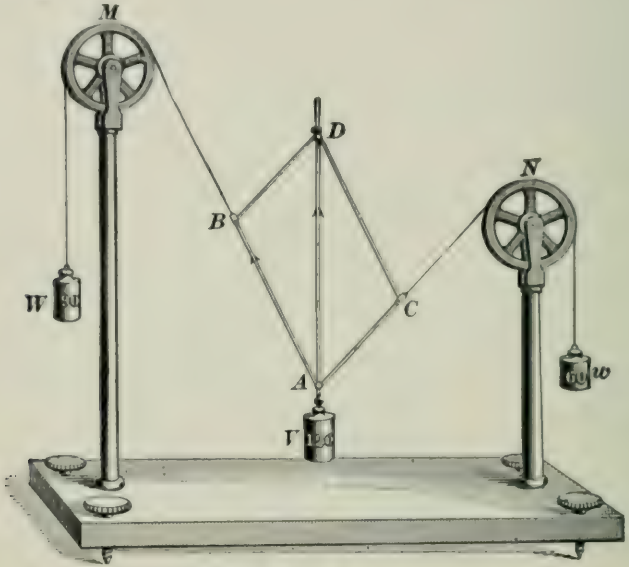

FIG. 113.

Satisfactory results of this kind will be secured when we have the proportion

$$AB : AC = W : w.$$

EXAMPLE.—If two forces act upon a body at a common point, both acting away from the body, and the angle between them is 80°, what is the value of the resultant, the magnitude of the two forces being 60 pounds and 90 pounds, respectively?

SOLUTION.—Draw two indefinite lines having an angle of 80° between them. With any convenient scale, say 10 pounds to the inch, measure off $AB = 60 \div 10 = 6$ inches, and $AC = 90 \div 10 = 9$ inches.

Through B, draw BD parallel to AC, and through C, draw CD parallel to AB, intersecting at D. Then draw AD, and AD will be the *resultant;* its *direction* is towards the point D, as shown by the arrow.

Measuring AD, we find that its length $= 11.7$ inches. Hence, $11.7 \times 10 = 117$ pounds. Ans.

Caution.—In solving problems by the graphical method, *use as large a scale as possible.* More accurate results are then obtained.

877. The above example might also have been solved by the method called the **triangle of forces,** which is as follows:

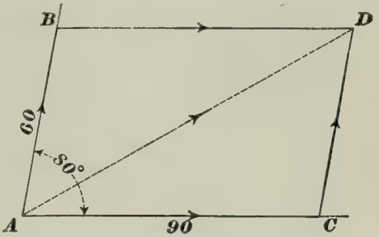

FIG. 114.

In Fig. 114, suppose that the two forces acted separately, first from A to B, and then from B to D, in the direction of the arrows.

Draw $A D$; then $A D$ is the *resultant* of the forces $A B$ and $A C$, since $B D = A C$; but $A D$ is a side of the triangle $A B D$. It will also be noticed that the direction of $A D$ is *opposed* to that of $A B$ and $B D$; hence, to find the **resultant** of two forces acting upon a body at a common point, by the method of triangle of forces:

Rule II.—*Draw the lines of action of the two forces as if each force acted separately, the lengths of the lines being proportional to the magnitude of the forces. Join the extremities of the two lines by a straight line, and it will be the resultant; its direction will be opposite to that of the two forces.*

NOTE.—When we speak of the resultant being opposed in direction to the other forces around the polygon, we mean that, starting from the point where we began to draw the polygon, and tracing each line in succession, the pencil will have the same general direction around the polygon, as if passing around a circle, from left to right, or from right to left, but that the closing line or resultant must have an *opposite direction*, that is, *the two arrow-heads must point towards the point of intersection of the resultant and the last side.*

878. When three or more forces act upon a body at a given point, their *resultant* may be found by the following rule:

Rule III.—*Find the resultant of any two forces; treat this resultant as a single force, and combine it with a third force to find a second resultant. Combine this second resultant with a fourth force, to find a third resultant, etc. After all the forces have been thus combined, the last resultant will be the resultant of all of the forces, both in magnitude and direction.*

EXAMPLE.—Find the resultant of all the forces acting on the point O in Fig. 115, the length of the lines being proportional to the magnitude of the forces.

SOLUTION.—Draw OE parallel and equal to AO, and EF parallel and equal to BO, then OF is the resultant of these two forces, and its direction is from O to F, opposed to OE and EF. Treat OF as if OE and EF did not exist, and draw FG parallel and equal to OC; OG will be the resultant of OF and FG; but OF is the resultant of OE and EF, hence, OG is the resultant of OE, EF, and FG, and likewise of AO, BO, and CO. The line FG, parallel to CO, could not be drawn from the point O to the right of OE, for in that case it would be opposed in direction to OF; but FG must have the same direction as OF, in order that the resultant may be opposed to both OF and FG.

For the same reason, draw GL parallel and equal to DO. Join O and L, and OL will be the *resultant* of all the forces AO, BO, CO, and DO (both in magnitude and direction), acting at the point O. If $L'O$ were drawn parallel and equal to OL, and having the same direction, it would represent the effect produced on the body by the combined action of the forces AO, BO, CO, and DO.

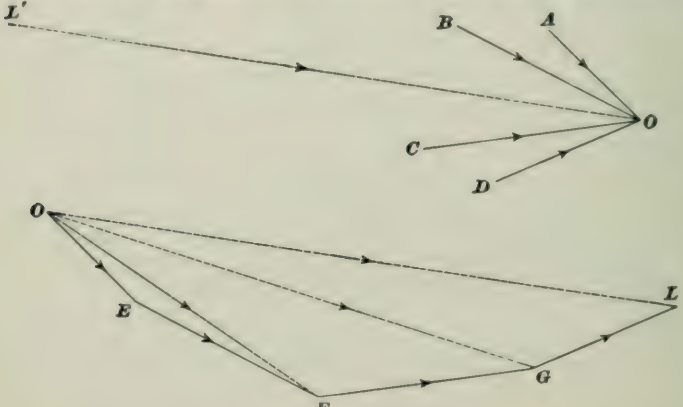

FIG. 115.

879. In the last figure, it will be noticed that OE, EF, FG, GL, and LO are sides of a polygon $OEFGL$, in which OL, the resultant, is the closing side, and that its direction is opposed to that of all the other sides. This fact is made use of in what is called the **method of the polygon of forces.**

To find the resultant of several forces acting upon a body at the same point :

Rule IV.—*Through a convenient point on the drawing, draw a line parallel to one of the forces, and having the same* *direction and magnitude. At the end of this line, draw another line parallel to a second force, and having the same direction and magnitude as this second force; at the end of the second line, draw a line parallel and equal in magnitude and direction to a third force. Thus continue until lines have been drawn parallel and equal in magnitude and direction to all of the forces.*

The straight line joining the free ends of the first and last lines will be the closing sides of the polygon; mark it opposite in direction to that of the other forces around the polygon, and it will be the resultant of all the forces.

EXAMPLE.—If five forces act upon a body at angles of 60°, 120°, 180°, 240°, and 270°, towards the same point, and their respective magnitudes are 60, 40, 30, 25, and 20 pounds, find the magnitude and direction of their resultant by the method of polygon of forces.*

SOLUTION.—From a common point O, Fig. 116, draw the lines of action of the forces, making the given angles with a horizontal line through O, and mark them as acting towards O, by means of arrowheads, as shown. Now, choose some convenient scale, such that the whole figure may be drawn in a space of the required size on the drawing. Choose any one of the forces, as AO, and draw OF parallel to it, and equal in length to 30 pounds on the scale. It must also act in the same direction as OA. At F, draw FG parallel to BO, and equal to 40 pounds. In a similar manner, draw GH, HI, and IK parallel to

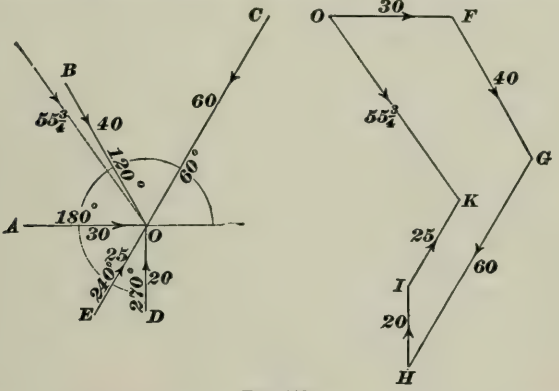

FIG. 116.

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*NOTE.—As stated in Art. 742, all angles are measured from a horizontal line, in a direction opposite to the movement of the hands of a watch (from around the circle to the left), from 1° or less up to 360°.

C O, *D O*, and *E O*, and equal to 60, 20, and 25 pounds, respectively. Join *O* and *K* by *O K*, and *O K* will be the resultant of the combined action of the five forces; its direction is opposite to that of the other forces around the polygon *O F G H I K*, and its magnitude = $55\frac{3}{4}$ pounds. Ans.

If the resultant *O K*, in Fig. 116, were to act alone upon the body in the direction shown by the arrow-head, with a force of $55\frac{3}{4}$ pounds, it would produce exactly the same effect upon a body as the combined action of the five forces.

If *O F*, *F G*, *G H*, *H I*, and *I K* represent the distances and directions that the forces would move the body, if acting separately, *O K* is the direction and distance of movement of the body when all the forces act together.

880. From what has been said before, it is seen that any number of forces acting on a body at the same point, or having their lines of action pass through the same point, can be replaced by a *single force* (resultant), whose line of action shall pass through that point.

881. Heretofore, it has been assumed that the forces acted upon a single point on the *surface* of the body, but it will make no difference where they act, so long as the lines of action of all the forces intersect at a *single point*, either within or without the body, only so that the resultant can be drawn through the *point of intersection*. If two forces act upon a body in the same straight line and in the same direction, their *resultant* is the *sum of the two forces*; but, if they act in opposite directions, their *resultant* is the *difference of the two forces*, and its direction is the same as that of the greater force. If they are equal and opposite, the *resultant* is *zero*, or one force just balances the other.

EXAMPLE.—Find the resultant of the forces whose lines of action pass through a single point, as shown in Fig. 117.

SOLUTION.—Take any convenient point *g*, and draw a line *g f*, parallel to one of the forces, say the one marked 40, making it equal in length to 40 pounds on the scale, and indicate its direction by the arrow-head. Take some other force—the one marked 37 will be convenient; the line *f e* represents this force. From the point *e*, draw a line parallel to some other force; say the one marked 29, and make it equal in magnitude and direction to it. So continue with the other forces, taking care that the general direction around the polygon is not changed. The last force drawn in the figure is *a b*, representing the force marked 25. Join the points *a* and *g*; then, *a g* is the resultant of all the forces shown in the figure. Its direction is from *g* to *a* opposed to the general direction of the others around the polygon. It does not matter in what order the different forces are taken, the resultant will be the same in magnitude and direction, if the work is done correctly.

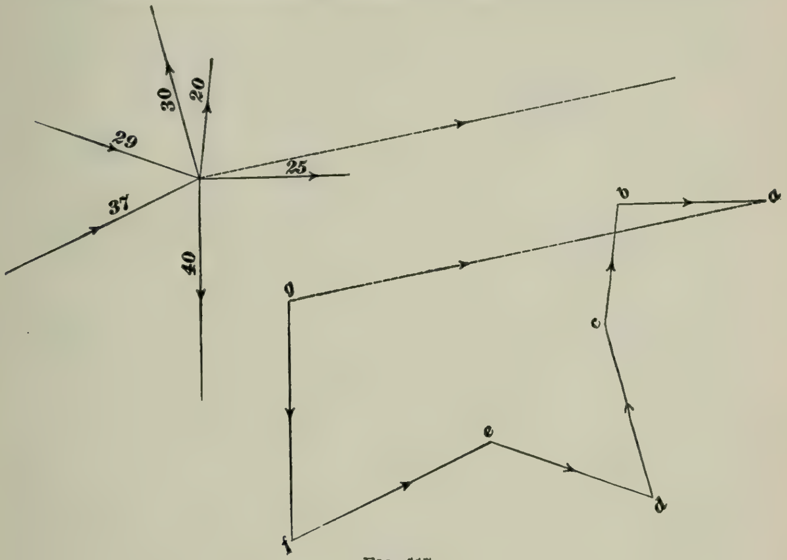

FIG. 117.

882. The various methods of finding the resultant of several forces are all grouped under one head : The **composition of forces.**

THE RESOLUTION OF FORCES.

883. Since two forces can be combined to form a single resultant force, we may also treat a single force as if it were the resultant of two forces, whose action upon a body will be the same as that of a single force. Thus, in Fig. 118, the force *O A* may be resolved into two forces, *O B* and *B A*, whose directions are opposed to *O A*.

If the force *O A* acts upon a body, moving or at rest upon a horizontal plane, and the resolved force *O B* is vertical, and *B A* horizontal, *O B*, measured to the same scale as

O A, is the magnitude of that part of *O A* which pushes the body *downwards*, while *B A* is the magnitude of that part of the force *O A* which is exerted in pushing the body in a *horizontal* direction. *O B* and *B A* are called the **components** of the force *O A*, and when these components are vertical and horizontal, as in the present case, they are called the *vertical component* and the *horizontal component* of the force *O A*.

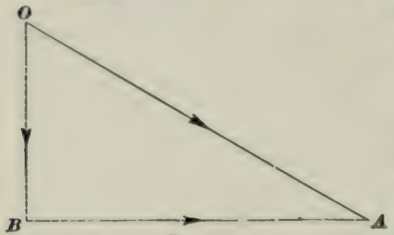

FIG. 118.

884. It frequently happens that the position, magnitude and direction of a certain force is known, and that it is desired to know the effect of the force in some direction, other than that in which it acts. Thus, in Fig. 118, suppose that *O A* represents, to some scale, the magnitude, direction, and line of action of a force acting upon a body at *A*, and that it is desired to know what effect *O A* produces in the direction *B A*. Now *B A*, instead of being horizontal, as in the cut, may have any direction. To find the value of the component of *O A* which acts in the direction *B A*, we employ the following rule:

Rule V.—*From one extremity of the line representing the given force, draw a line parallel to the direction in which it is desired that the component shall act; from the other extremity of the given force, draw a line perpendicular to the component first drawn, and intersecting it. The length of the component, measured from the point of intersection to the intersection of the component with the given force, will be magnitude of the effect produced by the given force in the required direction.*

Thus, suppose *O A*, Fig. 118, represents a force acting upon a body resting upon a horizontal plane, and it is desired to know what *vertical pressure O A* produces on the body. Here the desired direction is vertical; hence, from one extremity, as *O*, draw *O B* parallel to the desired direction (vertical in this case), and from the other extremity, draw AB perpendicular to OB, and intersecting OB at B. Then OB, when measured to the same scale as OA, will be the value of the vertical pressure produced by OA.

Example.—If a body weighing 200 pounds rests upon an inclined plane whose angle of inclination to the horizontal is 18°, what force does it exert perpendicular to the plane, and what force does it exert parallel to the plane, tending to slide downwards?

Solution.—Let ABC, Fig. 119, be the plane, the angle A being equal to 18°, and let W be the weight. Draw a vertical line $FD =$ 200 pounds, to represent the magnitude of the weight. Through F, draw FE parallel to AB, and through D draw DE perpendicular to EF, the two lines intersecting at E. FD is now resolved into two components, one, FE, tending to pull the weight downwards, and the other, ED, acting as a perpendicular pressure on the plane.

Since FD is perpendicular to AC, and ED is perpendicular to AB, the angle $D =$ angle $A = 18°$.

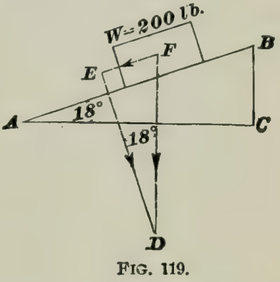

Fig. 119.

Hence, $FE = 200 \times \sin 18° = 200 \times .30902 = 61.804$ pounds, and $ED = 200 \times \cos 18° = 200 \times .95106 = 190.212$ pounds.

$$\left.\begin{array}{l}\text{Force parallel to the plane} = 61.804 \text{ pounds.}\\ \text{Force perpendicular to the plane} = 190.212 \text{ pounds.}\end{array}\right\} \text{Ans.}$$

DYNAMICS.

885. Dynamics may be defined as that branch of Mechanics which deals with bodies moving with a variable velocity. In Elementary Mechanics we shall consider only falling bodies and centrifugal force.

GRAVITATION.

886. Every body in the universe exerts a certain attractive force on every other body, which tends to draw the two bodies together. This attractive force is called **gravitation.**

If a body is held in the hand, a downward pull is felt, and if let go of, it will fall to the ground. This pull is commonly called *weight*, but it really is the attraction between the earth and the body.

887. Force of gravity is a term used to denote the attraction between the earth and bodies upon or near its surface. It always acts in a straight line between the center of the body and the center of the earth. The force of gravity varies at points on the earth's surface.

It is slightly less on the top of a high mountain than at the level of the sea. For this reason, the weight of a body also varies. But if the weight of a body at any place be divided by the force of gravity at that place, the result is called the *mass* of the body.

888. The **mass of a body** is the measure of the actual amount of matter that it contains, and is *always the same.*

If the mass of the body be represented by m, its weight by W, and the force of gravity at the place where the body was weighed, by g, we have

$$\text{mass} = \frac{\text{weight of body}}{\text{force of gravity}}, \text{ or } m = \frac{W}{g}. \qquad \textbf{(10.)}$$

889. Law of Gravitation :—

The force of attraction by which one body tends to draw another body towards it, is directly proportional to its mass, and inversely proportional to the square of the distance between their centers.

890. Laws of Weight :—

Bodies weigh most at the surface of the earth. Below the surface, the weight decreases as the distance to the center decreases.

Above the surface the weight decreases as the square of the distance increases.

ILLUSTRATION.—If the earth's radius is 4,000 miles, a body that weighs 100 pounds at the surface will weigh nothing at the center, since it is attracted in every direction with equal force. At 1,000 miles from the center, it will weigh 25 pounds, since

$$4{,}000 : 1{,}000 = 100 : 25.$$

At 2,000 miles from the center, it will weigh 50 pounds, since

$$4{,}000 : 2{,}000 = 100 : 50.$$

At 3,000 miles from the center, it will weigh 75 pounds, and at the surface, or 4,000 miles from the center, it will weigh 100 pounds. If carried still higher, say 1,000 miles from the surface, or 5,000 miles from the center of the earth, it will weigh 64 pounds, since

$$5{,}000^2 : 4{,}000^2 = 100 : 64.$$

At 4,000 miles from the surface, it will weigh 25 pounds, since

$$8{,}000^2 : 4{,}000^2 = 100 : 25.$$

891. Formulas for Gravity Problems:—

Let W = weight of body at the surface;

w = weight of a body at a given distance above or below the surface;

d = distance between the center of the earth and the center of the body;

R = radius of the earth = 4,000 miles.

Formula for weight when the body is below the surface:

$$w\,R = d\,W. \qquad \textbf{(11.)}$$

Formula for weight when the body is above the surface:

$$w\,d^2 = W\,R^2. \qquad \textbf{(12.)}$$

EXAMPLE.—How far below the surface of the earth will a 25-pound ball weigh 9 pounds?

SOLUTION.—Use formula **11**, $w\,R = d\,W$.

Substituting the values of R, W, and w, we have

$$9 \times 4{,}000 = d \times 25, \text{ or}$$

$$d = \frac{9 \times 4{,}000}{25} = 1{,}440 \text{ miles from the center. Ans.}$$

EXAMPLE.—If a body weighs 700 pounds at the surface of the earth, at what distance above the earth's surface will it weigh 112 pounds?

SOLUTION.—Use formula **12**, $w\,d^2 = W\,R^2$.

Substituting the values of R, W, and w, we have

$$112 \times d^2 = 700 \times 4{,}000^2, \text{ or}$$

$$d = \sqrt{\frac{700 \times 4{,}000^2}{112}} = 10{,}000 \text{ miles.}$$

Therefore, 10,000 − 4,000 = 6,000 miles above the earth's surface. Ans.

EXAMPLE.—The top of Mt. Hercules was said to be 32,000 feet, say 6 miles above the level of the sea. If a body weighs 1,000 pounds at sea-level, what would it weigh if carried to the top of the mountain?

SOLUTION.— $w\,d^2 = W\,R^2$, or $w \times 4{,}006^2 = 1{,}000 \times 4{,}000^2$; whence,

$$w = \frac{4{,}000^2 \times 1{,}000}{4{,}006^2} = 997 \text{ pounds. Ans.}$$

EXAMPLES FOR PRACTICE.

1. How much would 1,000 tons of coal weigh one mile below the surface? Ans. 1,999,500 lb.

2. How much would the coal in example 1 weigh one mile above the surface? Ans. 1,999,000 lb., nearly.

3. How far above the earth's surface would it be necessary to carry a body in order that it may weigh only half as much? Ans. 1,656.854 miles, nearly.

4. A man weighs 160 pounds at the surface; how much will he weigh 50 miles below the surface? Ans. 158 lb.

5. If a body weighs 100 pounds 400 miles above the earth's surface, how much will it weigh at the surface? Ans. 121 lb.

NOTE.—*Use 4,000 miles as the radius of the earth.*

FALLING BODIES.

892. If a leaden ball and a piece of paper are dropped from the same height, the ball would strike the ground first. This is not because the leaden ball is the heavier, but because the resistance of the air has a greater retarding effect upon the paper than upon the ball. If we place this same leaden ball and a piece of paper in a glass tube, Fig. 120, from which all of the air has been exhausted, it would be found that when the tube was inverted, both would drop to the bottom in exactly the same time. This experiment proves that it was only the resistance of the air that caused the ball to reach the ground first, in the former experiment. This resistance of the air may be nearly equalized by making the two bodies of the same shape and size. For example, if a wooden and an iron ball, having equal diameters, were dropped from the same height, they would strike the ground at almost exactly the same instant, although the iron ball might be ten times as heavy as the wooden ball.

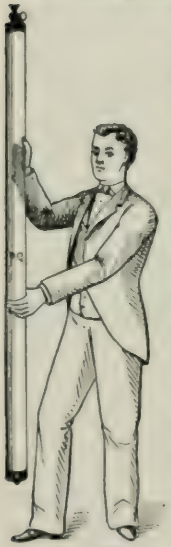

FIG. 120.

893. Suppose there were several leaden balls, as shown in Fig. 121, at *a*; it is obvious that if they were dropped together, all would strike the ground at the same time. If the balls were melted together into one ball, as *b*, they would still fall together, and strike the ground in the same time as before.

FIG. 121.

Since a number of horses cannot run a mile in less time than a single horse, so 100 pounds can fall no further in a given time than one pound can.

894. Acceleration is the rate of change of velocity. If a force acts upon a body free to move, then, according to the first law of motion, it will move forever with the same velocity unless acted upon by another force.

Suppose that, at the end of one second, the same force were to act again, the velocity at the end of the second second would be twice as great as at the end of the first second. If the same force were to act again, the velocity at the end of the third second would be three times that at the end of the first second. So, if a constant force acts upon a body free to move, the velocity of the body at the end of any time will be the velocity at the end of one second, multiplied by the number of seconds.

895. This constant force is called a **constant accelerating force,** or **constant retarding force,** according as the velocity is constantly *increased* or *decreased.*

If a body is dropped from a high tower, the velocity with which it approaches the ground will be constantly increased or accelerated; for the attraction of the earth, or force of gravity, is constant, and acts upon the body as a constant accelerating force. It has been found by careful experiments that this force of gravity, or constant accelerating force, on a freely falling body, is equivalent to giving the body a velocity of 32.16 feet in one second; it is always denoted by g. As was mentioned before, g varies at different points of the earth, being 32.0902 at the equator, and 32.2549 at the poles. Its value for this latitude (about 41° 25′ north) is very nearly 32.16, and this value should always be used in solving problems. It has also been found by

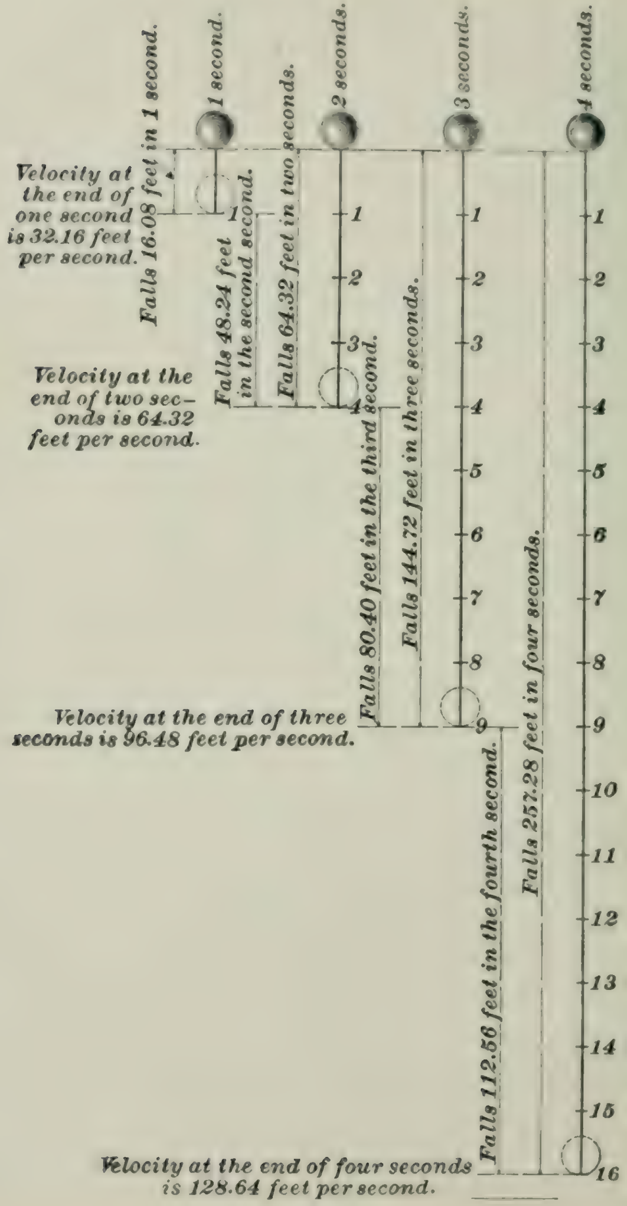
1 second.
2 seconds.
3 seconds.
4 seconds.
Falls 16.08 feet in 1 second.
Velocity at the end of one second is 32.16 feet per second.
Falls 48.24 feet in the second second.
Falls 64.32 feet in two seconds.
Velocity at the end of two seconds is 64.32 feet per second.
Falls 80.40 feet in the third second.
Falls 144.72 feet in three seconds.
Velocity at the end of three seconds is 96.48 feet per second.
Falls 112.56 feet in the fourth second.
Falls 257.28 feet in four seconds.
Velocity at the end of four seconds is 128.64 feet per second.

FIG. 122.

experiment that a freely falling body starting from rest will have fallen 16.08 feet at the end of the first second; 64.32 feet at the end of the second second; 144.72 feet at the end of the third second; 257.28 feet at the end of the fourth second, etc., all of which are shown in the diagram, Fig. 122.

Since $\frac{64.32}{16.08} = 4 = 2^2$; $\frac{144.72}{16.08} = 9 = 3^2$; $\frac{257.28}{16.08} = 16 = 4^2$, and 2^2, 3^2, 4^2 are the squares of the number of seconds during which the body falls, it is easy to see that the space through which a body free to move will fall in a given time is equal to 16.08 multiplied by the square of the time in seconds.

Since $16.08 = \frac{32.16}{2} = \frac{1}{2}g$, the space $= \frac{1}{2}g \times$ square of time in seconds.

896. Formulas for Falling Bodies:—

Let g = force of gravity = constant accelerating force due to the attraction of the earth;
t = number of seconds the body falls;
v = velocity at the end of the time t;
h = distance that a body falls during the time t.

$$v = gt. \qquad \textbf{(13.)}$$

That is, the velocity acquired by a freely falling body at the end of t seconds equals 32.16, multiplied by the time in seconds.

Example.—What is the velocity of a body after it has fallen four seconds, assuming that the air offered no resistance?

Solution.—Using formula **13**,

$$v = gt = 32.16 \times 4 = 128.64 \text{ feet per second. Ans.}$$

$$t = \frac{v}{g}. \qquad \textbf{(14.)}$$

That is, the number of seconds during which a body must have fallen to acquire a given velocity equals the given velocity in feet per second, divided by 32.16.

Example.—A falling body has a velocity of 192.96 feet per second; how long had it been falling at that instant?

Solution.—Using formula **14**,

$$t = \frac{v}{g} = \frac{192.96}{32.16} = 6 \text{ seconds. Ans.}$$

$$h = \frac{v^2}{2g}. \qquad \textbf{(15.)}$$

That is, the height from which a body must fall to acquire a given velocity equals the square of the given velocity, divided by 2 × 32.16.

EXAMPLE.—From what height must a stone be dropped to acquire a velocity of 24,000 feet per minute?

SOLUTION.— 24,000 ÷ 60 = 400 feet per second. Using formula **15**,

$$h = \frac{v^2}{2g} = \frac{400^2}{2 \times 32.16} = \frac{160,000}{64.32} = 2,487.56 \text{ feet. Ans.}$$

$$v = \sqrt{2gh.} \qquad \textbf{(16.)}$$

That is, the velocity that a body will acquire in falling through a given height equals the square root of the product of twice 32.16, and the given height.

EXAMPLE.—A body falls from a height of 400 feet; what will be its velocity at the end of its fall?

SOLUTION.—Using formula **16**,

$$v = \sqrt{2gh} = \sqrt{2 \times 32.16 \times 400} = 160.4 \text{ feet per second. Ans.}$$

$$h = \tfrac{1}{2} gt^2. \qquad \textbf{(17.)}$$

That is, the distance a body will fall in a given time equals 32.16 ÷ 2, multiplied by the square of the number of seconds.

EXAMPLE.—How far will a body fall in 10 seconds?

SOLUTION.—Using formula **17**,

$$h = \tfrac{1}{2} gt^2 = \tfrac{1}{2} \times 32.16 \times 10^2 = 1,608 \text{ feet. Ans.}$$

$$t = \sqrt{\frac{2h}{g}}. \qquad \textbf{(18.)}$$

That is, the time it will take a body to fall through a given height equals the square root of twice the height divided by 32.16.

EXAMPLE.—How long will it take a body to fall 4,116.48 feet?

SOLUTION.—Using formula **18**,

$$t = \sqrt{\frac{2 \times 4,116.48}{32.16}} = 16 \text{ seconds. Ans.}$$

897. A body thrown vertically upwards starts with a certain velocity called the **initial velocity.** In this case gravity acts as a constant retarding force. The formulas given above will also apply in this case.

Example.—If a cannon ball is shot vertically upwards with an initial velocity of 2,000 feet per second, (*a*) how high will it go? (*b*) How long a time must elapse before it reaches the earth again?

Solution.—(*a*) Using formula **15**,

$$h = \frac{v^2}{2g} = \frac{2{,}000^2}{2 \times 32.16} = 62{,}189 \text{ feet, nearly,} = 11.778 \text{ miles. Ans.}$$

To find the time it takes to reach a height of 62,189 feet, use formula **14.**

$$t = \frac{v}{g} = \frac{2{,}000}{32.16} = 62.19 \text{ seconds.}$$

Since it will take the same length of time to fall to the ground, the total time will be 62.19 × 2 = 124.38 seconds = 2 minutes 4.38 seconds. Ans.

PROJECTILES.

898. Any body thrown into the air is a **projectile,** and is acted upon by three forces—the original or initial force, the force of gravity, and the resistance of the air. We shall here consider only those projectiles which are thrown horizontally.

899. The **range** is the horizontal distance between the starting point and the point where the body strikes the ground. In Fig. 123, suppose that *A* represents the starting point of the projectile, and that it is shot horizontally outwards in the direction of the arrow with a velocity of 70 feet per second. Now, if the resistance of the air be neglected, the velocity in the horizontal direction will be uniform, and the projectile will pass over equal spaces in equal times. Let *A 1* represent 70 feet, or the space passed over in one second. At the end of five seconds, if gravity had not acted upon the projectile, it

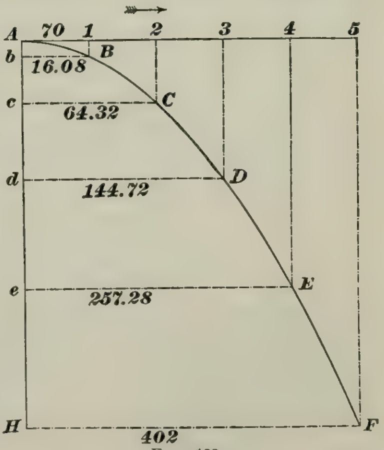

Fig. 123.

would have been at 5, but as gravity has acted, it *falls* 16.08 feet the first second; at the end of the second second it has fallen 64.32 feet, etc.

Let *A b* represent the fall in one second—that is, 16.08 feet, drawn to the same scale as *A 1*, which represents 70 feet. Now, complete the parallelogram *A 1 B b*, and *B* will be the point which the projectile has reached at the end of one second. If *A c* represents 64.32 feet, and the parallelogram *A 2 C c* is completed, the projectile will be at *C* at the end of the second second. Proceeding in this manner, find the points *D*, *E*, and *F*, the positions of the projectile at the end of *3*, *4*, and *5* seconds, respectively. Drawing the curve *A B C D E F* through the points thus found, it represents the path of the projectile. This curve is called **a parabola.**

The distance *H F* is the *range*, and, as is easily seen, *equals the time in seconds multiplied by the original velocity in feet per second.*

900. If the height *A H* and the initial velocity are given, and it is desired to find the range *H F*, *calculate the time that it will take to fall through a height equal to the given height, and multiply the time thus found by the initial velocity.*

Example.—A cannon ball is fired in a horizontal direction with an initial velocity of 1,500 feet per second. If the mouth of the cannon is 25 feet above the ground, what is its range?

Solution.—Applying formula **18,** Art. **896,**

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 25}{32.16}} = 1.247 \text{ seconds, nearly.}$$

$$\text{Range} = v\,t = 1{,}500 \times 1.247 = 1{,}870.5 \text{ feet. Ans.}$$

Example.—A projectile has an initial velocity of 90 feet per second. If it is desired to strike an object 15 feet away, how far below the horizontal line of direction must the object be located?

Solution.—The object must be located as far below as the distance that the body would fall, through the action of gravity, during the time it would take in passing over a distance of 15 feet at a velocity of 90 feet per second.

Hence, $15 \div 90 = \frac{1}{6}$ of a second. Applying formula **17,** Art. **896,** $h = \frac{1}{2} g t^2 = \frac{1}{2} \times 32.16 \times (\frac{1}{6})^2 = .447$ foot, nearly, $= 5.36$ inches. Ans.

EXAMPLES FOR PRACTICE.

1. A body starts from a state of rest, and falls freely for nine seconds; how far will it fall? Ans. 1,302.48 ft.

2. What velocity must a body have in order to carry it upwards 500 feet, vertically? Ans. 179.33 ft. per sec.

3. A baseball is thrown vertically upwards to a height of 200 feet; how long a time must elapse before it strikes the ground? Ans. 7.05 sec.

4. What will be the velocity of a freely falling body at the end of 6 seconds? Ans. 192.96 ft. per sec.

5. A baseball is thrown horizontally 5 feet above the ground, with a velocity of 80 feet per second; what is its range? Ans. 44.61 ft.

6. A leaden bullet falls from a tower 100 feet high; with what velocity will it strike the ground? Ans. 80.2 ft. per sec.

7. A bullet is dropped from a high tower. If it takes 4¼ seconds to reach the ground, how high is the tower? Ans. 290.445 ft.

8. A freely falling body has a velocity of 400 feet per second; how long has it been falling? Ans. 12.438 sec.

CENTRIFUGAL FORCE.

901. If a body be fastened to a string and whirled so as to give it a circular motion, there will be a pull on the string, which will be greater or less according as the velocity increases or decreases. The cause of this pull on the string will now be explained.

Suppose that the body is revolved horizontally, so that the action of gravity upon it will always be the same. According to the first law of motion, a body put in motion tends to move in a straight line unless acted upon by some other force, causing a change in the direction. When a body moves in a circle the force that causes it to move in a circle instead of a straight line is exactly equal to the tension of the string. If the string were cut, the pulling force that drew it away from the straight line would be removed and the body would then "fly off at a tangent"—that is, it would move in a straight line tangent to the circle, as shown in Fig. 124.

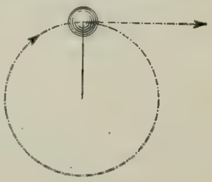

FIG. 124.

902. Since, according to the third law of motion, every action has an equal and opposite reaction, we call that force which acts as an equal and opposite force to the pull of the string the **centrifugal force,** and it acts *away* from the center of motion.

903. The other force or tension of the string is called the **centripetal force,** and it acts *towards* the center of motion. It is evident that these two forces acting in opposite directions tend to pull the string apart, and, if the velocity be increased sufficiently, the string will break. It is also evident that no body can revolve without generating centrifugal force. The value of the centrifugal force of any revolving body, expressed in pounds, is

$$F = .00034\ W R N^2, \qquad \textbf{(19.)}$$

in which $F =$ centrifugal force;
$W =$ total weight of body in pounds;
$R =$ radius, usually taken as the distance between the center of motion and the center of gravity of the revolving body, in feet;
$N =$ number of revolutions per *minute.*

904. In calculating the centrifugal force tending to burst a fly-wheel, it is the usual practice to consider one-half the rim of the wheel only, and not to take the arms and hub of the wheel into account. In this case, R is taken as the *distance between the inside edge of the rim and the center of the shaft.*

NOTE.—The general formula for centrifugal force is $F = \frac{m v^2}{R}$, where $m =$ the mass of the revolving body, $v =$ velocity of center of gravity of body in feet per second, and $R =$ radius, as above. Formula **19,** Art. **903,** is easily derived from this. Thus: $m = \frac{W}{g}$; $N = \frac{60\, v}{2 \pi R}$, or $v = \frac{2 \pi R N}{60}$; hence, $F = \frac{m}{R} v^2 = \frac{W}{g R}\left(\frac{2 \pi R N}{60}\right)^2 = .00034\ W R N^2$.

EXAMPLE.—What would be the centrifugal force tending to burst a cast-iron fly-wheel whose outside diameter was 10 feet, width of face 20 inches, and thickness of rim 6 inches, turning at the rate of 80 revolutions per minute?

SOLUTION.—First calculate the weight of one-half the rim. The diameter of the rim $= 10 \times 12 = 120$ inches; the diameter of the circle midway between the inside and outside diameters of the rim $=$

120 − 6 = 114 inches. The number of cubic inches in the rim = $114 \times 3.1416 \times 20 \times 6 = 42,977$ cubic inches. $42,977 \times .261 \times \frac{1}{2} = 5,608.5$ pounds = weight = W. $R = \frac{10}{2} - \frac{1}{2} = 4\frac{1}{2}$ feet. $N = 80$.

Hence, $F = .00034\, W R N^2 = .00034 \times 5,608.5 \times 4\frac{1}{2} \times 80^2 = 54,918.4$ pounds. Ans.

STATICS.

905. Statics may be defined as that branch of Mechanics which treats of bodies at rest or of bodies moving with a *uniform* velocity, when these bodies are acted upon by forces. A body is in **static equilibrium** when the resultant of *all* of the forces acting upon the body is zero.

MOMENTS OF FORCES.

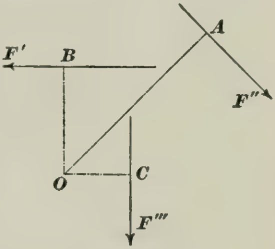

FIG. 125.

906. If from any point O, Fig. 125, a perpendicular be drawn to the line of action of a force, the product of the magnitude of the force and the length of the perpendicular is called the **moment of the force** *about the point O.*

Thus, in the figure, the moment of the force F' about the point O is $F' \times O B$; of the force F'' about the point O is $F'' \times O A$, and of F''' is $F''' \times O C$.

907. The use of the moment will be explained further on, when the necessity arises for using it. The point O is called the **center of moments.**

908. When two *equal* forces act in parallel lines, but in opposite directions, they constitute what is called a **couple.**

909. In Fig. 126, the equal and parallel forces F' and F'', acting in opposite directions (one up and the other down), form a couple. It is easy to see that if they were joined by a connection, as $A B$, that they would tend to turn $A B$ about the point C, midway between F' and F''. The moment

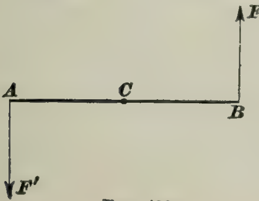

FIG. 126.

of a couple about *any* point is always the same, and is equal to *the product of one of the equal forces into the perpendicular distance between the two forces.*

Thus, the moment of the couple in the figure equals F' or F'' multiplied by $A\,B$. An example of a couple would be a wrench applied to a nut. Here, two opposite and parallel sides of the wrench act in parallel, but opposite, directions, against two parallel sides of the nut.

CENTER OF GRAVITY.

910. *The* **center of gravity** *of a body is that point at which the body may be balanced, or it is the point at which the whole weight of a body may be considered as concentrated.*

In a moving body, the line described by its center of gravity is always taken as the path of the body. In finding the distance that a body has moved, the distance that the center of gravity has moved is taken.

911. The definition of the center of gravity of a body may be applied to a system of bodies, if they are considered as being connected at their centers of gravity.

If w and W, Fig. 127, be two bodies of known weights, their center of gravity will be at C. The point C may be readily determined, as follows: Take C as the center of moments; then, since the weights are to balance each other, the moment of W about C must equal the moment of w about C; or, in other words, $W \times C\,W = w \times C\,w$.

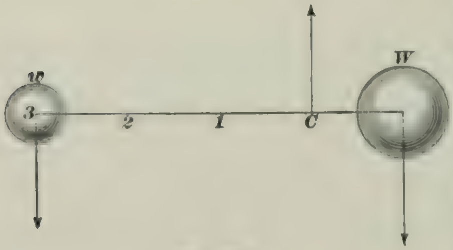

FIG. 127.

If the distance between the centers of gravity of W and w is known, it is very easy to find $C\,w$ and $C\,W$. For

Let $l =$ the distance wW between the centers of the bodies;

$l_1 =$ the short arm CW;
$w =$ weight of small body;
$W =$ weight of large body.

Then, since $wC = l - l_1$, we have, taking the moments about the point C,

$$Wl_1 = w(l - l_1) = wl - wl_1 \text{; whence,}$$
$$Wl_1 + wl_1 = (W + w)l_1 = wl, \text{ or}$$
$$l_1 = \frac{wl}{W + w}. \qquad \textbf{(20.)}$$

EXAMPLE.—In Fig. 127, $w = 10$ pounds, $W = 30$ pounds, and the distance between their centers of gravity is 36 inches; where is the center of gravity of both bodies situated?

SOLUTION.—Applying formula **20**,

$$CW = l_1 = \frac{wl}{W + w} = \frac{10 \times 36}{30 + 10} = 9 \text{ inches;}$$

hence, the center of gravity is 9 inches from the center of the larger body. Ans.

The general method for finding the short arm CW is, then, as follows: *Multiply the weight of the smaller body by the distance between the centers of the two bodies, and divide this product by the sum of the weights of the two bodies.*

912. It is now very easy to extend this principle, to the finding of the center of gravity of any number of bodies when their weights and the distances apart of their centers of gravity are known, by applying the principle of finding the resultant of several forces; that is, by finding the center of gravity of two of the bodies, as W_1 and W_4 in Fig. 128, at C_1.

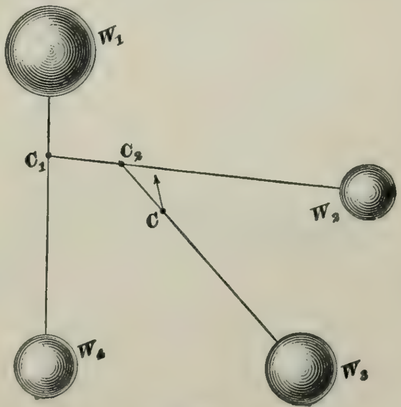

FIG. 128.

Assume that the weight of both bodies is concentrated at C_1, and find the center of gravity of this combined weight at C_1 and of W_2 to be at C_2; then, find that the center of gravity of the combined weights of W_1, W_4, and W_2 (concentrated at C_2) and W_3 to be at C, and C will be the center of gravity of the four bodies.

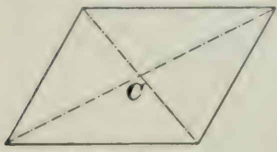

FIG. 129.

913. To find the center of gravity of any parallelogram : *Draw the two diagonals, Fig. 129, and their point of intersection C will be the center of gravity.*

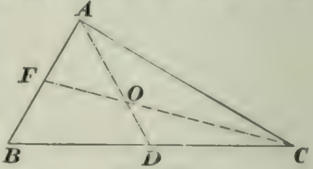

FIG. 130.

914. To find the center of gravity of a triangle, as $A\,B\,C$, Fig. 130 : *From any vertex, as A, draw a line to the middle point D of the opposite side B C. From one of the other vertexes, as C, draw a line to F, the middle point of the opposite side A B ; the point of intersection, O, of these two lines, is the center of gravity.*

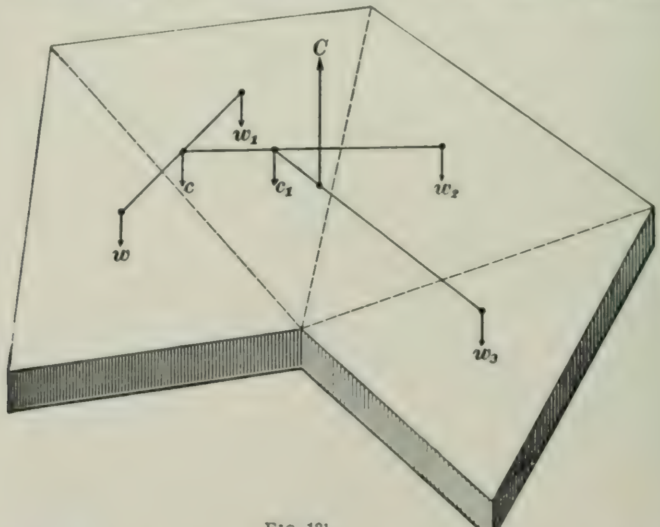

FIG. 131.

It is also true that the distance $D\,O = \frac{1}{3}\,D\,A$, and that

$F\,O = \frac{1}{3}\,F\,C$, and the center of gravity could have been found by drawing from any vertex a line to the middle point of the opposite side, and measuring back from that side $\frac{1}{3}$ of the length of the line.

The center of gravity of any regular plane figure is the same as the geometrical center.

915. To find the center of gravity of any irregular plane figure, but of uniform thickness throughout, divide one of the parallel surfaces into triangles, parallelograms, circles, ellipses, etc., according to the shape of the figure; find the area and center of gravity of each part separately, and combine the centers of gravity thus found, as in the case of more than two bodies whose weights were known, except that the area of each part is used instead of their weights. See Fig. 131.

916. Center of Gravity of a Solid.—In a body free to move, the center of gravity will lie in a vertical plumb line drawn through the point of support. Therefore, to find the position of the center of gravity of an irregular solid, as the crank, Fig. 132, suspend it at some point, as *B*, so that it will move freely. Drop a plumb line from the point of suspension, and mark its direction. Suspend the body at another point, as *A*, and repeat the process. The

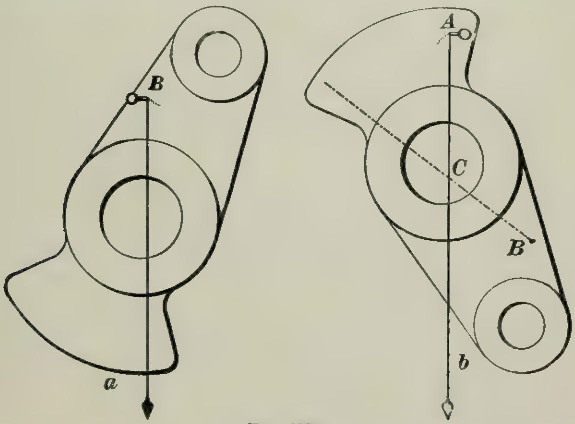

FIG. 132.

intersection of the two lines will be directly over the center of gravity.

Since the center of gravity depends wholly upon the shape and weight of a body, it may be without the body, as in the case of a circular ring, whose center of gravity is at the center of the circumference of the ring.

EQUILIBRIUM.

917. When a body is at rest, all of the forces which act upon it are said to *balance* one another, or to be in **equilibrium.** The most important of the forces is gravity, which acts upon every molecule of the body.

918. There are three states of equilibrium: **Stable, unstable**, and **neutral.**

919. A body is in **stable equilibrium** when, if slightly displaced from its position of rest, *it tends to return to that position.*

For example, a cube, a cone resting on its base, a pendulum, etc.

If a body is in **stable equilibrium,** *its center of gravity is raised when it is displaced.*

920. A body is in **unstable equilibrium** when, if slightly displaced from its position of rest, *it tends to fall farther from that position.*

For example, a cone standing upon its point, an egg balanced upon its end, etc.

Any movement, however slight, lowers *the center of gravity* when the body is in unstable equilibrium.

921. A body is in **neutral equilibrium** when it has *no tendency to move either way,* in the direction of its motion, after being slightly displaced.

For example, a sphere of uniform density; a cone resting on its side.

922. A vertical line drawn through the center of gravity of a body is called the **line of direction.** So long as the line of direction falls within the base, the body will stand. When the line of direction falls without the base, the body will fall.

Let $A\ C\ B$, Fig. 133, be a cylinder whose base is oblique to the center line $B\ O\ D$; and let O be the center of gravity of this cylinder.

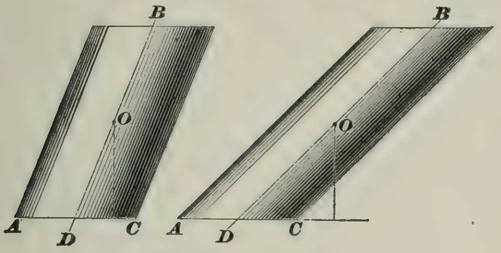

FIG. 133.

So long as the perpendicular through O falls bebetween A and C, the cylinder will stand, but the instant that it falls without the base, the cylinder will fall.

The center of gravity of a body always tends to seek its lowest point.

EXAMPLES FOR PRACTICE.

1. There are three weights in a straight line. The first weighs 40 lb.; the second, 16 lb., and the third, 50 lb. Distance between the first and second is 6 ft., and between the second and third, 10 ft. Where is the center of gravity? Ans. 8 ft. 5.434 in. from the 40-lb. weight.

2. Find the perpendicular distance between the center of gravity and the longer side of a triangle whose sides are 7 ft., 10 ft., and 15 ft. long. Solve graphically. Ans. 1.31 ft.

3. A rectangle, 2 ft. long and 1 ft. wide, has equal weights of 50 lb. each, suspended from two of its diagonally opposite corners. A weight of 60 lb. and another of 80 lb. are suspended from the other two corners. Supposing the rectangle to be without weight, where is the center of gravity? Ans. On the diagonal joining the 60-lb. and 80-lb. weights, 1.118 in. from the center.

4. Find the center of gravity of a quadrilateral whose sides are 14, 15, 16, and 18 in. long, the angle between the 18 and 16 in. sides being 45°. Give the perpendicular distance from the 18 in. side. Ans. 5.46 in.

SIMPLE MACHINES.

THE LEVER.

923. A **lever** is a bar capable of being turned about a pin, pivot or point, as in Figs. 134, 135 and 136.

924. The object W to be lifted is called the **weight**; the force used P is called the **power**; and the point or pivot F is called the **fulcrum.**

925. That part of the lever between the weight and the fulcrum, or $F\,b$, is called the **weight arm**, and the part between the power and the fulcrum, or $F\,c$, is called the **power arm.**

926. Take the fulcrum, or point F, as the center of moments; then, in order that the lever shall be in equilibrium, the moment of P about F, or $P \times F\,c$, must equal the moment of W about F, or $W \times F\,b$. That is, $P \times F\,c = W \times F\,b$, or, in other words, *the power multiplied by the power arm equals the weight multiplied by the weight arm.*

927. If F be taken as the center of a circle, and arcs be described through b and c, it will be seen that, if the weight arm is moved through a certain angle, the power arm will move through the same angle; also, that the vertical distance that W moves will be the sine of this angle, in a circle whose radius is the weight arm, and that the vertical distance that P moves will be the sine of the same angle in a circle whose radius is the power arm. From this it is seen that the power arm is proportional to the distance through which the power moves, and the weight arm is proportional to the distance through which the weight moves.

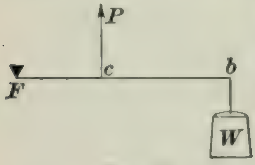

FIG. 134.

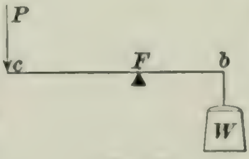

FIG. 135.

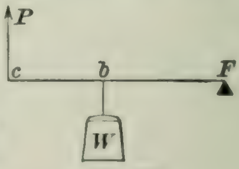

FIG. 136.

Hence, instead of writing $P \times F\,c = W \times F\,b$, we might have written it $P \times$ distance through which P moves $= W \times$ distance through which W moves. This is the general law of all machines, and can be applied to any mechanism, from the simple lever up to the most complicated arrangement. Stated in the form of a rule, it is as follows:

Rule VI.—*The power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves.*

Example.—If the weight arm of a lever is 6 inches long and the power arm is 4 feet long, how great a weight can be raised by a force of 20 pounds at the end of the power arm?

Solution.—4 feet = 48 inches. Hence, $20 \times 48 = W \times 6$, or $W = 160$ pounds. Ans.

Example.—(*a*) What is the ratio between the power and the weight in the last example? (*b*) In the last example, if P moves 24 inches, how far does W move? (*c*) What is the ratio between the two distances?

Solution.—(*a*) $20 : 160 = 1 : 8$; that is, the weight moved is 8 times the power. Ans.

(*b*) $20 \times 24 = 160 \times x$. $x = \frac{480}{160} = 3$ inches, the distance that W moves. Ans.

(*c*) $3 : 24 = 1 : 8$, or the ratio is $1 : 8$. Ans.

928. The law which governs the straight lever also governs the bent lever; but care must be taken to determine the true lengths of the lever arms, which are in every case *the perpendicular distances from the fulcrum to the line of direction of the weight or power.*

Thus, in Figs. 137, 138, 139, and 140, $F\,c$ in each case represents the power arm, and $F\,b$ the weight arm.

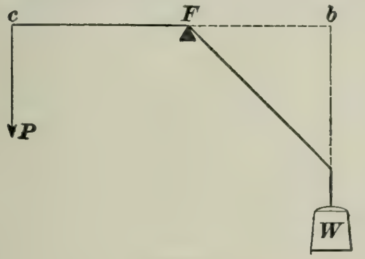

Fig. 137.

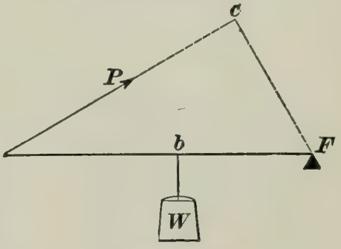

Fig. 138.

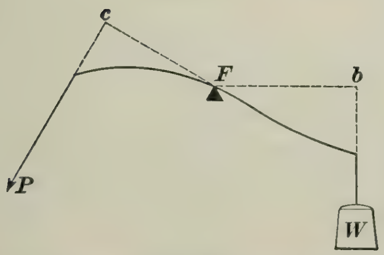

Fig. 139.

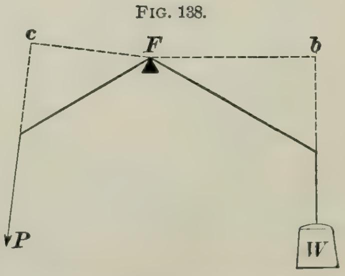

Fig. 140.

929. The Compound Lever.—A **compound lever** is a series of single levers arranged in such a manner that when a force is applied to the first it is communicated to the second, and from this to the third, and so on.

Fig. 141 shows a compound lever. It will be seen that when a power is applied to the first lever at P it will be communicated to the second lever at P, from this to the third lever at P, and thus raise the weight W.

The weight which the power of the first lever could raise acts as the power of the second, and the weight which this could raise by means of the second lever acts as the power of the third lever, and so on, no matter how many single levers make up the compound lever.

In this case, as in every other, the power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves.

Hence, if we move the P end of the lever, say, 4 inches, and the W end moves $\frac{1}{5}$ of an inch, we know that the ratio between P and W is the same as the ratio between $\frac{1}{5}$ and 4; that is, 1 to 20, and, hence, that 10 pounds at P would balance 200 pounds at W, without measuring the lengths of the different lever arms. If the lengths of the lever arms are known, the ratio between P and W may be readily obtained from the following rule:

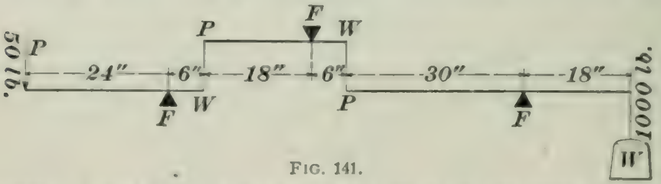

FIG. 141.

Rule VII.—*The continued product of the power and each power arm equals the continued product of the weight and each weight arm.*

EXAMPLE.—If, in Fig. 141, the power arms $P F = 24$ inches, 18 inches, and 30 inches, and weight arms $W F = 6$ inches, 6 inches, and 18 inches, (*a*) how great a force must be applied at the free end P to raise 1,000 pounds at W? (*b*) What is the ratio between P and W?

SOLUTION.— $P \times 24 \times 18 \times 30 = 1,000 \times 6 \times 6 \times 18$,

$$\text{or } P = \frac{648,000}{12,960} = 50 \text{ pounds. Ans.}$$

$$50 : 1,000 = 1 : 20, \text{ or } P : W = 1 : 20. \text{ Ans.}$$

THE WHEEL AND AXLE.

930. The **wheel and axle** consists of *two cylinders of different diameters, rigidly connected*, so that they turn together about a common axis, as in Fig. 142. Then, as before, $P \times$ distance through which it moves $= W \times$ distance through which it moves; and, since these distances are proportional to the radii of the power cylinder and weight cylinder, $P \times F c = W \times F b$.

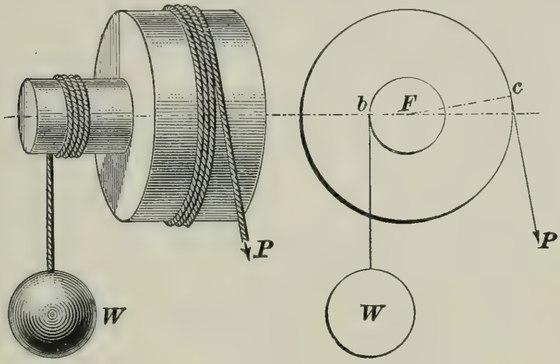

FIG. 142.

It is not necessary that an entire wheel be used; an arm, projection, radius, or anything which the power causes to revolve in a circle, may be considered as the wheel. Consequently, if it is desired to hoist a weight with a windlass, Fig. 143, the force is applied to the handle of the crank, and the distance between the center line of the crank-handle and the axis of the drum corresponds to the radius of the wheel.

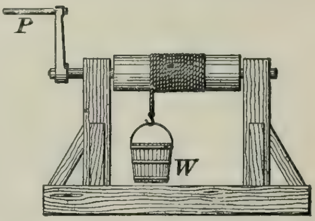

FIG. 143.

EXAMPLE.—If the distance between the center line of the handle and the axis of the drum, in Fig. 143, is 18 inches, and the diameter of the drum is 6 inches, what force will be required at P to raise a load of 300 pounds?

SOLUTION.— $P \times 18 = 300 \times \frac{6}{2}$, or $P = 50$. Ans.

931. Wheelwork.—A combination of wheels and axles, as in Fig. 144, is called a **train.** The wheel in a train to which motion is imparted from a wheel on another shaft, by such means as a belt or gearing, is called the **driven wheel** or **follower;** the wheel which imparts the motion is called the **driver.**

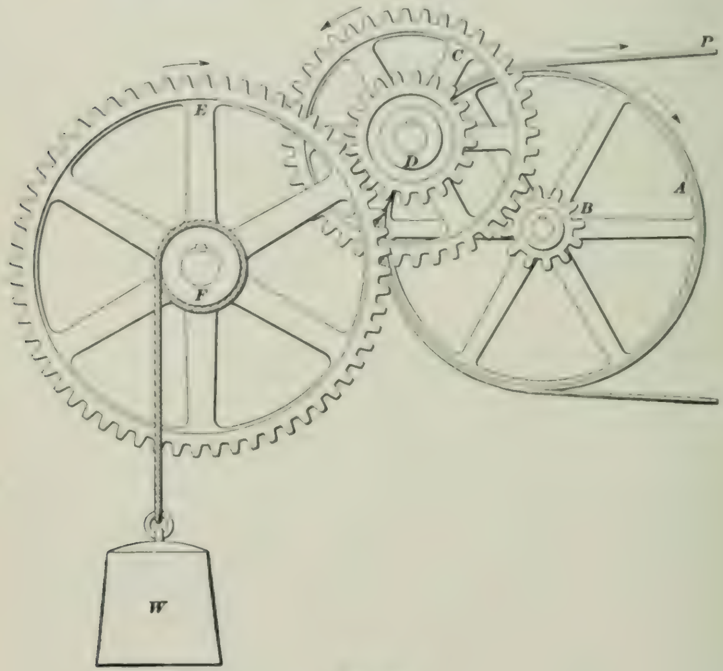

FIG. 144.

932. It will be seen that the wheel and axle bears the same relation to the train that the simple lever does to the compound lever; that is, *the continued product of the power and the radii of the driven wheels equals the continued product of the weight, the radius of the drum that moves the weight, and the radii of the drivers.*

EXAMPLE.—If the radius of the wheel *A* is 20 inches, of *C*, 15 inches, and of *E*, 24 inches; if the radius of the drum *F* is 4 inches, of the pinion D, 5 inches, and of the pinion B, 4 inches, how great a weight will a force of 1 pound at P raise?

SOLUTION.— $1 \times 20 \times 15 \times 24 = W \times 4 \times 5 \times 4$, or $W = \frac{7,200}{80} = 90$ pounds. Ans.

933. Hence, also, if W were raised one inch, P would move 90 inches, or P would have to move 90 inches to raise W one inch. It is now clear that another great law has made itself manifest, and that is that, *whenever there is a gain in power without a corresponding increase in the initial force, there is a loss in speed.*

In the last example, if P were to move the entire 90 inches in one second, W would move only 1 inch in one second. The same principle may be applied to any machine.

THE PULLEY.

934. A **pulley** is a wheel turning on an axle, over which a cord, chain, or band is passed in order to transmit the force through the cord, chain, or band.

935. The frame which supports the axle of the pulley is called the **block.**

936. A **fixed pulley** is one whose block is not movable, as in Fig. 145. In this case, if the weight W be lifted by pulling down P, the other end of the cord W will evidently move the same distance upwards that P moves downwards; hence, P must equal W.

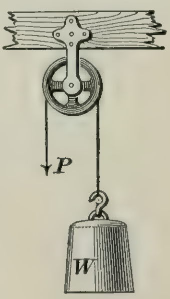

FIG. 145.

937. A **movable pulley** is one whose block is movable, as in Fig. 146. One end of the cord is fastened to the beam, and the weight is suspended from the pulley, the other end of the cord being drawn up by the application of a force P. A little consideration will show that if P moves through a certain distance, say 1 foot, W will move through *half* that distance, or 6 inches; hence, a pull of 1 pound at *P* will lift 2 pounds at *W*.

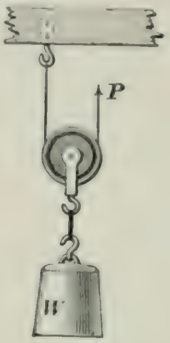

FIG. 146.

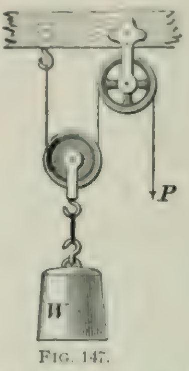

FIG. 147.

The same would also be true if the free end of the cord were passed over a *fixed pulley*, as in Fig. 147, in which case the fixed pulley merely changes the direction in which *P* acts, so that a weight of 1 pound hung on the free end of the cord will balance 2 pounds hung from the *movable pulley*.

938. A combination of pulleys, as shown in Fig. 148, is sometimes used. In this case there are three movable and three fixed pulleys, and the amount of movement of *W*, owing to a certain movement of *P*, is readily found.

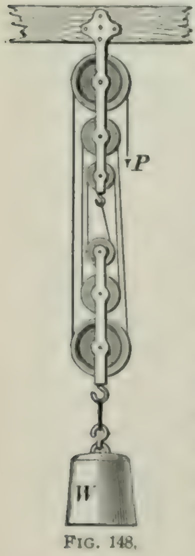

FIG. 148.

It will be noticed that there are *six parts* of the rope, not counting the free end; hence, if the movable block be lifted 1 foot, *P* remaining in the same position, there would be 1 foot of slack in each of the six parts of the rope, or *six feet* in all. Therefore, *P* would have to move 6 feet in order to take up this slack, or *P* moves six times as far as *W*. Hence, 1 pound at *P* will support 6 pounds at *W*, since the *power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves.* It will also be noticed that there are three movable pulleys, and that 3 × 2 = 6.

Rule VIII.—*In any combination of pulleys where one continuous rope is used a load on the free end will balance a weight on the movable block as many times as great as the load on the free end as there are parts of the rope supporting the load—not counting the free end.*

The above law is good, whether the pulleys are side by side, as in the ordinary *block and tackle*, or whether they are arranged as in the figure.

EXAMPLE.—In a block and tackle having five movable pulleys, how great a force must be applied to the free end of the rope to raise 1,250 pounds?

SOLUTION.—Since there are 5 movable pulleys, there are 10 parts of the rope supporting them, and 1 pound on the free end will balance 10 pounds on the movable block; therefore, the ratio of P to W is 1 : 10, and $P = \frac{1{,}250}{10} = 125$ pounds. Ans.

939. In Fig. 149 is shown an arrangement called a **differential pulley.** It will be seen that if a force be applied at P, so as to pull the point P down to D, the rope or chain will wind up on the large pulley A, and unwind from the smaller pulley B, and since C is a movable pulley, the weight W will move an amount equal to one-half the difference between the amount of winding on A and unwinding on B.

Let the radius of A be represented by R, and of B by r; then, when P is pulled down to D, a point E on the pulley A will move through a certain angle, $E\,O\,F$, the length of the arc $E\,F$ being equal to the distance $P\,D$. Any point on the pulley B, which is fastened to A, will turn through the same angle; and the difference between the arc $K\,L$, through which this point turns, and the arc $E\,F$ will be proportional to the difference of the radii R and r.

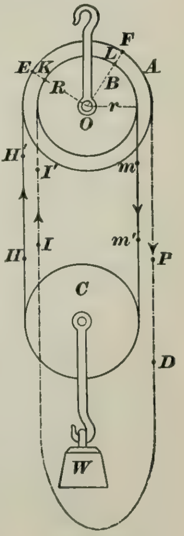

FIG. 149.

When P moves down to D, the point H on the other side will move up through the same distance to H'. The point I will move up to I', a distance equal to the length of the arc $K L$, and m will fall through the same distance to m', thus causing the weight W to be *raised one half the difference between m m' and $P D$.* The ratio between the distances through which W and P move will be

$$\frac{\text{arc } E F - \text{arc } K L}{2} : E F, \text{ or } \frac{R - r}{2} : R.$$

$$\text{Hence, } W \times \frac{R - r}{2} = P \times R, \text{ or } W = \frac{2 P R}{R - r}. \qquad \textbf{(21.)}$$

EXAMPLE.—If $R = 7$ inches, and $r = 6\frac{1}{2}$ inches, how much weight can be raised at W with a force of 50 pounds at P?

SOLUTION.— $W = \frac{2 P R}{R - r} = \frac{2 \times 50 \times 7}{7 - 6\frac{1}{2}} = 1,400$ pounds. Ans.

THE INCLINED PLANE.

940. An **inclined plane** is a slope or a flat surface, making an angle with a horizontal line.

Three cases may arise in practice with the inclined plane:

I. When the power acts parallel to the plane, as in Fig. 150.

II. When the power acts parallel to the base, as in Fig. 151.

III. When the power acts at an angle to the plane, or to the base, as in Fig. 152.

Case I.—In Fig. 150, the relation existing between the power and the weight is easily found. The weight ascends a distance equal to c b, or the height of the inclined plane, while the power descends through a distance equal to a b, or the length of the inclined plane. Therefore, the *power multiplied by the length of the inclined plane equals the weight multiplied by the height of the inclined plane.* Hence, if the length $a b = 40$ feet, and the height $c b = 20$ feet, $W \times 20 = P \times 40$, or 1 pound at P will balance 2 pounds at W.

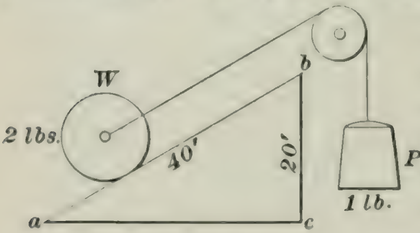

FIG. 150.

Case II.—In Fig. 151, the power is supposed to act parallel to the base for any position of W, the pulley being shifted up and down; therefore, while W is moving from the level $a\ c$ to b, or through the height $c\ b$ of the inclined plane, P will move a distance downwards, relative to the axis of the pulley, equal to the length of the base $a\ c$. Hence, when the power acts parallel to the base, $W \times$ *height of the inclined plane* $= P \times$ *length of base.*

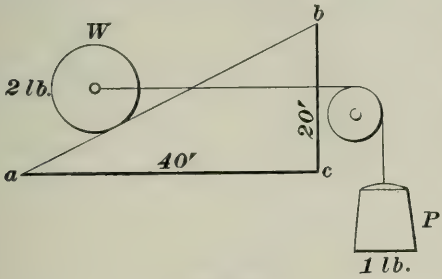

FIG. 151.

If the length of the base is 40 feet, and the height of the inclined plane is 20 feet, $W \times 20 = P \times 40$, and 1 pound at P will balance 2 pounds at W.

Case III.—For Fig. 152 no rule can be given. The ratio of the power to the weight must be determined for every position of W by means of the triangle or parallelogram of forces.

This case will be explained by means of an example, as it affords a splendid illustration of the principle of *resolution of forces.*

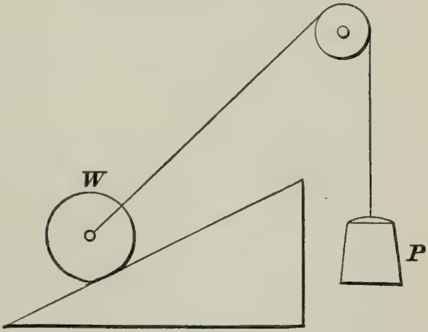

FIG. 152.

EXAMPLE.—In Fig. 153, a body W is shown resting on an inclined plane $A\ B$, whose dimensions are marked on the cut; the weight P acts to pull the body up the plane by means of the rope r and pulley p. It is required to find what the weight of P must be in order to start W up the plane. Suppose W weighs 120 pounds, and that friction is neglected. It is also required to find the perpendicular pressure which W exerts against the plane.

SOLUTION.—Through the point a, the center of gravity of W, draw $a\,b$ vertical, and make it of such a length as to represent 120 pounds to a convenient scale, say 60 pounds = 1 inch. Drawing $a\ c$ and $c\ b$, respectively, parallel and perpendicular to the plane, ac represents the magnitude of the force which must be exerted parallel to AB in order to put the body in equilibrium—i. e., to balance the force which gravity exerts in pulling the body down the plane. If the rope r were parallel to AB, ac would represent the weight of P; but, since r makes an angle with the plane, P will not be equal to ac. To find what the weight of P must be, draw ad parallel to ac, but indicate it as acting in the opposite direction, or from a to d instead of from a to c. Now treat ad as though it were a *component* of the force acting in the rope—i. e., draw de perpendicular to ad, instead of perpendicular to ae. The reason for this is that if de were drawn perpendicular to ae it could be resolved into components, one of which would be parallel to ad, a result which we wish to avoid; in other words, we want de perpendicular to the plane. The line ae, measured to the same scale as ab, will give the value of P. Measuring it, its length is .89″; hence, $P = .89 \times 60 = 53.4$ pounds. Ans.

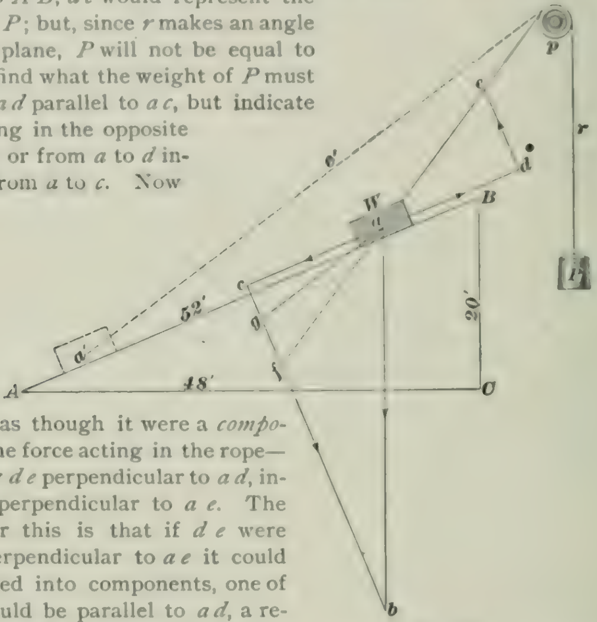

FIG. 153.

To determine the perpendicular pressure against the plane, it will be noticed that ab equals the pressure due to gravity. Since cb and de are both perpendicular to AB, they are parallel, and since de acts in the opposite direction to cb, the actual pressure against the plane is given by the difference between cb and de. Making cf equal to de, fb represents the perpendicular pressure against the plane when the force P $(= ae)$ acts as shown. The length of fb is 1.39″; hence, the perpendicular pressure is $1.39 \times 60 = 83.4$ pounds. Ans.

Since ca and ad are parallel and equal, and cf and de are also parallel and equal, it follows that af and ae must also be parallel and equal. Consequently, the force P might have been found by drawing af parallel to the direction in which the pull on the rope acts, and bf perpendicular to the plane AB. Thus, suppose that the weight occupies the position shown by the dotted lines. Then, drawing ag

parallel to $a'e'$, ag represents the weight of P, and gb represents the perpendicular pressure of the body W against the plane. Measuring ag, its length is .79″; hence, $P = .79 \times 60 = 47.4$ pounds. Measuring gb, its length is 1.65″; hence, the perpendicular pressure $= 1.65 \times 60 = 99$ pounds.

941. The Wedge.—The **wedge** is a movable inclined plane, and is used for moving a great weight a short distance. A common method of moving a heavy body is shown in Fig. 154.

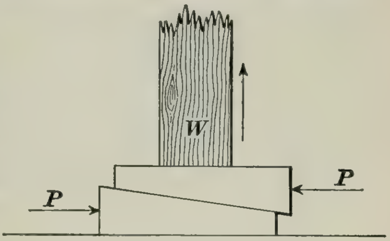

FIG. 154.

Simultaneous blows of equal force are struck on the heads of the wedges, thus raising the weight W. The laws for wedges are the same as for Case II of the inclined plane.

THE SCREW.

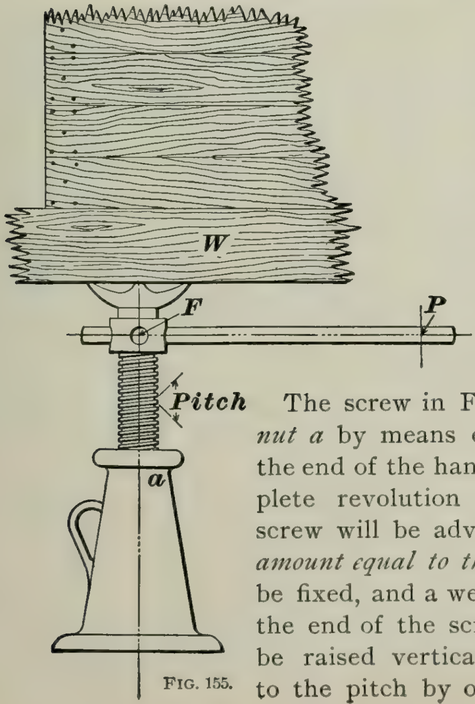

FIG. 155.

942. A **screw** is a cylinder with a spiral groove winding around its circumference. This spiral is called the **thread** of the screw. The distance that the thread is drawn back or advanced in one turn of the screw is called the **pitch** of the screw. The screw in Fig. 155 is turned in a *nut* a by means of a force applied at the end of the handle P. For one complete revolution of the handle, the screw will be advanced lengthwise *an amount equal to the pitch.* If the nut be fixed, and a weight be placed upon the end of the screw, as shown, it will be raised vertically a distance equal to the pitch by one revolution of the screw. During this revolution the force at P will move through a distance equal to the circumference, whose radius is $P F$. Hence, $W \times$ pitch of thread $= P \times$ circumference of P.

943. Single-threaded screws of less than 1 inch pitch are generally classified by the number of threads they have in 1 inch of their length. In such cases, *one inch divided by the number of threads equals the pitch;* thus, the pitch of a screw that has 8 threads per inch is $\frac{1}{8}''$; one of 32 threads per inch is $\frac{1}{32}''$, etc.

EXAMPLE.—It is desired to raise a weight by means of a screw having 5 threads per inch. The force applied is forty pounds at a distance of 14 inches from the center of the screw; how great a weight can be raised?

SOLUTION.—Diameter of the circumference passed through by the force $= 14 \times 2 = 28$ inches. Therefore, $W \times \frac{1}{5} = 40 \times 28 \times 3.1416$, or $W = 17{,}593$ pounds. Ans.

VELOCITY RATIO.

944. The ratio of the distance that the power moves to the distance which the weight moves on account of the movement of the power is called the **velocity ratio.**

Thus, if the power is moving 12 inches while the weight is moving 1 inch, the velocity ratio is 12 to 1, or 12; that is, P moves 12 times as fast as W.

945. If the velocity ratio is known, the weight which any machine can raise equals the *power multiplied by the velocity ratio.* If the velocity ratio is 8.7 to 1, or 8.7, $W = 8.7 \times P$, since $W \times 1 = P \times 8.7$.

NOTE.—In all of the preceding cases, including the last, friction has been neglected.

FRICTION.

946. **Friction** is the resistance that a body meets from the surface on which it moves.

947. The **ratio** between the *resistance* to the motion of a body due to friction and the *perpendicular* pressure between the surfaces is called the **coefficient of friction.**

If a weight W, as in Fig. 156, rests upon a horizontal plane, and has a cord fastened to it passing over a pulley a, from which a weight P is suspended, then, if P is just sufficient to start W, the ratio of P to W, or $\frac{P}{W}$, is the *coefficient of friction* between W and the surface it slides upon.

The weight W is the perpendicular pressure, and P is the force necessary to overcome the resistance to the motion of W due to friction.

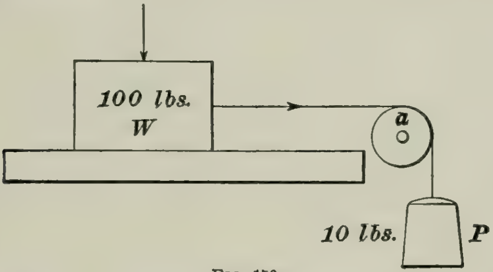

FIG. 156.

If $W = 100$ pounds and $P = 10$ pounds, the coefficient of friction for this particular case would be $\frac{P}{W} = \frac{10}{100} = .1$.

948. Laws of Friction :—

I. *Friction is directly proportional to the perpendicular pressure between the two surfaces in contact.*

II. *Friction is independent of the extent of the surfaces in contact when the total perpendicular pressure remains the same.*

III. *Friction increases with the roughness of the surfaces.*

IV. *Friction is greater between surfaces of the same material than between those of different materials.*

V. *Friction is greatest at the beginning of motion.*

VI. *Friction is greater between soft bodies than between hard ones.*

VII. *Rolling friction is less than sliding friction.*

VIII. *Friction is diminished by polishing or lubricating the surfaces.*

Law I shows why the friction is so much greater on journals after they begin to heat than before. The heat causes the journal to expand, thus increasing the pressure between the journal and its bearing, and, consequently, increasing the friction.

Law II states that, no matter how small the surface may be which presses against another, if the perpendicular pressure is the same, the friction will be the same. Therefore, large surfaces are used where possible; not to reduce the friction, but to reduce the wear and diminish the liability of heating.

For instance, if the perpendicular pressure between a journal and its bearing is 10,000 pounds, and the coefficient of friction is .2, the amount of friction is $10{,}000 \times .2 = 2{,}000$ pounds.

Suppose that one-half the area of the surface of the journal is 80 square inches; then, the amount of friction for each square inch of bearing is $2{,}000 \div 80 = 25$ pounds.

If half the area of the surface had been 160 square inches, the friction would have been the same—that is, 2,000 pounds; but the friction per square inch would have been $2{,}000 \div 160 = 12\frac{1}{2}$ pounds, just one-half as much as before, and the wear and liability to heat would be one-half as great also.

COEFFICIENTS OF FRICTION.

TABLE 17.

Description of Surfaces in Contact.	Disposition of Fibers.	State of the Surfaces.	Coefficient of Friction.
Oak on oak	Parallel	Dry	.48
Oak on oak	Parallel	Soaped	.16
Wrought iron on oak......	Parallel	Dry	.62
Wrought iron on oak.......	Parallel	Soaped	.21
Cast iron on oak	Parallel	Dry	.49
Cast iron on oak	Parallel	Soaped	.19
Wrought iron on cast iron..	—	Slightly Unctuous	.18
Wrought iron on bronze....	—	Slightly Unctuous	.18
Cast iron on cast iron	—	Slightly Unctuous	.15

EFFICIENCY.

949. The force which is required to raise a weight, or overcome an equal resistance any machine, is always *greater than this weight or resistance, divided by the velocity ratio of the machine.*

Thus, if there were no friction, a machine whose velocity ratio was 5 would, by an application of 100 pounds of force, raise a weight of 500 pounds.

Now, suppose that the friction in the machine is equivalent to 10 pounds of force, then it would take 110 pounds of force to raise 500 pounds.

If, in the above illustration, friction were neglected, 110 pounds $\times$ 5 = 550 pounds, or the weight that 110 pounds would raise; but, owing to the frictional resistance, it only raised 500 pounds; therefore, we have for the ratio between the two $\frac{500}{550} = .91$. That is, $500 : 550 = .91 : 1$.

950. This ratio between the weight actually raised and the applied force multiplied by the velocity ratio is called the **efficiency of the machine.**

Let F = the force applied to the machine;
V = the velocity ratio of the machine;
W = the weight actually lifted, or equivalent resistance overcome;
E = the efficiency of the machine.

$$\text{Then, } \mathbf{E} = \frac{W}{F\,V}. \qquad \textbf{(22.)}$$

Example.—In a machine having a combination of pulleys and gears, the velocity ratio of the whole is 9.75. A force of 250 pounds just lifts a weight of 1,626 pounds; what is the efficiency of the machine?

Solution.—$\mathbf{E} = \frac{W}{F\,V} = \frac{1{,}626}{250 \times 9.75} = .6671$, or 66.71%. Ans.

Since the total amount of friction varies with the load, it follows that the efficiency will also vary for different loads.

Example.—A pulley block with a velocity ratio of 7 has an efficiency of 38% for a load of 3,000 pounds; what power is required to start the load?

Solution.—$\mathbf{E} = \frac{W}{F\,V}$, or $.38 = \frac{3{,}000}{7\,F}$

Hence, $F = \frac{3{,}000}{.38 \times 7} = 1{,}128$ pounds, nearly. Ans.

EXAMPLES FOR PRACTICE.

1. A wedge is caused to move a weight vertically by means of a screw, which pulls the wedge horizontally on its base. If the screw has 5 threads per inch and the handle is 10 inches long, what force will be necessary to apply to the handle to raise a weight of 1,400 pounds, the height of the wedge being 8 inches, and the length, 14 inches?
Ans. 2.546 lb., nearly.

2. If the distance from the fulcrum to the point at which a force of 135 pounds is applied to a lever is 4 feet, and the distance from the fulcrum to the weight is 1½ inches, how great a weight will the force lift? Ans. 4,320 lb.

3. It is desired to raise a weight of 600 pounds by means of a block and tackle having two fixed and two movable pulleys; what force must be applied at the free end of the rope? Ans. 150 lb.

4. If, in the last example, the free end of the rope be attached to a windlass whose drum is 5 inches in diameter, and which has a handle situated 15 inches from the axis of the drum, what force will be necessary to raise the weight? Ans. 25 lb.

5. It is required to pull a wagon up an inclined plane one mile long. The height of the plane being 120 feet, and the weight of the wagon and load 2,816 pounds, what force will be necessary? Ans. 64 lb.

6. In Fig. 144, the radius of the wheel A is 32 inches, of C, 18 inches, and of E, 30 inches; the radius of the drum F is 10 inches, of the pinion D, 4 inches, and of the pinion B, 6 inches. If P moves 4 feet 6 inches, how far will the weight W move? Ans. ¾ in.

7. In the last example, how great a force must be applied at P to raise a weight of 2,160 pounds? Ans. 30 lb.

8. In example 4, if the friction be taken as 22% of the load lifted, what force will be necessary? What will be the efficiency?

Ans. { 30.5 lb.
{ 82%, nearly.

WORK AND ENERGY.

951. Work *is the overcoming of resistance continually occurring along the path of motion.*

Mere motion is not work, but if a body in motion constantly overcomes a resistance, it does work.

952. The **unit of work** *is one pound raised vertically one foot*, and is called one **foot-pound.** All work is measured by this standard. A horse going up hill does an amount of work equal to his own weight, plus the weight of the wagon and contents, plus the frictional resistances reduced to an equivalent weight, multiplied by the vertical height of the hill. Thus, if the horse weighs 1,200 pounds, the wagon and contents 1,200 pounds, and the frictional resistances equal 400 pounds, then, if the vertical height of the hill is 100 feet, the work done is equal to $(1{,}200 + 1{,}200 + 400) \times 100 = 280{,}000$ foot-pounds.

953. If the force necessary to overcome the resistance be represented by F, the space through which the resistance acts by S, and the work done by U, then $U = F\,S$.

If $W =$ the weight of a body, and $h =$ the height through which it is raised, $U = W\,h$. Hence, the work done

$$U = F\,S = W\,h. \qquad \textbf{(23.)}$$

954. The *total* amount of work is independent of time, whether it takes one minute or one year in which to do it; but, in order to compare the work done by different machines with a common standard, time must be considered.

When time is considered, the unit of time is always *one minute*, and the unit which measures the capacity of any contrivance for producing work then becomes one **foot-pound per minute.** This unit is called the **unit of power.** The term *power* as here used has a different meaning from that previously given to it, which simply meant force acting to produce motion in simple machines. The student should carefully distinguish between the two. There will be no difficulty in doing this, as the wording of the sentence in which it occurs will always show whether force or work per minute is meant. The power of a machine may always be determined by *dividing the work done in foot-pounds by the time in minutes required to do the work;* i. e.,

$$\text{Power} = \frac{F\,S}{T}, \qquad \textbf{(24.)}$$

in which F and S have the same values as in formula **23,** and $T =$ the time in minutes. Hence, if a certain machine does, say, 10,000 foot-pounds of work in 10 minutes, its *power* is $10{,}000 \div 10 = 1{,}000$ foot-pounds per minute; if another machine does the same work in 5 minutes, its power is $10{,}000 \div 5 = 2{,}000$ foot-pounds per minute—just twice as much. Hence, we say that the power of the second machine is twice that of the first, and understand thereby that if both machines work for the same length of time the second machine will do twice as much work as the first machine.

955. Since the unit of power is very small, and would lead to the use of very large numbers in expressing the power of large machines, the common standard to which all work is reduced is the horsepower, which equals 33,000 units of power.

One **horsepower** *is 33,000 foot-pounds per minute; in other words, it is 33,000 pounds raised vertically one foot in one minute, or 1 pound raised vertically 33,000 feet in one minute, or any combination that will give 33,000 foot-pounds in one minute by multiplying the resistance in pounds by the distance in feet through which it is overcome, and dividing by the time in minutes.*

Thus, 110 pounds raised vertically 5 feet in one second, is a horsepower; for, since one second $= \frac{1}{60}$ of a minute, $110 \times 5 \div \frac{1}{60} = 33,000$ foot-pounds in one minute. The abbreviation for horsepower is H. P.

EXAMPLE.—If the coefficient of friction is .3, how many horsepower will it require to draw a load of 10,000 pounds on a level surface a distance of one mile in one hour?

SOLUTION.—$10,000 \times .3 = 3,000$ pounds = the force necessary to overcome the resistance (resistance of the air is neglected). One mile = 5,280 feet; one hour = 60 minutes.

$$\text{Power} = \frac{FS}{T} = \frac{3,000 \times 5,280}{60} = 264,000 \text{ foot-pounds per minute.}$$

$$\text{Horsepower} = \frac{264,000}{33,000} = 8 \text{ H.P. Ans.}$$

956. Energy is a term used to express *the ability of an agent to do work.* Work cannot be done without motion, and the work that a moving body is capable of doing in being brought to rest is called the **kinetic energy** of the body.

Kinetic energy means the actual energy of a body in motion. The work which a moving body is capable of doing in being brought to rest is exactly the same as the kinetic energy developed by it when falling in a vacuum through a height sufficient to give it the same velocity.

Let $W =$ the weight of the body in pounds;

$v =$ its velocity in feet per second;

$h =$ the height in feet through which the body must fall to produce the velocity v;

$m =$ the mass of the body $= \frac{W}{g}$. (See formula **10**, Art. **888.**)

957. The work necessary to raise a body through a height h is Wh. The velocity produced in falling through a height h is $v = \sqrt{2gh}$, and $h = \frac{v^2}{2g}$. (See formulas **15** and **16**, Art. **896.**)

Therefore, work $= Wh = W\frac{v^2}{2g} = \frac{1}{2} \times \frac{W}{g} \times v^2 = \frac{1}{2}mv^2$, or

$$Wh = \tfrac{1}{2}mv^2. \qquad \textbf{(25.)}$$

In other words, if the weight of the body and its velocity are given, the work necessary to bring it to rest is equal to one-half the product of the mass and the square of the velocity in feet per second. This is the kinetic energy of a moving body.

If a body, whose weight is 64.32 pounds, is moving with a velocity of 20 feet per second, the kinetic energy is $\frac{1}{2}mv^2 = \frac{W}{2g} \times v^2 = \frac{64.32}{2 \times 32.16} \times 20^2 = 400$ foot-pounds.

958. The height through which a body would have to fall in a vacuum to gain a velocity of 20 feet per second is $h = \frac{v^2}{2g} = \frac{400}{2 \times 32.16} = \frac{400}{64.32}$; the work done in raising a body through this height is $Wh = 64.32 \times \frac{400}{64.32} = 400$ foot-pounds, the same result as before. The distinction between kinetic energy and work, then, is this: *A body moving with a certain velocity, and meeting with no resistance, is not doing any work*, but has a certain amount of kinetic energy stored up in it, which depends upon the velocity and mass of the body.

If a resistance be interposed just sufficient to stop the body, the body will then do an amount of *work* equal to its kinetic energy.

Example.—If a body weighing 25 pounds falls from a height of 100 feet, how much work can be done?

Solution.—Work $= Wh = 25 \times 100 = 2{,}500$ foot-pounds. Ans.

Example.—A body weighing 50 pounds has a velocity of 100 feet per second; what is its kinetic energy?

Solution.—Kinetic energy $= \frac{1}{2} m v^2 = \frac{W v^2}{2 g} = \frac{50 \times 100^2}{2 \times 32.16} = 7{,}773.63$ foot-pounds. Ans.

Example.—In the last example, how many horsepower would be required to give the body this amount of kinetic energy in 3 seconds?

Solution.—1 H.P. $= 33{,}000$ pounds raised one foot in one minute. If 7,773.63 foot-pounds of work are done in 3 seconds, in one second there would be done $\frac{7{,}773.63}{3} = 2{,}591.21$ foot-pounds of work, and in one minute, $2{,}591.21 \times 60 = 155{,}472.6$ foot-pounds.

The number of horsepower developed would be

$$\text{H.P.} = \frac{155{,}472.6}{33{,}000} = 4.7113 \text{ H.P.} \quad \text{Ans.}$$

959. ***Potential energy*** *is latent energy; it is the energy which a body at rest is capable of giving out under certain conditions.*

If a stone be suspended by a string from a high tower, it has potential energy due to its position. If the string be cut, the stone will fall to the ground, and during its fall its potential energy will change into kinetic energy, so that, at the instant it strikes the ground, its potential energy is wholly changed into kinetic energy.

At a point equal to one-half the height of the fall, the potential and kinetic energies were equal. At the end of the first quarter, the potential energy was ¾, and the kinetic energy ¼; at the end of the third quarter, the potential energy was ¼, and the kinetic energy ¾.

A pound of coal has a certain amount of potential energy. When the coal is burned, the potential energy is liberated, and changed into kinetic energy in the form of heat. The kinetic energy of the heat changes water into steam, which thus has a certain amount of potential energy. The steam acting on the piston of an engine causes it to move through a certain space, thus overcoming a resistance, changing the potential energy of the steam into kinetic energy, and thus doing work.

Potential energy, *then, is the energy stored within a body, which may be liberated and produce motion, thus generating kinetic energy, and enabling work to be done.*

960. The principle of **conservation of energy** teaches that energy, like matter, can never be destroyed. If a clock is put in motion, the potential energy of the spring is changed into the kinetic energy of motion, which turns the wheels, thus producing friction. The friction produces heat, which dissipates into the surrounding air, but still the energy is not destroyed—it merely exists in another form. The potential energy in coal was received from the sun, in the form of heat, ages ago, and has laid dormant for millions of years.

FORCE OF A BLOW.

961. The *average* **force of a blow** may be determined as follows:

In driving a nail into a piece of wood with a hammer, the head of the hammer must be capable of exerting, in a very short time, a force equal to that of a load sufficiently heavy to produce by its weight a movement of the nail into the wood, equal to the movement of the nail produced by the hammer; in other words, *the striking force, multiplied by the distance that the nail is driven into the wood, must equal the kinetic energy of the hammer.*

Suppose that the velocity of the hammer, as it strikes the nail, is 30 feet per second, that the weight of the head is 2 pounds, and that the nail is driven into the wood $\frac{1}{4}$ of an inch. Let the resistance offered by the wood, which is the same as the striking force of the hammer, be represented by F; then, since $\frac{1}{4}$ inch $= \frac{1}{48}$ of a foot,

$$F \times \tfrac{1}{48} = \tfrac{1}{2} m \times v^2 = \frac{2}{2 \times 32.16} \times 30^2 = 28, \text{ nearly.}$$

Therefore, $F = 28 \times 48 = 1{,}344$ pounds.

Had the penetration been $\frac{1}{8}$ of an inch, instead of $\frac{1}{4}$ of an inch, the value of F would have been twice as large, or 2,688 pounds. This is as it should be, since the work done is the same in both cases, while the distance through which the force acts is only half as great in the second case as in the first case. Hence, in order that the work may be the same, the force must be doubled.

Example.—If the head of a drop-hammer, weighing 400 pounds, falls from a height of 10 feet, and compresses a piece of cold iron .01 of an inch, what was the striking force of the hammer?

Solution.—.01 of an inch = $\frac{.01}{12}$ of a foot.

$$F \times \frac{.01}{12} = W h = 400 \times 10 = 4{,}000.$$

Hence, $F = \dfrac{4{,}000 \times 12}{.01} = 4{,}800{,}000$ lb. Ans.

DENSITY AND SPECIFIC GRAVITY.

962. The **density of a body** is its mass divided by its volume in cubic feet.*

Let D be the density; then, the density of a body is,

$$D = \frac{m}{V}. \quad \text{Since } m = \frac{W}{g},\ D = \frac{W}{gV}. \qquad \textbf{(26.)}$$

Example.—Three cubic feet of cast iron weigh 1,350 pounds; what is the density of cast iron?

Solution.—$D = \dfrac{W}{gV} = \dfrac{1{,}350}{32.16 \times 3} = 13.992.$

Example.—A cubic foot of water weighs 62.42 pounds; (*a*) what is its density? (*b*) What is the ratio between the density of cast iron and the density of water?

Solution.—(*a*) $D = \dfrac{W}{gV} = \dfrac{62.42}{32.16 \times 1} = 1.941 =$ density. Ans.

(*b*) $\dfrac{13.992}{1.941} = 7.21 =$ ratio. Ans.

963. The **specific gravity of a body** *is the ratio between its weight and the weight of a like volume of water.*

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*Note.—Some writers define density as the weight of a unit of volume of the material. When English measures are used, the density of any material, according to this definition, is the weight of a cubic foot of the material in pounds.

964. Since gases are so much lighter than water, it is usual to take the specific gravity of a gas as the ratio between the weight of a certain volume of the gas and the weight of the same volume of air.

EXAMPLE.—A cubic foot of cast iron weighs 450 pounds; what is its specific gravity, a cubic foot of water weighing 62.42 pounds?

SOLUTION.—According to the definition,

$$\frac{450}{62.42} = 7.21. \quad \text{Ans.}$$

NOTE.—Notice that this answer is the same as that of the preceding example; hence, the specific gravity of a body is also the *ratio of the density of a body to the density of water.*

965. The specific gravities of different bodies are given in printed tables; hence, if it is desired to know the weight of a body that cannot be conveniently weighed, *calculate its cubical contents, and multiply the specific gravity of the body by the weight of a like volume of water, remembering that a cubic foot of water weighs 62.42 pounds.*

EXAMPLE.—How much will 3,214 cubic inches of cast iron weigh? Take its specific gravity as 7.21.

SOLUTION.—Since 1 cubic foot of water weighs 62.42 pounds, 3,214 cubic inches weigh

$$\frac{3,214}{1,728} \times 62.42 = 116.098 \text{ pounds.}$$

Then, $116.098 \times 7.21 = 837.067$ pounds. Ans.

EXAMPLE.—What is the weight of a cubic inch of cast iron?

SOLUTION.—$\frac{62.42}{1,728} \times 7.21 = .26044$ pound. Ans.

One cubic foot of pure distilled water at a temperature of 39.2° Fahrenheit weighs 62.425 pounds, but *the value usually taken in making calculations is 62.5 pounds.*

EXAMPLE.—What is the weight in pounds of 7 cubic feet of oxygen?

SOLUTION.—One cubic foot of air weighs .08073 pound, and the specific gravity of oxygen is 1.1056, compared with air; hence,

$$.08073 \times 1.1056 \times 7 = .62479 \text{ pound, nearly.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. A man jumps from a railroad train traveling at the rate of 60 miles per hour; if he weighs 160 lb., what is the kinetic energy of his body? Ans. 19,263.68 ft.-lb.

2. A hammer strikes a nail with a velocity of 40 ft. per sec.; if the head weighs 1½ lb., and the nail is driven $\frac{5}{16}$ in., what is the force of the blow? Ans. 1,433 lb., nearly.

3. A load of 20,000 lb. is pulled up an inclined plane 1½ miles long in 5 minutes; if the height of the plane is 800 ft., and the frictional resistances are 3% of the load, what is the horsepower required?

Ans. 125.62 H. P.

4. If a cubic foot of a certain substance weighs 162½ lb., what is its specific gravity? Ans. 2.6.

5. A mixture of lead and tin has a specific gravity of 9.28; what is the weight of a cubic inch? Ans. 5.37 oz.

6. A weight of 11,625 lb. is raised vertically 10 ft. in 3 minutes, by means of a block and tackle and a windlass; the frictional resistances being 26%, how many horsepower were required? Ans. 1.48 H. P.

7. A body weighing 24,062.5 lb. is drawn on a horizontal surface at the rate of 600 ft. per min.; the coefficient of friction being 8%, what horsepower will be necessary to overcome the resistance of friction?

Ans. 35 H. P.

8. A solid cast iron sphere, 12″ in diameter, falls through a height of 50 ft.; what will be its kinetic energy on striking? Ans. 11,781 ft.-lb.

966. The table of specific gravities gives the specific gravities of a variety of substances likely to be met with in ordinary practice. The weights per cubic foot are calculated on a basis of 62.5 pounds of water per cubic foot.

HYDROMECHANICS.

HYDROSTATICS.

967. **Hydrostatics** treats of liquids at rest under the action of forces.

968. Liquids are very nearly *incompressible.* A pressure of 15 pounds per square inch compresses water less than $\frac{1}{20000}$ of its volume.

969. Fig. 157 represents two cylindrical vessels of exactly the same size. The vessel *a* is fitted with a wooden block of the same size as the cylinder, and can move in it; the vessel *b* is filled with water, whose depth is the same as the length of the wooden block in *a*. Both vessels are fitted with air-tight pistons *P* whose areas are each 10 square inches.

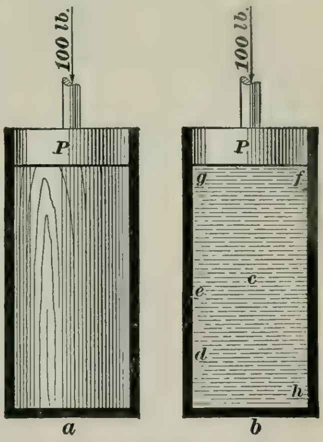

FIG. 157.

Suppose, for convenience, that the weights of the cylinders, pistons, block, and water be neglected, and that a force of 100 pounds be applied to both pistons. The pressure per square inch will be $\frac{100}{10} = 10$ pounds. In the vessel *a*, this pressure will be transmitted to the bottom of the vessel, and will be 10 pounds per square inch; it is easy to see that there will be no pressure on the sides. In the vessel *b*, an entirely different result is obtained. The pressure on the bottom will be the same as in the other case—that is, 10 pounds per square inch—but, owing to the fact that the molecules of the water are perfectly free to move, this pressure of 10 pounds per square inch is *transmitted in every direction with the same intensity*; that is to say, the pressure at any point, *c*, *d*, *e*, *f*, *g*, *h*, etc., due to the force of 100 pounds is exactly the same, and equals 10 pounds per square inch.

This may be easily proven experimentally by means of an apparatus like that shown in Fig. 158. Let the area of the piston *a* be 20 square inches; of *b*, 7 square inches; of *c*, 1 square inch; of *d*, 6 square inches; of *e*, 8 square inches, and of *f*, 4 square inches.

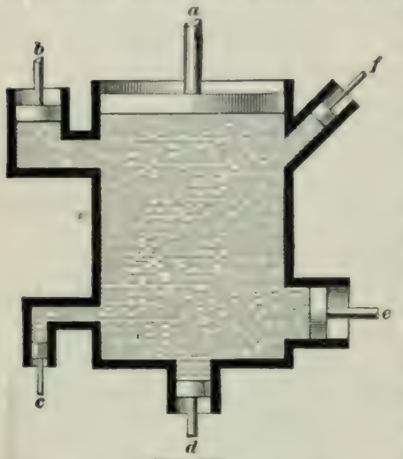

FIG. 158.

If the pressure due to the weight of the water be neglected, and a force of 5 pounds be applied at *c* (whose area is 1 square inch), a pressure of 5 pounds per square inch will be transmitted in all directions; and in order that there shall be no movement, a force of $6 \times 5 = 30$ pounds must be applied at *d*, 40 pounds at *e*, 20 pounds at *f*, 100 pounds at *a*, and 35 pounds at *b*.

If a force of 99 pounds were applied to *a*, instead of 100 pounds, the piston *a* would rise, and the other pistons *b*, *c*, *d*, *e*, and *f* would move inwards; but, if the force applied to *a* were 100 pounds, they would all be in equilibrium. Had 101 pounds been applied at *a*, the pressure per square inch would be $\frac{101}{20} = 5.05$ pounds, which would be transmitted in all directions; and, since the pressure due to *c* is only 5 pounds per square inch, it is now evident that the piston *a* will move downwards, and the pistons *b*, *c*, *d*, *e*, and *f* will be forced outwards.

The whole may be summed up as follows:

970. Rule.—*The pressure per unit of area exerted anywhere upon a mass of liquid is transmitted undiminished in* *all directions, and acts with the same force upon all surfaces in a direction at* **right angles** *to those surfaces.*

This law was first discovered by Pascal, and is the most important in Hydromechanics. Its meaning should be thoroughly understood.

EXAMPLE.—If the area of the piston *e* in Fig. 158 were 8.25 square inches, and a force of 150 pounds were applied to it, what forces would have to be applied to the other pistons to keep the water in equilibrium, assuming that their areas were the same as given before?

SOLUTION.—$\frac{150}{8.25}$ = 18.182 lb. per sq. in., nearly.

$$\left.\begin{array}{l} 20 \times 18.182 = 363.64 \text{ lb.} = \text{force to balance } a. \\ 7 \times 18.182 = 127.274 \text{ lb.} = \text{force to balance } b. \\ 1 \times 18.182 = 18.182 \text{ lb.} = \text{force to balance } c. \\ 6 \times 18.182 = 109.092 \text{ lb.} = \text{force to balance } d. \\ 4 \times 18.182 = 72.728 \text{ lb.} = \text{force to balance } f. \end{array}\right\} \text{Ans.}$$

971. ***The pressure due to the weight of a liquid*** *may be downwards, upwards, or sideways.*

972. Downward Pressure.—In Fig. 159, the pressure on the bottom of the vessel *a* is, of course, equal to the weight of the water it contains. If the area of the bottom of the vessel *b*, and the depth of the liquid contained in it, are the same as in the vessel *a*, the pressure on the bottom of *b* will be the same as on the bottom of *a*. Suppose the bottoms of the vessels *a* and *b* are 6 inches square, and that the part *c d*, in the vessel *b*, is 2 inches square, and that both vessels are filled with water. Then, the weight of 1 cubic inch of water being $\frac{62.5}{1,728}$ = .03617 pound, and the number of cubic inches in *a*, $6 \times 6 \times 24 = 864$ cubic inches, the weight of the water is $864 \times .03617 = 31.25$ pounds. Hence, the total pressure on the bottom of the vessel *a* is 31.25 pounds, or $\frac{31.25}{36} = .868$ pound per square inch.

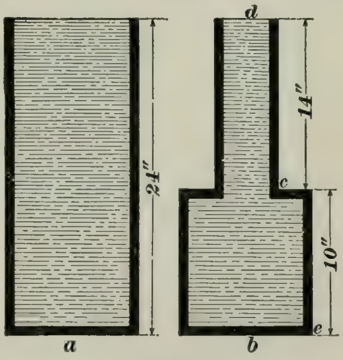

FIG. 159.

The pressure in *b*, due to the weight contained in the part *b c*, is 6 × 6 × 10 × .03617 = 13.02 pounds.

The weight of the part contained in *c d* is 2 × 2 × 14 × .03617 = 2.0255 pounds, and the weight per square inch of area in *c d* is $\frac{2.0255}{4} = .5064$ pound.

According to Pascal's law, this weight (pressure) is transmitted equally in all directions, therefore, every square inch of the large part of the vessel *b* will be subjected to a pressure of .5064 pound. The area of the part *b c* is 6 × 6 = 36 square inches, and the total pressure due to the weight of the water in the small part will be .5064 × 36 = 18.23 pounds. Hence, the total pressure on the bottom of *b* will be 13.02 + 18.23 = 31.25 pounds, the same result as in the case of the vessel *a*.

If an additional pressure of ten pounds per square inch were applied to the upper surface of both vessels, the total pressure on their bottoms would be 31.25 + (6 × 6 × 10) = 31.25 + 360 = 391.25 pounds.

In case this pressure were obtained by means of a weight placed on a piston, as shown in Figs. 157 and 158, the weight for the vessel *a* would be 6 × 6 × 10 = 360 pounds, and for the vessel *b*, 2 × 2 × 10 = 40 pounds.

973. The General Law for the Downward Pressure upon the Bottom of any Vessel:

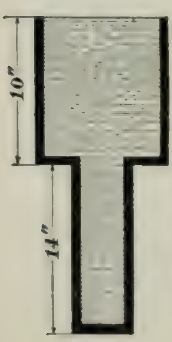

FIG. 160.

Rule.—*The pressure upon the bottom of a vessel containing a fluid is independent of the shape of the vessel, and is equal to the weight of a prism of the fluid whose base has the same area as the bottom of the vessel, and whose altitude is the distance between the bottom and the upper surface of the fluid, plus the pressure per unit of area upon the upper surface of the fluid multiplied by the area of the bottom of the vessel.*

974. Suppose that the vessel *b*, in Fig. 159, were inverted, as shown in Fig. 160, the pressure upon the bottom will still be .868 pound per square inch, but it will require a weight of 3,490 pounds to be placed upon the piston at the upper surface to make the pressure on the bottom 391.25 pounds, instead of a weight of 40 pounds, as in the other case.

EXAMPLE.—A vessel filled with salt water, having a specific gravity of 1.03, has a circular bottom 13 inches in diameter. The top of the vessel is fitted with a piston 3 inches in diameter, on which is laid a weight of 75 pounds; what is the total pressure on the bottom, if the depth of the water is 18 inches?

SOLUTION.—The weight of 1 cubic inch of the water is $\frac{62.5 \times 1.03}{1,728} =$ .037254 lb.

$13 \times 13 \times .7854 \times 18 \times .037254 = 89.01$ pounds = the pressure due to the weight of the water.

$\frac{75}{3 \times 3 \times .7854} = 10.61$ lb. per sq. in. due to the weight on the piston.

$13 \times 13 \times .7854 \times 10.61 = 1,408.29$ lb. = pressure on the bottom due to the weight.

Total pressure $= 1,408.29 + 89.01 = 1,497.3$ lb. Ans.

975. Upward Pressure.—In Fig. 161 is represented a vessel of exactly the same size as that represented in Fig. 160. There is no upward pressure on the surface *c* due to the weight of the water in the large part *c d*, but there is an upward pressure on *c* due to the weight of the water in the small part *b c*. The pressure per square inch due to the weight of the water in *b c* was found to be .5064 pound (see Art. **972**); the area of the upper surface *c* of the large part *c d* is evidently $(6 \times 6) - (2 \times 2) = 36 - 4 = 32$ square inches, and the total upward pressure due to the weight of the water is $.5064 \times 32 = 16.2$ pounds.

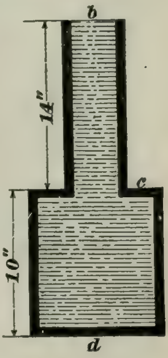

FIG. 161.

If an additional pressure of 10 pounds per square inch were applied to a piston fitting the top of the vessel, the total upward pressure on the surface *c* would be $16.2 + (32 \times 10) = 336.2$ pounds.

976. General Law for Upward Pressure:

Rule.—*The upward pressure on any submerged horizontal surface equals the weight of a prism of the liquid whose base* *has an area equal to the area of the submerged surface, and whose altitude is the distance between the submerged surface and the upper surface of the liquid, plus the pressure per unit of area on the upper surface of the fluid multiplied by the area of the submerged surface.*

EXAMPLE.—A horizontal surface 6 inches by 4 inches is submerged in a vessel of water 26 inches below the upper surface. If the pressure on the water is 16 pounds per square inch, what is the total upward pressure on the horizontal surface?

SOLUTION.—6 × 4 × 26 × .03617 = 22.57 lb., or the upward pressure due to the weight of the water.

6 × 4 × 16 = 384 lb., or the upward pressure due to the outside pressure of 16 lb. per sq. in.

The total upward pressure = 384 + 22.57 = 406.57 lb. Ans.

977. Lateral (Sideways) Pressure.—Suppose the top of the vessel shown in Fig. 162 is 10 inches square, and that the projections at *a* and *b* are 1 inch × 1 inch, and 10 inches long.

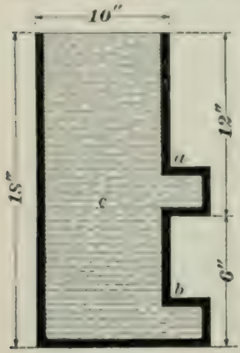

FIG. 162.

The pressure per square inch on the bottom of the vessel, due to the weight of the liquid, is 1 × 1 × 18 × the weight of a cubic inch of the liquid.

The pressure at a depth equal to the distance of the upper surface *b* below the top of the vessel is 1 × 1 × 17 × the weight of a cubic inch of the liquid.

Since both of these pressures are transmitted in every direction, they are also transmitted laterally (sideways), and the *pressure per unit of area on the projection b is a mean between the two*, and equals 1 × 1 × 17½ × the weight of a cubic inch of the liquid.

To find the lateral pressure on the projection *a*, imagine that the dotted line *c* is the bottom of the vessel, then the conditions will be the same as in the preceding case, except that the depth is not so great.

The lateral pressure on *a* is thus seen to be 1 × 1 × 11½ × the weight of a cubic inch of the liquid.

General Law for Lateral Pressure:

978. Rule.—*The pressure upon any vertical surface due to the weight of a liquid is equal to the weight of a prism of the liquid whose base has the same area as the vertical surface, and whose altitude is the depth of the center of gravity of the vertical surface below the level of the liquid.*

Any additional pressure is to be added as in the previous cases.

EXAMPLE.—A well 3 feet in diameter and 20 feet deep is filled with water; what is the pressure on a strip of the wall 1 inch wide, the center of which is 1 foot from the bottom? What is the pressure on the bottom? What is the upward pressure per square inch, 2 feet 6 inches from the bottom?

SOLUTION.—$1 \times 36 \times 3.1416 = 113.1$ sq. in. = area of the strip. Depth of center of gravity $= 20 - 1 = 19$ ft.

$113.1 \times 19 \times 12 \times .03617 = 932.71$ lb. = total pressure upon the strip. Ans.

The pressure per square inch is $\frac{932.71}{113.1} = 8.247$ lb., nearly.

$36 \times 36 \times .7854 \times 20 \times 12 \times .03617 = 8{,}836$ lb. = the pressure on the bottom. Ans.

$20 - 2.5 = 17.5$. $1 \times 17.5 \times 12 \times .03617 = 7.596$ lb. = the upward pressure per square inch, 2 ft. 6 in. from the bottom, since 2 ft. 6 in. = 2.5 ft. Ans.

979. The effects of the lateral pressure are illustrated in Fig. 163. *e* is a tall vessel having a stop-cock near its base, and arranged to float upon the water, as shown. When this vessel is filled with water, the lateral pressures at any two points of the surface of the vessel, and opposite to each other, are equal. Being equal, and acting in opposite directions, they destroy each other (see Art. **881**), and no motion

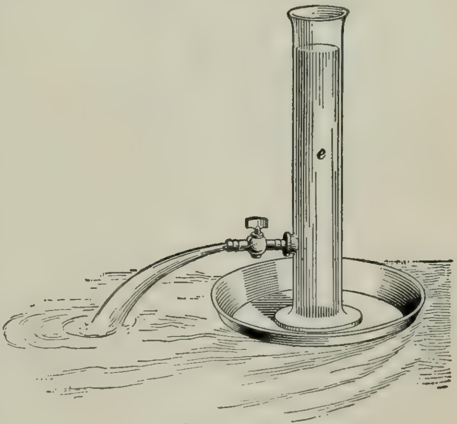

FIG. 163.

can result; but, if the stop-cock be opened, there will be no resistance to that pressure acting on the surface equal to the area of the opening, the pressure which formerly acted causing the water to flow out, while its equal and opposite pressure will cause the vessel to move backward through the water in a direction opposite to that of the spouting water.

Since the pressure on the bottom of a vessel due to the weight of the liquid is dependent only upon the height of the liquid, and not upon the shape of the vessel, it follows that if a vessel has a number of radiating tubes, as Fig. 164, the water in each tube will be on the same level no matter what may be the shape of the tubes. For, if the water were higher in one tube than in the others, the downward pressure on the bottom due to the height of the water in this tube would be greater than that due to the height of the water in the other tubes.

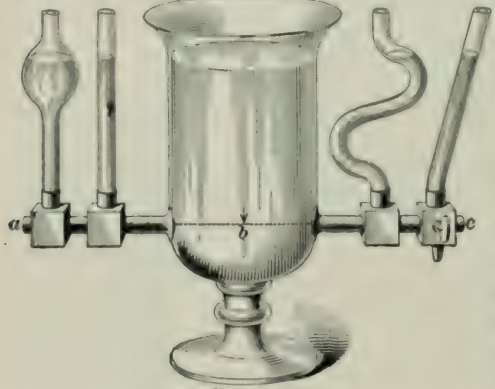

FIG. 164.

Consequently, the upward pressure would also be greater; the equilibrium would be destroyed, and the water would flow from this tube into the vessel, and rise in the other tubes until it was at the same level in all, when it would be in equilibrium. This principle is expressed in the familiar saying, "*Water seeks its level.*"

This explains why city water reservoirs are located on high elevations, and why water on leaving the hose-nozzle spouts so high.

If there was no resistance by friction and air, the water would spout to a height equal to the level of the water in the reservoirs. If a long pipe was attached to the nozzle, whose length was equal to the vertical distance between the nozzle and the level of the water in the reservoir, the water would just reach the end of the pipe. If the pipe was lowered slightly, the water would trickle out.

Fountains, canal locks, and artesian wells are examples of the application of this principle.

EXAMPLE.—The water level in a city reservoir is 150 feet from the level of the street; what is the pressure of the water per square inch on the hydrant?

SOLUTION.—$1 \times 150 \times 12 \times .03617 = 65.106$ lb. per sq. in. Ans.

980. The weight of a column of water 1 inch square and 1 foot high is $.03617 \times 12 = .434$ pound, nearly. Hence, if the depth (head) is given, the pressure per square inch may be found by multiplying the depth in feet by .434. The constant .434 is the one ordinarily employed in practical calculations.

981. In Fig. 165, let the area of the piston *a* be 1 square inch; of *b*, 40 square inches. According to Pascal's law, 1 pound placed upon *a* will balance 40 pounds placed upon *b*.

Suppose that *a* moves downwards 10 inches; then, 10 cubic inches of water will be forced into the tube *b*. This will be distributed over the entire area of the tube *b* in the form of a cylinder, whose cubical contents must be 10 cubic inches, whose base has an area of 40 square inches, and whose altitude must be $\frac{10}{40} = \frac{1}{4}$ of an inch; that is, a movement of 10 inches of the piston *a* will cause a movement of $\frac{1}{4}$ of an inch of the piston *b*.

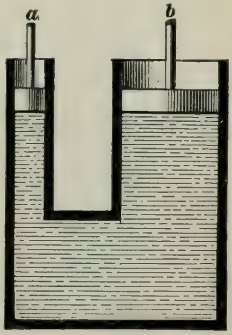

FIG. 165.

Here is the old principle of machines: *The power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves.*

Hence, if 1 pound on the piston *a* represents the power P, the equivalent weight W on *b* may be obtained from the equation $P \times 10 = W \times \frac{1}{4}$, or $10 = \frac{1}{4} W$, and $W = 40$.

Another familiar fact is also recognized, for the velocity ratio of P to W is 10 : $\frac{1}{4}$, or 40. Since in any machine the weight equals the power multiplied by the velocity ratio, $W = P \times 40$, and when $P = 1$ pound, $W = 40$ pounds.

982. This principle is made use of in the hydraulic press represented in Fig. 166. As the man depresses the lever O, he forces down the piston a upon the water in the cylinder A. The water is forced through the bent tube d into the cylinder in which the large piston C works, and causes it to rise, thus lifting the platform K, and compressing the bales. If the area of a be 1 square inch, and that of C be 100 square inches, the velocity ratio will be 100.

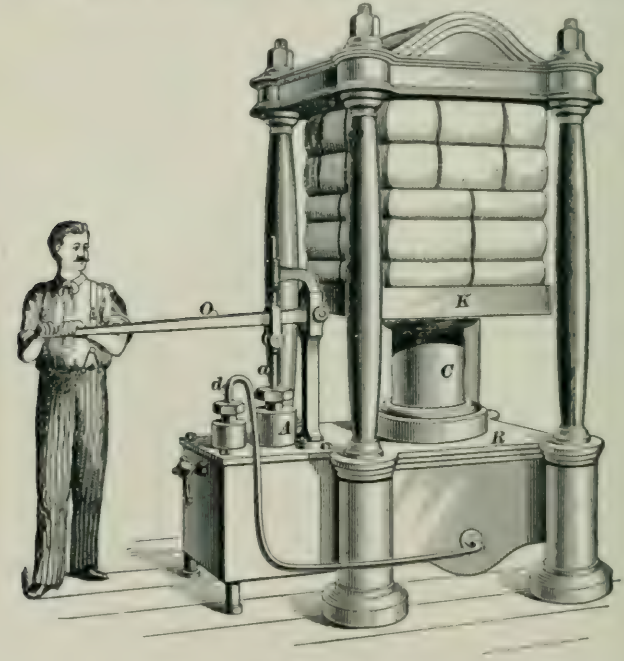

FIG. 166.

If the length of the lever between the hand and the fulcrum is 10 times the length between the fulcrum and the piston a, the velocity ratio of the lever will be 10, and the total velocity ratio of the hand to the piston C will be 1,000.

Hence, a force of 100 pounds applied by the hand will raise $100 \times 1{,}000 = 100{,}000$ pounds. But, if the average movement of the hand per stroke is 10 inches, it will require $\frac{1{,}000}{10} = 100$ strokes to raise the platform 1 inch, and it is again seen that what is gained in power is lost in speed.

Applications of this principle are seen in the hydraulic machines used for forcing locomotive drivers on their axles, etc., and for testing the strength of boiler shells.

983. Example.—A suspended vertical cylinder is tested for the tightness of its heads by filling it with water. A pipe whose inside diameter is ¼ of an inch, and whose length is 20 feet, is screwed into a hole in the upper head, and then filled with water; what is the pressure per square inch on each head, if the cylinder is 40 inches in diameter and 60 inches long?

Solution.—Area of heads $= 40^2 \times .7854 = 1{,}256.64$ sq. in.

Pressure per square inch on the bottom head, due to the weight of the water in the cylinder, $= 1 \times 60 \times .03617 = 2.17$ lb. $(\frac{1}{4})^2 \times .7854 = .04909$ sq. in., the area of the pipe.

$.04909 \times 20 \times 12 \times .03617 = .426$ lb. = the weight of water in pipe = the pressure on a surface area of .04909 sq. in.

The pressure per square inch due to the water in the pipe is $\frac{1}{.04909} \times .426 = 8.68$ lb. per sq. in. upon the upper head. Ans.

The pressure per square inch on the lower head is $8.68 + 2.17 = 10.85$ lb. per sq. in. Ans.

Example.—In the last example, if the pipe is fitted with a piston weighing ¼ of a pound, and a 5-pound weight is laid upon it, what is the pressure per square inch upon the upper head?

Solution.—In addition to the pressure of .426 pound on the area of .04909 sq. in., there is now an additional pressure upon this area of $5 + \frac{1}{4} = 5.25$ lb., and the total pressure upon this area is $.426 + 5.25 = 5.676$ lb.

The pressure per square inch is $\frac{1}{.04909} \times 5.676 = 115.6$ lb. Ans.

984. Pressure upon Oblique Surfaces.—Heretofore, the pressure upon horizontal and vertical surfaces only has been considered. The pressure upon sides which are oblique to the bottom will now be considered.

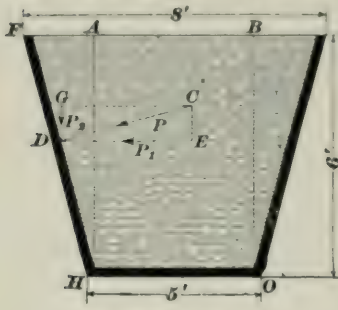

FIG. 167.

According to the law of Pascal, the pressure which a fluid exerts upon a surface is at right angles to the surface; hence, in Fig. 167, the pressure acts in a direction indicated by the arrow-head on the line $C D$, and the lateral pressure on every square unit of area of the side $F H$ will be in the direction $C D$. According to the law for lateral pressure (Art. **978**), the total perpendicular pressure upon the side $F H$ will be: Area of side $F H \times \frac{1}{2} A H \times$ the weight of a cubic inch of the fluid.

Let the line $C D$ represent this perpendicular pressure and call it P. Now, resolve P into two components, one, P_1, acting horizontally, and the other, P_2, acting vertically. The angle $C D E = F H A$, for $C D$ is perpendicular to $F H$, and $E D$ is perpendicular to $A H$. (See Art. **692.**) Therefore, $P_1 = P \times \cos C D E = P \times \cos F H A$; but the cosine of $F H A$ is numerically equal to $A H$, which equals the projection of $F H$ on a line at right angles to $E D$.

The angle $G D C = E C D = A F H$, since $C D$ is perpendicular to $F H$, and $G D$ is perpendicular to $A F$; consequently, the vertical component $P_2 = P \cos G D C = P \cos A F H$. But the cosine $A F H$ is numerically equal to $A F$, which equals the projection of $F H$ upon a line at right angles to $G D$. Hence, the rule for finding the pressure exerted by a fluid in any direction upon a surface is:

Rule.—*The pressure exerted by a fluid in any direction on a plane surface is equal to the weight of a prism of the fluid whose base is the projection of the surface at right angles to the direction considered, and whose height is the depth of the center of gravity of the surface below the level of the liquid.*

985. It can be proven by higher mathematics that the preceding rule holds good for any surface, no matter what its shape may be; that is, whether flat, spherical, cylindrical, or any shape whatever. Consequently, if a cylinder is filled with water, and a pressure is applied, the total pressure upon any half-section of the cylinder, as $A\ C\ B$ (Fig. 168), tending to separate one half from the other, is equal to the *projected area of the half cylinder* (*or the inside diameter times the length of the cylinder*), *multiplied by the depth of the center of gravity of the cylinder* (*or* $\frac{1}{2}\ d$), *multiplied by the weight of a cubic unit of water, plus the diameter of the shell multiplied by the pressure per square inch multiplied by the length of the cylinder.*

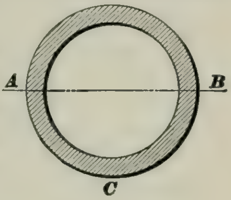

FIG. 168.

If $d =$ the inside diameter in inches and $l =$ the length of the cylinder in inches, the pressure due to the weight of the water (when the cylinder is horizontal and filled with water) upon the half cylinder $A\ C\ B = d \times l \times \frac{d}{2} \times$ the weight of a cubic inch of water $= l \times \frac{d^2}{2} \times$ the weight of a cubic inch of water.

The pressure due to an additional pressure P_1 pounds per square inch $= l \times d = P_1$. Thus, if a cylinder 60 inches long and 40 inches in diameter, lying horizontally, is filled with water, the pressure on any half-section, as $A\ C\ B$, due to the weight of the water, will be found as follows:

$$60 \times \frac{40^2}{2} \times .03617 = 1{,}736.16 \text{ pounds.}$$

If there were an additional pressure per square inch of 50 pounds, the total pressure would be $60 \times 40 \times 50 + 1{,}736.16 = 121{,}736.16$ pounds.

If the cylinder were vertical instead of horizontal, the depth of the center of gravity would evidently be $\frac{1}{2}\ l$ below the surface, and the pressure tending to separate one half from the other, due to the weight of the water, would be $d \times l \times \frac{l}{2} \times$ weight of a cubic inch of water $= d \times \frac{l^2}{2} \times$ weight of a cubic inch of water.

Any additional pressure should be calculated as in the previous case.

986. Example.—What would be the pressure due to the weight of the water, if the cylinder in the last example were vertical?

Solution.— $40 \times \frac{60^2}{2} \times .03617 = 2,604.24$ lb. Ans.

In the case of a sphere, the projected area is evidently the area of a circle whose diameter is the same as the diameter of the sphere. Hence, the pressure upon a hemisphere due to the weight of the water will be $d^2 \times .7854 \times \frac{d}{2} \times$ the weight of a cubic inch of water $= \frac{d^3}{2} \times .7854 \times$ the weight of a cubic inch of water.

The pressure upon a hemisphere whose diameter is 20 inches, when filled with water, due to the weight of the water only, will be $\frac{20^3}{2} \times .7854 \times .03617 = 113.63$ pounds.

For an additional pressure P in pounds per square inch, the pressure on the hemisphere due to this will be $d^2 \times .7854 \times P$.

Example.—If the ends of the vessel shown in Fig. 167 make right angles with the bottom, and the distance between them, or length of the vessel, is 12 feet, what are the horizontal, vertical, and perpendicular pressures against the sides, both making equal angles with the bottom?

Solution.—Apply rule, Art. **984.** In finding the horizontal pressure, the *direction considered* is that of the line ED; that is, horizontal. The projected area of the surface whose edge is FH projected at right angles to ED is $AH \times$ length of vessel $= 6 \times 12 = 72$ sq. ft. Depth of center of gravity $= 6 \div 2 = 3$ ft. Hence, horizontal pressure $= 6 \times 12 \times 3 \times 62.5 = 13,500$ lb. Ans.

Note.—We multiply by 62.5, because all dimensions are in feet.

In a similar manner, the vertical pressure is found. Thus, *direction considered* is that of the line GD. Projected area of surface FH, when projected at right angles to GD, $= FA \times$ length of vessel. $FA = \frac{8-5}{2} = 1\frac{1}{2}$ ft. Vertical pressure $= 1\frac{1}{2} \times 12 \times 3 \times 62.5 = 3,375$ lb. Ans.

To find the perpendicular pressure, first find the length of the side FH. $FH = \sqrt{6^2 + (1\frac{1}{2})^2} = 6.1847$ ft. Perpendicular pressure $= 6.1847 \times 12 \times 3 \times 62.5 = 13,916$ lb., nearly. Ans.

Example.—A sphere having a diameter of 30 inches is filled with water, and subjected to an additional pressure of 75 pounds per square inch; what is the total pressure tending to separate one vertical half-section of the sphere from its opposite half?

Solution.—The pressure due to the weight of the water is $\frac{30^3}{2} \times .7854 \times .03617 = 383.5$ lb.

$30^2 \times .7854 \times 75 = 53{,}014.5$ lb. = additional pressure.

Total pressure tending to separate any two halves $= 53{,}014.5 + 383.5 = 53{,}398$ lb. Ans.

BUOYANT EFFECTS OF WATER.

987. In Fig. 169 is shown a 6-inch cube entirely submerged in water. The lateral pressures are equal, and act in opposite directions. The upward pressure $= 6 \times 6 \times 21 \times .03617$; the downward pressure $= 6 \times 6 \times 15 \times .03617$, and the difference $= 6 \times 6 \times 6 \times .03617 =$ the volume of the cube in cubic inches × the weight of 1 cubic inch of water. That is, the upward pressure exceeds the downward pressure by the weight of a volume of water equal to the volume of the body.

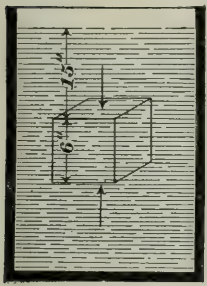

Fig. 169.

This excess of upward pressure over the downward pressure acts against gravity; consequently, *if a body be immersed in a fluid, it will lose in weight an amount equal to the weight of the fluid it displaces.* This is called the **principle of Archimedes,** because it was first stated by him.

988. Archimedes' principle may be experimentally demonstrated with the beam scales, as shown in Fig. 170.

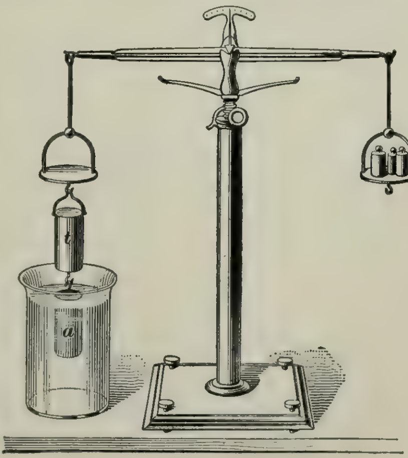

Fig. 170.

From one scale-pan, suspend a hollow cylinder of metal *t*, and below that a solid cylinder *a* of the same size as the hollow part of the upper cylinder. Put weights in the other scale-pan until they exactly balance the two cylinders. If *a* be immersed in water, the scale-pan containing the weights will descend, showing that *a* has lost some of its weight. Now, fill *t* with water, and the volume of water that can be poured into *t* will equal that displaced by *a*. The scale-pan that contains the weights will gradually rise until *t* is filled, when the scales balance again.

989. If the immersed body be lighter than the liquid, the upward pressure will cause it to rise and extend partly out of the liquid, until the weight of the body and the weight of the liquid displaced are equal. If the immersed body is heavier than the liquid, the downward pressure plus the weight of the body will be greater than the upward pressure, and the body will fall downwards until it touches bottom or meets an obstruction. If the weights of equal volumes of the liquid and the body are equal, the body will remain stationary, and be in equilibrium in any position or at any depth beneath the surface of the liquid.

An interesting experiment in confirmation of the above facts may be performed as follows: Drop an egg into a glass jar filled with fresh water. The mean density of the egg being a little greater than that of water, the egg will fall to the bottom of the jar. Now, dissolve salt in the water, stirring it so as to mix the fresh and salt water. The salt water will presently become denser than the egg, and the egg will rise. Now, if fresh water be poured in until the egg and water have the same density, the egg will remain stationary in any position that it may be placed below the surface of the water.

EXAMPLES FOR PRACTICE.

1. The diameter of the plunger of a hydraulic press used in an engineering establishment is 12″. Water is forced into the cylinder of the press by means of a small pump having a plunger whose diameter is ¾″, and stroke is 4″. What pressure is exerted by the large plunger, when the force acting on the small plunger is 125 pounds?

Ans. 32,000 lb.

2. If the small plunger, in the last example, makes 96 working strokes per minute, (*a*) how long will it take the large plunger to move 8″? (*b*) What is the velocity ratio?

Ans. $\begin{cases} (a)\ 5\tfrac{1}{3} \text{ min.} \\ (b)\ 256 : 1. \end{cases}$

3. A vertical pipe, 88 feet high, is filled with water; (*a*) what is the pressure on bottom? (*b*) If the diameter of the pipe is 8″, what is the total pressure on a strip, $2\frac{1}{2}''$ high, whose center of gravity is 21 feet from the bottom?

Ans. $\begin{cases} (a)\ 38.2 \text{ lb., nearly.} \\ (b)\ 1{,}827.03 \text{ lb.} \end{cases}$

4. A sphere, 16″ in diameter, is submerged in water with its center of gravity 43 ft. 8″ below the surface. What is (*a*) the total lateral pressure? (*b*) the total pressure?

Ans. $\begin{cases} (a)\ 7{,}620.8 \text{ lb.} \\ (b)\ 15{,}241.6 \text{ lb.} \end{cases}$

SPECIFIC GRAVITY.

990. In Art. **963** it was stated that the specific gravity of a body was the ratio between the weight of the body and the weight of an equal volume of water, but no methods were given for finding this ratio. Some of these methods will now be explained.

991. Archimedes' principle affords an easy and accurate method of finding the specific gravity of solids not easily soluble in water. Weigh the body in air; then, weigh the body in water, suspending it by a string, and attaching the string to a scale-pan in place of the two cylinders shown in Fig. 170. *The difference between the two weights will be the weight of an equal volume of water. The ratio of the weight in air to the difference thus found will be the specific gravity.* The abbreviation for specific gravity is Sp. Gr.

Let W be the weight of the solid in air and W' the weight in water; then, $W - W'$ = the weight of a volume of water equal to the volume of the solid, and

$$\text{Sp. Gr.} = \frac{W}{W - W'}. \qquad \textbf{(27.)}$$

EXAMPLE.—A body in air weighs $36\frac{1}{4}$ ounces and in water 30 ounces; what is its specific gravity?

SOLUTION.—Substituting in formula **27**,

$$\text{Sp. Gr.} = \frac{W}{W - W'} = \frac{36\frac{1}{4}}{36\frac{1}{4} - 30} = \frac{36\frac{1}{4}}{6\frac{1}{4}} = 5.8. \quad \text{Ans.}$$

992. If the body be lighter than water, a piece of iron or other heavy substance must be attached to it, sufficiently heavy to sink both. *Then, weigh both bodies in air and both in water. Weigh both separately in air, and weigh the heavier body in water. Subtract the weights of the bodies in air and in water, and the result will be the weight of a volume of the water equal to the volume of the two bodies. Find the difference of the weights of the heavy body in air and in water, and the result will be the weight of a volume of water equal to the volume of the heavy body. Subtract this last result from the former, and the result will be the weight of a volume of water equal to the volume of the light body. The weight of the light body in air divided by the weight of its equal volume of water is the specific gravity of the light body.*

Let W = weight of both bodies in air;
W' = weight of both bodies in water;
w = weight of light body in air;
W_1 = weight of heavy body in air;
W_2 = weight of heavy body in water.

Then, the specific gravity of the light body is given by

$$\text{Sp. Gr.} = \frac{w}{(W - W') - (W_1 - W_2)}. \qquad \textbf{(28.)}$$

Example.—A piece of cork weighs 4.8 ounces in air. A piece of cast iron weighs 36 ounces in air and 31 ounces in water. The weight of the iron and cork together in water is 15.8 ounces; what is the specific gravity of the cork? Of the cast iron?

Solution.—Substituting in formula **28** the values given,

$$\text{Sp. Gr.} = \frac{w}{(W - W') - (W_1 - W_2)} = \frac{4.8}{(40.8 - 15.8) - (36 - 31)} = \frac{4.8}{20} =$$

.24, the specific gravity of the cork. Ans.

By formula **27**, $\text{Sp. Gr.} = \frac{W}{W - W'} = \frac{36}{36 - 31} = 7.2$, the specific gravity of the iron. Ans.

993. To find the specific gravity of a liquid:

Weigh an empty flask; fill it with water, then weigh it and find the difference between the two results; this will equal the weight of the water. Then, weigh the flask filled with the liquid, and subtract the weight of the flask; the *result is the weight of a volume of the liquid equal to the volume of the water. The weight of the liquid divided by the weight of the water is the specific gravity of the liquid.*

Let W = the weight of the flask and liquid;
W' = the weight of the flask and water;
w = the weight of the flask.

Then,

$$\text{Sp. Gr.} = \frac{W - w}{W' - w}. \qquad \textbf{(29.)}$$

EXAMPLE.—If the weight of the flask is 8 ounces, the weight when filled with water is 33 ounces, and when filled with alcohol 28 ounces, what is the specific gravity of the alcohol?

SOLUTION.—Substituting in formula **29,**

$$\text{Sp. Gr.} = \frac{W - w}{W' - w} = \frac{28 - 8}{33 - 8} = .8. \quad \text{Ans.}$$

994. The method of finding the specific gravity of gases is about the same as that just given for liquids, the air being pumped out of the flask by an air pump. As there are great difficulties attending the operation, the method will not be described.

995. Instruments called **hydrometers** are in general use for determining quickly and accurately the specific gravities of liquids and some forms of solids. They are of two kinds—viz.:

(1.) *Hydrometers of constant weight, as* **Beaume's.**
(2.) *Hydrometers of constant volume, as* **Nicholson's.**

996. A hydrometer of constant weight is shown in Fig. 171. It consists of a glass tube, near the bottom of which are two bulbs. The lower and smaller bulb is loaded with mercury or shot, so as to cause the instrument to remain in a vertical position when placed in the liquid. The upper bulb is filled with air, and its volume is such that the whole instrument is lighter than an equal volume of water.

The point to which the hydrometer sinks when placed in water is usually marked, the tube being graduated above and below in such a manner that the specific gravity of the liquid can be read

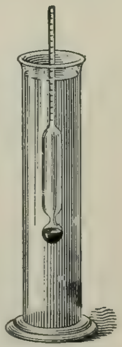

FIG. 171.

directly. It is customary to have two instruments, one with the zero point near the top of the stem for use in liquids heavier than water, and the other with the zero point near the bulb for use in liquids lighter than water.

These instruments are more commonly used for determining the degree of concentration or dilution of certain liquids, as acids, alcohol, milk, solutions of sugar, etc., rather than their actual specific gravities. They are then known as *acidometers*, *alcoholometers*, *lactometers*, *saccharometers*, etc., according to the use to which they are put.

997. Nicholson's Hydrometer.—This instrument is shown in Fig. 172. It consists of a hollow cylinder carrying at its lower end a basket d, heavy enough to keep the apparatus upright when placed in water. At the top of the cylinder is a vertical rod, to which is attached a shallow pan a for holding weights, etc. The cylinder must be of such size that the apparatus may be so much lighter than water that a certain weight W must be placed in the pan to sink it to a given point c on the rod. The body whose specific gravity it is desired to find must weigh less than W. It is placed in the pan a, and enough weight w is added to sink the point c to the water level. It is evident that the weight of the given body is $W - w$. The body is now removed from the pan a and placed in the basket d, an additional

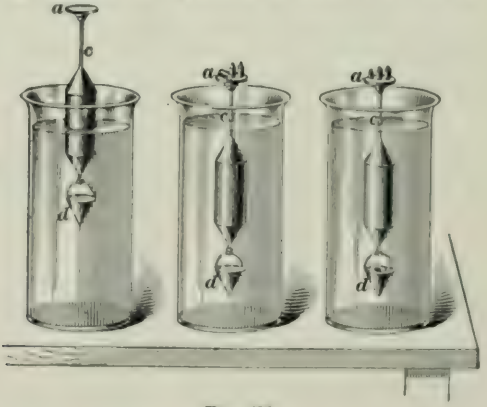

FIG. 172.

weight being added to sink the point c to the water level. Represent the weight now in the pan by W'. The difference $W' - w$ is the weight of a volume of water equal to the volume of the body. Hence,

$$\text{Sp. Gr.} = \frac{W - w}{W' - w}. \qquad \textbf{(30.)}$$

EXAMPLE.—The weight necessary to sink the hydrometer to the point c is 16 ounces; the weight necessary when the body is in the pan a is 7.3 ounces, and when the body is in the basket d, 10 ounces; what is the specific gravity of the body?

SOLUTION.—$\text{Sp. Gr.} = \frac{W - w}{W' - w} = \frac{16 - 7.3}{10 - 7.3} = \frac{8.7}{2.7} = 3.222.$ Ans.

998. Archimedes' principle gives a very easy and accurate method of finding the volume of an irregularly shaped body. Thus, subtract its weight in water from it's weight in air, and divide by .03617; the result will be in cubic inches, or divide by 62.5 and the result will be in cubic feet.

If the specific gravity of the body is known, its cubical contents can be found by dividing its weight by its specific gravity, and then dividing again by either .03617 or 62.5.

EXAMPLE.—A certain body has a specific gravity of 4.38 and weighs 76 pounds; how many cubic inches are there in the body?

SOLUTION.— $\frac{76}{4.38 \times .03617} = 479.72$ cu. in. Ans.

999. Since the weight of a cubic foot of water varies for different temperatures, and with the amount of impurities it contains, it is necessary to have some standard when getting the specific gravity. This standard is pure distilled water at its maximum density, which occurs at a temperature of 39.2° Fahrenheit. At this temperature water weighs 62.425 pounds per cubic foot; but for ordinary calculations it is customary to take it as weighing 1,000 ounces, or 62.5 pounds, per cubic foot.

CAPILLARY ATTRACTION.

1000. If a clean glass rod be placed vertically in water, the water will be drawn up around the tube. (See a, Fig. 173.) If the same rod be placed in mercury, the liquid will be depressed instead of raised. On examination, it will be found that water *wets the glass, while mercury does not.* If the rod be greased and placed in water, the surface of the water will be depressed about the rod. If a clean lead or zinc strip be placed in mercury, the surface of the mercury will be *raised* about the strip.

In the last two cases, the greased rod came out *dry*, no water adhering to it, while the mercury did adhere to the lead or zinc strip, which came out *wet.*

In general, *all liquids that will wet the solids placed in them will be lifted, while those that do not will be pushed down.*

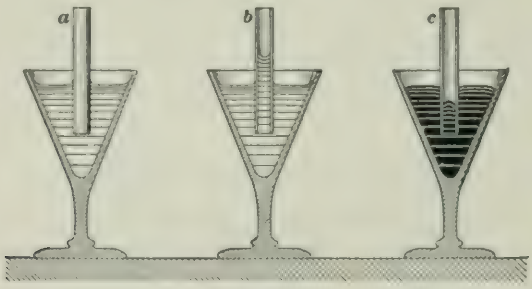

FIG. 173.

These phenomena are called **capillary attraction,** because they are best shown in very fine or hair-like tubes. In Fig. 173, *b* is a glass tube in water, and *c* is a glass tube in mercury. The surface of the water in the tube *b* is *concave*, while the surface of the mercury in the tube *c*, is *convex*.

1001. The amount which a liquid will ascend or be depressed, varies inversely as the diameter of the tube. Thus, water will rise twice as far in a tube $\frac{1}{32}$ of an inch in diameter as in one $\frac{1}{16}$ of an inch in diameter.

1002. There are many illustrations of capillary action. It is capillary attraction that aids the ascent of sap in the pores of plants. It lifts the oil between the fibers of a lamp wick to the place of combustion. It enables cloth and sponges to take up moisture. It causes blotting paper to absorb ink ; but when paper is *sized*, its pores are filled, and the ink dries by evaporation. It is capable of exerting great force, as is shown in the effects produced by the swelling of wood and other substances when kept wet. Dry wooden wedges driven into a groove cut around a cylinder of stone, and occasionally wet, will cause it to break asunder. As the pores between the fibers of a rope run around it in spiral lines, the swelling produced by wetting a tight rope will cause the fibers to shorten, and to contract the rope with immense force.

EXAMPLES FOR PRACTICE.

1. If a certain quantity of red lead weighs 5 pounds in air, and 4.441 pounds in water, what is its specific gravity? Ans. 8.94 +.

2. A piece of iron weighing 1 pound in air and .861 pound in water is attached to a piece of wood weighing 1 pound in air. When both bodies are placed in water they weigh .2 pound. What is (*a*) the specific gravity of the iron? (*b*) Of the wood? Ans. $\begin{cases} (a)\ 7.194. \\ (b)\ .602. \end{cases}$

3. An empty flask weighed 13 oz.; when filled with water, it weighed 22 oz., and when filled with sulphuric acid, 29.56 oz. What was the specific gravity of the acid? Ans. 1.84.

4. How many cubic feet of brick, having a specific gravity of 1.9, are required to weigh 260 pounds? Ans. 2.19 cu. ft., nearly.

HYDROKINETICS.

THE MEAN VELOCITY.

1003. Hydrokinetics, also called **hydrodynamics** and **hydraulics,** treats of water in motion. The velocity is not the same at all points of the flow, unless all cross-sections of the pipe, or canal, are equal. That *velocity*, which being *multiplied by the area of the cross-section of the stream* will equal the total quantity *discharged*, is called the **mean velocity.**

Let Q = the quantity in cubic feet which passes any section in 1 second;

A = the area of the section in square feet;

v_m = the mean velocity in feet per second.

Then,
$$Q = A\,v_m, \qquad \textbf{(31.)}$$
and
$$v_m = \frac{Q}{A}. \qquad \textbf{(32.)}$$

EXAMPLE.—The area of a certain cross-section of a stream is 27.9 square inches; the mean velocity of the water through this section is 51 feet per second; what is the quantity discharged, in cubic feet?

SOLUTION.—$Q = A v_m = \frac{27.9}{144} \times 51 = 9.9$ cu. ft. per sec. Ans.

NOTE.—1 square foot = 144 square inches.

EXAMPLE.—In the last example, what would the mean velocity have been had the area of the cross-section been 36 square inches, to discharge the same quantity?

SOLUTION. $v_m = \frac{Q}{A} = \frac{9.9}{\frac{36}{144}} = \frac{9.9 \times 144}{36} = 39.6$ ft. per sec. Ans.

VELOCITY OF EFFLUX.

1004. If a small aperture is made in a vessel containing water, the velocity with which the water issues from the vessel is the same as if it had fallen from the level of the surface to the level of the aperture, all resistances being neglected. This velocity is called the **velocity of efflux.**

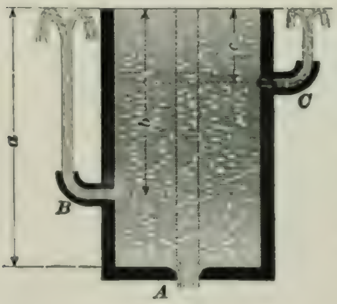

FIG. 174.

1005. The *vertical height* of the level surface of the water above the horizontal line through the center of the aperture, is called the **head.** In Fig. 174, a is the head for the aperture A; b is the head for the aperture B, and c is the head for the aperture C.

Let v = the velocity of efflux in feet per second;

h = the head in feet at the aperture considered;

W = the weight of the water in pounds flowing through this aperture per second.

Were it not for the resistance of the air, friction, and the effect of the falling particles, the issuing water would spout to the level of the water in the vessel; that is, to a height equal to its head. The kinetic energy of the issuing water will be expressed by $\frac{W v^2}{2 g}$. The work it can do will be $W h$.

Since the kinetic energy equals the work, $\frac{Wv^2}{2g} = Wh$, or $v = \sqrt{2gh}$; that is, *the velocity of efflux is the same as if the same weight of water had fallen through a height equal to its head.*

EXAMPLE.—A small orifice is made in a pipe 50 feet below the water level; what is the velocity of the issuing water?

SOLUTION.—$v = \sqrt{2gh} = \sqrt{2 \times 32.16 \times 50} = 56.7$ ft. per sec. Ans.

1006. From the above formula, as in the laws of falling bodies, $h = \frac{v^2}{2g}$. Here, h is called the *head due to the velocity v.* Consequently, if the velocity of efflux is known, the head can be found.

EXAMPLE.—An issuing jet of water has a velocity of 60 feet per second; what is the head that causes it to flow with this velocity?

SOLUTION.—$h = \frac{v^2}{2g} = \frac{60^2}{2 \times 32.16} = 55.97$ feet. Ans.

1007. Suppose that a tall vessel be fitted with a piston, and that it has an orifice near the bottom fitted with a stopcock. If an additional pressure be applied to the piston, it is evident that the velocity of efflux will be increased.

Let p be the pressure per unit of area at the level of the water, due to the additional pressure on the piston. If the unit of area is one square inch, the height of a column of water that will cause a pressure equal to p is $\frac{p}{.434}$ feet.

If the unit of area is 1 square foot, the height of a column of water is $\frac{p}{62.5}$ feet. Denote this height corresponding to the additional pressure by h_1. The original head of the water in the vessel is h; hence, $h_1 + h =$ the total head, and the velocity of efflux, when the cock is opened, will be

$$v = \sqrt{2g(h_1 + h)}. \qquad \textbf{(33.)}$$

1008. The total head, $h_1 + h$, is called the **equivalent head,** and must, in all cases, be reduced to feet before substituting in the formula.

EXAMPLE.—The area of a piston fitting a vessel filled with water is 27.36 square inches. The total pressure on the piston is 80 pounds; the weight of the piston is 25 pounds, and the head of the water at the level of the orifice is 6 feet 10 inches; what is the velocity of the efflux, assuming that there are no resistances?

SOLUTION.—80 + 25 = 105 lb. = the total pressure on the upper surface of the liquid. $\frac{105}{27.36} = 3.8377$ lb. per sq. in.

$\frac{3.8377}{.434} = 8.8426$ feet = head in feet due to the pressure of 105 pounds $= h_1$. 6 ft. 10 in. = 6.8333 ft. $= h$.

$v = \sqrt{2g(h_1 + h)} = \sqrt{2g(8.8426 + 6.8333)} = \sqrt{2 \times 32.16 \times 15.6759} =$ 31.75 ft. per sec. Ans.

1009. When water issues from the side of a vessel, it will be subjected to the same laws that govern projectiles. The range may be calculated in the same manner by taking the *velocity of efflux* as the *initial velocity* of the projectile.

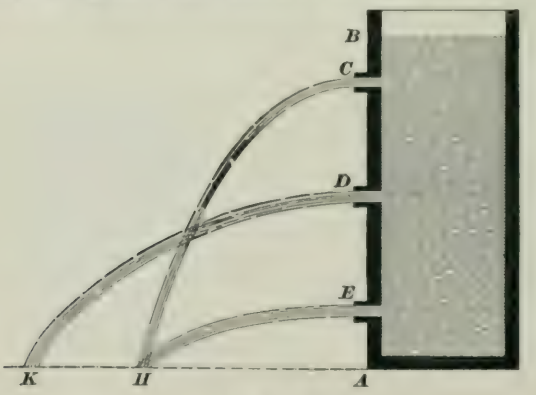

FIG. 175.

The range may be calculated more conveniently by the following formula:

$$R = \sqrt{4hy}, \qquad (34.)$$

in which R is the range, h is the head, or equivalent head, at the level of the orifice, and y is the vertical height of the orifice above the point where the water strikes, all dimensions being in feet. In Fig. 175, the upper surface of the water is free. For the orifice E, $h = BE$ and $y = EA$; for the orifice C, $h = BC$ and $y = CA$.

The greatest range is obtained when $h = y$; that is, when the orifice is half way between the upper surface of the water, and the level of the place where the stream strikes. If two orifices are situated equally distant from the middle orifice, giving the greatest range, as C and E in Fig. 175, the ranges of the water issuing from them will be equal.

EXAMPLE.—The vertical height above the ground of the surface of the water in a vessel is 12 feet. If an orifice is situated 4 feet from the upper surface, what is the range? What is the greatest range? Where is the other point of equal range?

SOLUTION.—$R = \sqrt{4hy} = \sqrt{4 \times 4 \times 8} = 11.31$ ft., nearly. Ans.

Greatest range $= \sqrt{4 \times 6 \times 6} = 12$ feet. Ans.

$6 - 4 = 2$; hence, the point of equal range is $6 + 2 = 8$ feet below the surface of the water.

PROOF.—Range $= \sqrt{4hy} = \sqrt{4 \times 8 \times 4} = 11.31$ ft., as before.

1010. When the water flows through an orifice of large size in the bottom of the vessel, compared with the area of the base, a different rule must be used from that given above. In Fig. 176, suppose that the area of the orifice in the bottom of the vessel is a, and that the area of the bottom is A; then, the velocity v is expressed by the formula

$$v = \sqrt{\frac{2gh}{1 - \frac{a^2}{A^2}}}. \qquad \textbf{(35.)}$$

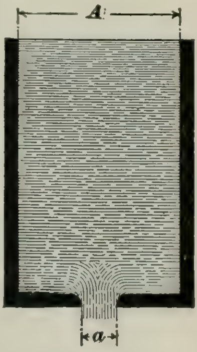

FIG. 176.

That is, *the velocity of efflux from the bottom of a vessel in feet per second, equals the square root of 2 g times the head, divided by 1 minus the ratio of the square of the area of the orifice to the square of the area of the bottom.*

1011. If the area of the cross-section of the base is more than 20 times the area of the orifice, use the formula $v = \sqrt{2gh}$. That is,

Rule.—*The velocity of efflux from a small orifice, when the cross-sectional area of the vessel is equal to, or more than, twenty times the area of the orifice, equals the square root of 2 g times the head.*

1012. Example.—A vessel has a rectangular cross-section, 11 × 14 inches, and the upper surface of the water is 14 feet above the bottom. If an orifice 4 inches square is made in the bottom of the vessel, what is the velocity of efflux?

Solution.—Area of the cross-section is 14 × 11 = 154 sq. in. Area of orifice is 4 × 4 = 16 sq. in. Since 154 ÷ 16 = 9⅝, the area of the base is less than 20 times the area of the orifice; hence, using formula **35**,

$$v = \sqrt{\frac{2\,g\,h}{1 - \frac{a^2}{A^2}}} = \sqrt{\frac{2 \times 32.16 \times 14}{1 - \frac{16^2}{154^2}}} = 30.17 \text{ ft. per sec.} \quad \text{Ans.}$$

Example.—If the orifice had been 2 inches square in the above example, what would the velocity of efflux have been? Also, if it had been 8 inches square?

Solution.—2 inches × 2 inches = 4 sq. in., or the area of the orifice. Since 154 ÷ 4 = 38½, the area of the base is greater than 20 times the area of the orifice; hence, using rule, Art. **1011**,

$$v = \sqrt{2\,g\,h} = \sqrt{2 \times 32.16 \times 14} = 30.008 \text{ feet per second.} \quad \text{Ans.}$$

8 inches × 8 inches = 64 square inches, or the area of the orifice in the second case.

$$v = \sqrt{\frac{2\,g\,h}{1 - \frac{a^2}{A^2}}} = \sqrt{\frac{2 \times 32.16 \times 14}{1 - \frac{64^2}{154^2}}} = 32.99 \text{ feet per second; practical-}$$

ly, 33 feet per second. Ans.

THE CONTRACTED VEIN.

1013. When water issues from an orifice in a thin plate (see Fig. 177), or from a square-edged* orifice (see Fig. 178), the stream is contracted at a short distance from the orifice,

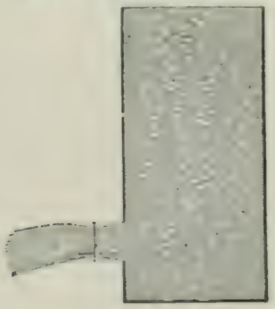

Fig. 177.

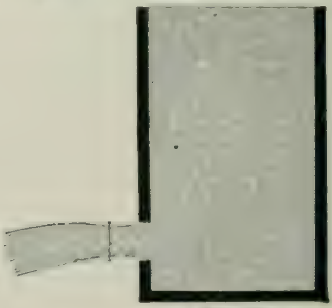

Fig. 178.

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*Note.—By square edged orifice is meant one whose *edges* are neither rounding nor tapering. The orifice itself may have any shape.

and expands again to the full size of the orifice. The point at which the contraction is greatest is at a distance from the orifice equal to about one-half of the diameter of the orifice. In consequence of this contraction, the *velocity of efflux* is slightly reduced from the theoretical value, and the *quantity discharged* is greatly reduced. This contraction is called the **contracted vein,** a name given to it by Sir Isaac Newton.

For ordinary purposes, the actual *velocity of efflux* may be taken as 98% of the theoretical values, calculated by the preceding rules.

The actual velocity of efflux from a small square-edged orifice is expressed by the formula

$$v = .98\sqrt{2\,g\,h}. \qquad \textbf{(36.)}$$

Example.—What is the actual velocity of discharge from a small square-edged orifice in the side of a vessel, if the head is 20 feet?

Solution.—Applying formula **36,**

$v = .98\sqrt{2\,g\,h} = .98\sqrt{2 \times 32.16 \times 20} = 35.15$ ft. per sec. Ans.

1014. The diameter of the contracted vein at its smallest section is about .8 of the diameter of the orifice, and its area is about $.8 \times .8 = .64$ of the area of the orifice. In Art. **1003,** it was stated that the quantity discharged in cubic feet per second was equal to the area of the section, multiplied by the mean velocity, or $Q = A\,v_m$. Now, if the theoretical velocity v be substituted for the mean velocity v_m, the formula becomes $Q = Av$, the theoretical value; *the actual value is the area of the contracted vein multiplied by the actual velocity of efflux*, or $Q = .64\,A \times .98\,v = .627\,A\,v$; that is, the actual discharge is about .627 of the theoretical discharge. This number, .627, is called the **coefficient of efflux.** The theoretical velocity is the velocity which a body would acquire by falling in a vacuum through a height equal to the head.

1015. The coefficient of efflux varies somewhat according to the head, and the size and shape of the orifice; but for square-edged orifices, or for orifices in thin plates, its average value may be taken as .615. Hence, *the actual quantity discharged is .615 times the theoretical amount*, or

$$Q = .615\,A\,v. \qquad \textbf{(37.)}$$

EXAMPLE.—The theoretical discharge from a certain vessel is 12.4 cubic feet per minute: what is the amount actually discharged per second?

SOLUTION.—Multiplying the theoretical discharge by the average coefficient of efflux, $12.4 \times .615 = 7.626$ cu. ft. per min.; $\frac{7.626}{60} = .1271$ cu. ft. per sec. Ans.

1016. If the water discharges through a short tube whose length is from $1\frac{1}{2}$ to 3 times the diameter of the orifice (see Fig. 179) the discharge is increased. From a large number of experiments made by different persons, the coefficient of efflux for a short tube may be taken as .815; that is, the actual discharge may be taken as .815 times the theoretical discharge through an orifice of the same size. If the inside edges of the tube are well rounded, and the tube is conical, as shown in Fig. 180, there will be no contraction, and the coefficient of efflux may be taken as .97, that is, the actual discharge through a tube of this form is .97 times the theoretical discharge through an orifice whose area is the same as the area of the end of the tube.

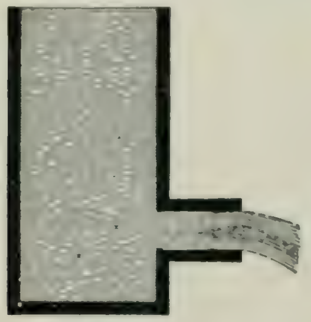


FIG. 179.

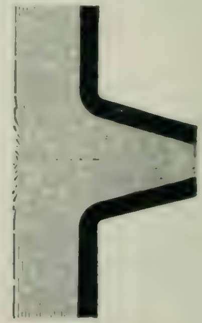


FIG. 180.

If in a compound mouthpiece or tube, such as shown in Fig. 181, *a b*, the narrowest part, be taken as the diameter of the orifice, the coefficient of efflux may be taken as 1.5526; that is, the actual discharge through a compound mouthpiece of this shape is 1.5526 times the theoretical discharge through an orifice whose area is the same as the area of the smallest section of the mouthpiece.

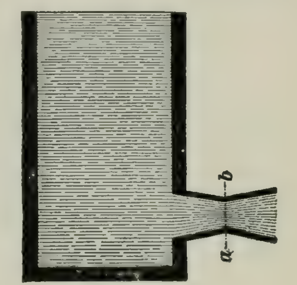


FIG. 181.

1017. When the upper surface of the water remains at the same height above the orifice, there is said to be a *constant head.* The velocity of efflux varies for different points in the orifice; it is greater at the bottom of the orifice than at the top, since the head is greater at the bottom than at the top. A mean velocity may be obtained by *dividing the quantity of water discharged in feet per second by the area of the orifice*, or $v_m = \frac{Q}{A}$.

The theoretical head necessary to give this velocity v_m is $h = \frac{v_m^2}{2g}$. Since the actual velocity is .98 of the theoretical velocity, the actual head is $h = \left(\frac{v_m}{.98}\right)^2 \div 2g = 1.04\frac{v^2}{2g}$.

That is, the actual head must be 1.04 times the theoretical head due to the mean velocity.

Let Q = theoretical number of cubic feet discharged per second;

v_m = theoretical mean velocity through orifice in feet per second $= Q \div A$;

A = area of orifice in square feet;

h = theoretical head necessary to give a mean velocity v_m;

Q_a = actual quantity discharged in cubic feet per second.

Then, for an orifice in a thin plate, or a square-edged orifice (the hole itself may be of any shape, triangular, square, circular, etc., but the edges must not be rounded), the actual quantity discharged is

$$Q_a = .615\ Q = .615\ A\ v_m = .615\ A\sqrt{2gh}. \qquad \textbf{(38.)}$$

That is, *the actual quantity discharged through a square-edged orifice or through a thin plate is .615 times the theoretical discharge, and equals .615 multiplied by the area of the orifice, multiplied by the mean velocity, or equals .615 multiplied by the area of the orifice, multiplied by the square root of 2 g times the theoretical head corresponding to the theoretical mean velocity.*

For a discharge through a short tube, as shown in Fig. 179,

$$Q_a = .815\ Q = .815\ A\ v_m = .815\ A\sqrt{2gh}. \qquad \textbf{(39.)}$$

For a discharge through a mouthpiece, as shown in Fig. 180,

$$Q_a = .97\ Q = .97\ A\ v_m = .97\ A\sqrt{2gh}. \qquad \textbf{(40.)}$$

For a discharge through the compound mouthpiece, as shown in Fig. 181, the area of the orifice being taken as the area of the smallest section,

$$Q_a = 1.5526\ Q = 1.5526\ A\ v_m = 1.5526\ A\sqrt{2gh}. \qquad \textbf{(41.)}$$

In these four formulas, it is taken for granted that there is a constant head.

WEIRS.

1018. The **weir** is a device universally used for measuring the discharge of water. It is a rectangular orifice through which the water flows.

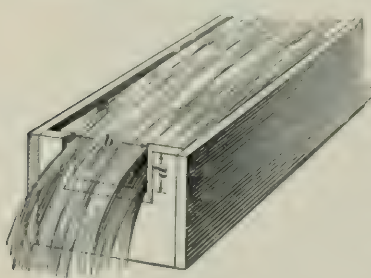


FIG. 182.

1019. In Fig. 182 is represented a weir in which the top of the weir (orifice) is level with the upper surface of the water flowing through it. By means of higher mathematics it has been found that the *theoretical mean velocity* v_m is equal to $\frac{2}{3}\sqrt{2gh}$.

1020. If d = the depth of the opening in feet, and b its breadth in feet, the area of the opening is $A = d \times b$, and the theoretical discharge is $Q = d \times b \times v_m = d\,b \times \frac{2}{3}\sqrt{2gh} = \frac{2}{3}\,b\,d\sqrt{2gd}$, the head for this case being taken as d.

The actual discharge is

$$Q_a = .615\,Q = .615 \times \tfrac{2}{3}\,b\,d\sqrt{2gd} = .41\,b\sqrt{2gd^3}. \qquad \textbf{(42.)}$$

That is, *the actual discharge through a weir in cubic feet per second whose top is on a level with the upper surface of the water, is equal to .41 multiplied by the breadth of the weir, multiplied by the square root of 2 g times the cube of the depth of the weir. All dimensions are to be taken in feet.*

Example.—A weir like that represented in Fig. 182 has a depth $d = 18$ inches and a breadth $b = 30$ inches; what is the actual discharge per minute in cubic feet?

Solution.—Applying formula **42**,

$Q_a = .41\,b\sqrt{2gd^3} = .41 \times \frac{30}{12} \times \sqrt{2 \times 32.16 \times (\frac{18}{12})^3} = 15.1$ cu. ft. per sec., or. $15.1 \times 60 = 906$ cu. ft. per min. Ans.

To obtain the mean velocity v_m, divide the actual discharge (which may be calculated by the last rule) by the area of the weir, or

$$v_m = \frac{Q_a}{A} = \frac{Q_a}{b\,d}. \qquad \textbf{(43.)}$$

Example.—What is the mean velocity in feet per second of the water in the last example?

Solution.—$v_m = \frac{Q_a}{b\,d} = \frac{15.1}{2\frac{1}{2} \times 1\frac{1}{2}} = \frac{15.1}{3.75} = 4.027$ ft. per sec. Ans.

1021. It should be kept in mind that a weir is but a rectangular opening. It is a special name given to a rectangular orifice. Some writers use the term **rectangular notch,** instead of weir.

1022. In Fig. 183 is represented a weir whose top is below the level of the upper surface of the water. If h_1 is the depth in feet of the top of the weir below the surface of the water, and h is the depth in feet of the bottom of the weir below the

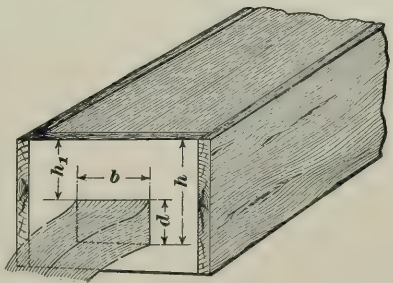


FIG. 183.

surface of the water, the actual discharge Q_a in cubic feet per second is

$$Q_a = .41\, b\sqrt{2\,g}\,(\sqrt{h^3} - \sqrt{h_1^3}). \qquad \textbf{(44.)}$$

That is, *the actual discharge through a weir whose top is h_1 feet, and whose bottom is h feet below the surface of the water, equals .41 times the breadth of the weir multiplied by the square root of 2 g times the difference between the square roots of the cubes of the depth of the bottom of the weir and the depth of the top-of the weir.*

Example.—A weir like that shown in Fig. 183 has a depth $d = 2$ feet and a breath $b = 3$ feet. The depth of the top below the surface of the water is 5 feet; what is the discharge in cubic feet per minute?

Solution.—Using formula **44**, $h_1 = 5$, $h = 5 + 2 = 7$, and $Q_a = .41\, b\sqrt{2\,g}\,(\sqrt{h^3} - \sqrt{h_1^3}) = .41 \times 3\sqrt{2 \times 32.16}\,(\sqrt{7^3} - \sqrt{5^3}) = 72.41$ cu. ft. per sec. $= 72.41 \times 60 = 4{,}344.6$ cu. ft. per min. Ans.

Example.—What is the mean velocity in the last example?

Solution.—Using formula **43**,

$$v_m = \frac{Q_a}{b\,d} = \frac{72.41}{2 \times 3} = 12.07 \text{ ft. per sec.} \quad \text{Ans.}$$

FLOW OF WATER IN PIPES.

1023. When water flows from one reservoir to another through a pipe, the velocity of efflux is considerably less than the theoretical velocity due to the head. This loss is due to several causes, but is principally caused by the friction of the water against the inside surface of the pipe. *This friction varies directly as the length of the pipes, and inversely as the diameter;* that is, the friction in a pipe 200 feet long is twice as much as in a pipe 100 feet long, and the friction in a pipe 4 inches in diameter is only half as much as in a pipe 2 inches in diameter, the velocity remaining the same in both cases. *The friction also varies nearly as the square of the velocity.*

THE MEAN VELOCITY OF DISCHARGE.

1024. For straight cylindrical pipes of uniform diameter,

Let v_m = mean velocity of discharge in feet per second;
h = total head in feet = the vertical distance between the level of the water in the reservoir and the point of discharge;
l = length of pipe in feet;
d = diameter of pipe in inches;
f = coefficient of friction.

Then, $$v_m = 2.315\sqrt{\frac{h\,d}{f\,l + \frac{1}{8}d}}. \qquad \textbf{(45.)}$$

That is, *the mean velocity of discharge equals 2.315 times the square root of the head in feet, multiplied by the diameter in inches, divided by the length in feet, multiplied by the coefficient of friction, plus one-eighth of the diameter of the pipe in inches.*

1025. The head is always taken as the vertical distance between an imaginary line through the point of discharge and the level of the water at the source, or point from which it is taken, and is always measured in feet. It matters not how long the pipe is, whether vertical or inclined, whether straight or curved, nor whether any part of the pipe goes below the level of the point of discharge or not, the head is always measured as stated above.

1026. EXAMPLE.—What is the mean velocity of efflux from a 6-inch pipe, 5,780 feet long, if the head is 170 feet? Take $f = .021$.

SOLUTION.— $v_m = 2.315\sqrt{\frac{h\,d}{f\,l + \frac{1}{8}d}} = 2.315\sqrt{\frac{170 \times 6}{.021 \times 5{,}780 + \frac{1}{8} \times 6}}$ = 6.69 ft. per sec. Ans.

1027. When the pipe is very long, compared with the diameter, as in the above example, the following formula may be used:

$$v_m = 2.315\sqrt{\frac{h\,d}{f\,l}}, \qquad \textbf{(46.)}$$

in which the letters have the same meaning as in the preceding formula.

EXAMPLE.—In the preceding example, calculate the value of v_m by using formula **46.**

SOLUTION.—Substituting the values given,

$$v_m = 2.315\sqrt{\frac{h\,d}{f\,l}} = 2.315\sqrt{\frac{170 \times 6}{.021 \times 5{,}780}} = 6.71 \text{ ft. per sec.} \quad \text{Ans.}$$

1028. Formula **46** may be used when the length of the pipe exceeds 10,000 times its diameter, since the difference between the values calculated by formulas **45** and **46** then becomes quite small.

THE ACTUAL HEAD.

1029. The actual head necessary to produce a certain velocity v_m may be calculated by the formula

$$h = \frac{f\,l\,v_m^2}{5.36\,d} + .0233\,v_m^2. \qquad \textbf{(47.)}$$

That is, *the total head in feet necessary to produce a velocity of efflux v_m, in a straight cylindrical pipe, is equal to the coefficient of friction multiplied by the length of the pipe in feet, multiplied by the square of the mean velocity of efflux in feet per second, divided by 5.36 times the diameter of the pipe in inches, plus .0233 times the square of the mean velocity.*

EXAMPLE.—A 7-inch pipe 6,000 feet long is required to deliver water with a velocity of 7 feet per second; what must be the necessary head? Assume $f = .026$.

SOLUTION.—Substituting the given values in formula **47,**

$$h = \frac{f\,l\,v_m^2}{5.36\,d} + .0233\,v_m^2 = \frac{.026 \times 6{,}000 \times 7^2}{5.36 \times 7} + .0233 \times 7^2 = 204.87 \text{ ft.,}$$
nearly. Ans.

THE QUANTITY DISCHARGED FROM PIPES.

1030. The formulas just given are made use of in ascertaining the quantity of water that will be discharged from a pipe in a given time, with a given head. This is readily found by means of formula **31,** $Q = A\,v_m$.

Since $A = .7854\,d^2$ = area of the inside of the pipe, the quantity discharged can be readily calculated as soon as v_m

is known. This method gives the discharge in cubic feet per second, when the diameter d is taken in feet.

One cubic foot contains 7.48 gallons; hence, when d is taken in feet, $Q = .7854\,d^2 v_m \times 7.48$ gallons. If d is taken in inches, $Q = \frac{.7854\,d^2}{144} v_m \times 7.48$, or

$$Q = .0408\,d^2 v_m. \qquad \textbf{(48.)}$$

That is, *the discharge in gallons per second equals .0408 times the square of the diameter of the pipe in inches, multiplied by the mean velocity of efflux in feet per second.*

EXAMPLE.—What is the discharge in gallons per minute from a 6-inch pipe, if the mean velocity of efflux is 5.6 feet per second?

SOLUTION.—$Q = .0408\,d^2 v_m = .0408 \times 36 \times 5.6 = 8.225$ gal. per sec., or, $8.225 \times 60 = 493.5$ gal. per min. Ans.

1031. If the diameter of the pipe and the discharge are known, the mean velocity can be readily found from the formula

$$v_m = \frac{24.51\,Q}{d^2}. \qquad \textbf{(49.)}$$

That is, *the mean velocity of discharge equals 24.51 times the number of gallons discharged per second, divided by the square of the diameter of the pipe in inches.*

EXAMPLE.—A 5-inch pipe is discharging 360 gallons per minute; what is the mean velocity of efflux?

SOLUTION.—$\frac{360}{60} = 6$ gal. discharged per sec. Applying formula **49**,

$$v_m = \frac{24.51 \times Q}{d^2} = \frac{24.51 \times 6}{25} = 5.882 \text{ ft. per sec. Ans.}$$

1032. If the head, the length of the pipe, and the diameter of the pipe, are given to find the discharge, use the formula

$$Q = .09445\,d^2 \sqrt{\frac{h\,d}{f\,l + \frac{1}{8}d}}. \qquad \textbf{(50.)}$$

That is, *the discharge in gallons per second equals .09445 times the square of the diameter of the pipe in inches, multiplied by the square root of the head in feet times the diameter* *of the pipe in inches, divided by the coefficient of friction times the length of the pipe in feet, plus one-eighth the diameter of the pipe in inches.*

1033. To find the value of f, calculate v_m by formula **46,** assuming that $f = .025$, and get the final value of f from the following table:

TABLE 18.

$v_m =$	0.1	0.2	0.3	0.4	0.5	0.6
$f =$	.0686	.0527	.0457	.0415	.0387	.0365
$v_m =$	0.7	0.8	0.9	1	1¼	1½
$f =$	.0349	.0336	.0325	.0315	.0297	.0284
$v_m =$	2	3	4	6	8	12
$f =$	.0265	.0243	.0230	.0214	.0205	.0193

NOTE.—The values given in Table 18 are calculated by the formula $f = .014392 + \frac{.017156}{\sqrt{v_m}}$.

EXAMPLE.—The length of a pipe is 6,270 feet, its diameter is 8 inches, and the total head at the point of discharge is 215 feet; how many gallons are discharged per minute?

SOLUTION.—Using formula **46,**

$$v_m = 2.315\sqrt{\frac{h\,d}{f\,l}} = 2.315\sqrt{\frac{215 \times 8}{.025 \times 6{,}270}} = 7.67 \text{ feet per second, nearly.}$$

For $v_m = 6, f = .0214$, and for $v_m = 8, f = .0205$. $.0214 - .0205 = .0009$ = difference for a difference in the v_m's of $8 - 6 = 2$ ft. per sec.

$7.67 - 6 = 1.67$. Hence, $2 : 1.67 :: .0009 : x$, or $x = .0007515$. Using but four decimal places $x = .0008$. Since the value of f is smaller for $v_m = 8$, than for $v_m = 6$, the value of f for $v_m = 7.67$ is $.0214 - .0008 = .0206$. Hence, using formula **50,**

$$Q = .09445\,d^2\sqrt{\frac{h\,d}{f\,l + \frac{1}{8}d}} = .09445 \times 8^2\sqrt{\frac{215 \times 8}{.0206 \times 6{,}270 + \frac{1}{8} \times 8}} = 21.98$$

gal. per sec., nearly, $= 21.98 \times 60 = 1{,}318$ gal. per min. Ans.

1034. If it is desired to find the head necessary to give a discharge of a certain number of gallons per second, through a pipe whose length and diameter are known, calculate the mean velocity of efflux by using formula **49.** Find the value of f from the table corresponding to this value of v_m; substitute these values of f and v_m in formula **47,** and calculate the head.

EXAMPLE.—A 4-inch pipe 2,000 feet long is to discharge 24,000 gallons of water per hour; what must be the head?

SOLUTION.—$\frac{24,000}{60 \times 60} = 6\frac{2}{3}$ gal. per sec. Using formula **49,**

$$v_m = \frac{24.51\,Q}{d^2} = \frac{24.51 \times 6\frac{2}{3}}{16} = 10.2 \text{ ft. per sec.}$$

In the table, $f = .0205$ for $v_m = 8$, and $.0193$ for $v_m = 12$. $.0205 - .0193 = .0012 =$ difference for a difference in the v_m's of $12 - 8 = 4$ ft. per sec. $10.2 - 8 = 2.2$. Hence, $4 : 2.2 :: .0012 : x$, or $x = .00066 = .0007$, when using but four decimal places. Then, $.0205 - .0007 = .0198 = f$ for $v_m = 10.2$.

Substituting in formula **47,**

$$h = \frac{f\,l\,v_m^2}{5.36\,d} + .0233\,v_m^2 = \frac{.0198 \times 2,000 \times 10.2^2}{5.36 \times 4} + .0233 \times 10.2^2 =$$

194.6 ft., nearly. Ans.

TO CALCULATE THE DIAMETER OF A PIPE.

1035. There is no simple accurate method known by which the diameter of a pipe may be calculated, that will give the exact discharge for any required head and length. All methods are approximations at best, but the following, which is based on formula **50,** is as accurate as any, and is better than most printed formulas.

Neglecting the fraction $\frac{1}{8}\,d$, in formula **50,** and solving for d,

$$d = 2.57\sqrt[5]{\frac{f\,l\,Q^2}{h}}.$$

Assuming that $f = .025$, for a trial value, the above equation then becomes

$$d = 1.229\sqrt[5]{\frac{l\,Q^2}{h}}. \qquad \textbf{(51.)}$$

Formula **50** may also be written

$$d = 2.57\sqrt[5]{\frac{(fl + \frac{1}{6}d)Q^2}{h}}. \qquad \textbf{(52.)}$$

1036. By aid of formulas **51** and **52**, the diameter of a pipe may be approximated as follows:

Rule.—*Find the value of d by formula* **51**; *substitute this value in formula* **49**, *and find the value of* v_m. *Then, find from the table the value of f corresponding to this value of* v_m. *Substitute the values of d and f, just found, in the right-hand member of formula* **52**, *and solve for d; the result will be the diameter of the pipe, nearly enough for all practical purposes.*

Example.—A pipe 2,000 feet long is required to discharge 24,000 gallons of water per hour. The head being 195 feet, what should be the diameter of the pipe?

Solution.—By formula **51** (remembering that Q is gallons per *second*),

$$d = 1.229\sqrt[5]{\frac{2{,}000 \times (6\frac{2}{3})^2}{195}} = 4.18'' +.$$

Substituting this value in formula **49**,

$$v_m = \frac{24.51 \times 6\frac{2}{3}}{4.18^2} = 9.352 \text{ feet per second.}$$

The value of f from the table for $v_m = 9.352$, is .0201. Substituting this value of f, and the value for d, found above, in formula **52**,

$$d = 2.57\sqrt[5]{\frac{(.0201 \times 2{,}000 + \frac{1}{6} \times 4.18)(6\frac{2}{3})^2}{195}} = 4.01'' +.$$

Hence, the diameter is 4″. Ans.

1037. If this value were again substituted in formula **49**, and the value of d again calculated by formula **52**, a still closer approximation to 4″ would be obtained; but such refinement is unnecessary, since the next larger size of pipe, as made by the manufacturers, is 4½″. It will be noticed that this example is the reverse of that in Art. **1034.** For most practical calculations, formula **51**, will give sufficiently exact results.

1038. In laying pipes, all bends and elbows should be avoided as much as possible. When they are absolutely necessary, they should be as large as the circumstances will permit, so as to change the direction gradually. Sudden changes in direction destroy the velocity very rapidly, and, consequently, reduce the discharge. A reduction or increase in the size of the pipe, owing to screwing on of branch pipes, smaller or larger than the main pipe, also reduces the velocity.

When bends are necessary, it is better to round them as shown in Fig. 184, than to have a sharp bend as shown in Fig. 185.

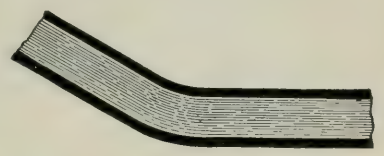


FIG. 184.

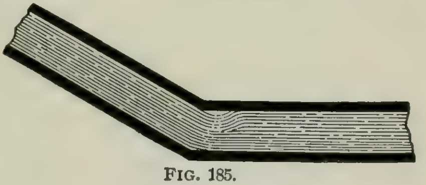


FIG. 185.

A bend at right angles, as shown in Fig. 186, is very destructive to the velocity. A rounded elbow, as shown in Fig. 187, should be used, and the radius should be made as large as possible.

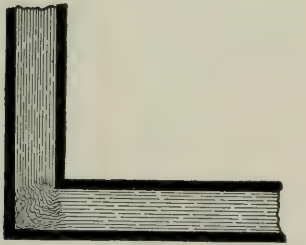


FIG. 186.

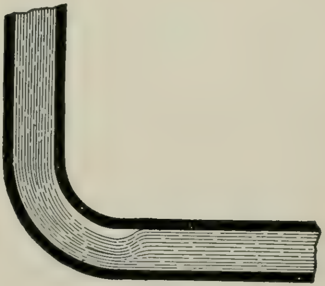


FIG. 187.

EXAMPLES FOR PRACTICE.

1. A weir whose top is level with the water has a depth of 13″ and a breadth of 20″; what is the actual discharge per minute?

Ans. 370.8 cu. ft.

2. A weir whose top is 4 feet below the surface of the water, is 32" broad and 14" deep; what is the actual discharge per minute?

Ans. 14,733.8 gal.

3. What is the mean velocity of efflux from a 5-inch pipe, 2¼ miles long, under a head of 65 feet? Ans. 2.392 ft. per sec.

4. In the last example, what is the discharge in gallons per minute?

Ans. 146.39 gal. per min.

5. What head is necessary in order that a 4-inch pipe, 1,200 feet long, may discharge 10,800 gallons of water per hour? Ans. 27.1 ft.

6. What must be the diameter of a pipe that is 8,700 feet long, in order that it may discharge 90,000 gallons of water per hour under a head of 180 feet. Give diameter to nearest inch next larger in size.

Ans. 10".

PNEUMATICS.

PROPERTIES OF AIR AND GASES.

1039. **Pneumatics** is that branch of Mechanics which treats of the properties of gases.

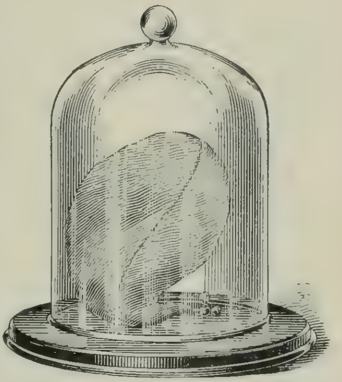


FIG. 188.

1040. The most striking feature concerning gases is that, *no matter how small the quantity may be, they will always fill the vessels which contain them.* If a bladder or football is partly filled with air and placed under a glass jar (called a **receiver**), from which the air has been exhausted, the bladder or football will immediately expand, as shown in Fig. 188. The force which a gas always exerts when confined in a limited space, is called **tension.** The word tension in this case means pressure, and is only used in this sense in reference to gases.

1041. As *water* is the most common type of fluids, so *air* is the most common type of gases. It was supposed by the ancients that air was imponderable, i. e., that it weighed nothing, and it was not until about the year 1650 that it was proven that air really had weight. A cubic inch of air, under ordinary conditions, weighs .31 grain, nearly. The ratio of the weight of air to water is about 1 : 774; that is, air is only $\frac{1}{774}$ as heavy as water. In Art. **989** it was shown that if a body was immersed in water, and weighed less than the volume of water displaced, the body would rise and extend partly out of the water. The same is true to a certain extent of air. If a vessel made of light material is filled with a gas lighter than air, so that the total weight of the vessel and gas is less than the weight of the volume of air which they displace, the vessel will rise. It is on this principle that balloons are made.

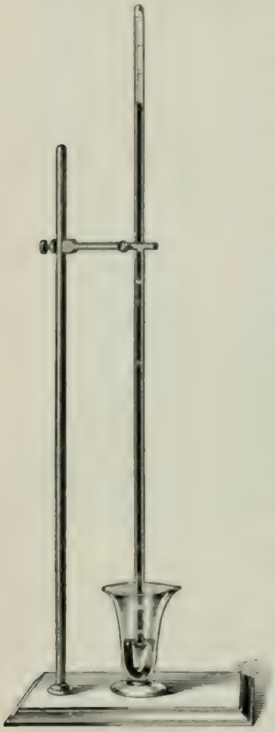


FIG. 189.

1042. Since air has weight, it is evident that the enormous quantity of air that constitutes the atmosphere must exert a considerable pressure upon the earth. This is easily proven by taking a long glass tube, closed at one end, and filling it with mercury. If the finger is placed over the open end, so as to keep the mercury from running out, and the tube is inverted and placed in a cup of mercury, as shown in Fig. 189, the mercury will fall, then rise, and after a few oscillations will come to rest at a height above the top of the mercury in the glass equal to about 30 inches. This height will always be the same under the same atmospheric conditions (allowance being made for the effects of capillary attraction). Now, if the atmosphere has weight, it must press upon the upper surface of the mercury in the glass with equal intensity upon every square unit, except upon that part of the surface occupied by the tube. According to Pascal's law (see Art. **970**), this pressure is transmitted in all directions. There being nothing in the tube, except the mercury, to counterbalance the upward pressure of the air, the mercury falls in the tube until it exerts a downward pressure on the upper surface of the mercury in the cup sufficiently great to counterbalance the upward pressure produced by the atmosphere. In order that there shall be equilibrium, the pressure of the air per unit of area on the upper surface of the mercury in the glass must equal the pressure (weight) exerted per unit of area by the mercury inside of the tube. Suppose that the area of the inside of the tube is one square inch; then, since mercury is 13.6 times as heavy as water, the weight of the mercurial column is $.03617 \times 13.6 \times 30 = 14.7574$ pounds. The actual height of the mercury is a little less than 30 inches, and the actual weight of a cubic inch of distilled water is a little less than .03617 pound. When these considerations are taken into account, the average weight of the mercurial column at the level of the sea is 14.69 pounds, or, as it is usually expressed, 14.7 pounds. Since this weight, exerted upon 1 square inch of the liquid in the glass, just produced equilibrium, it is plain that the pressure of the outside air is 14.7 pounds upon every square inch of surface.

1043. Vacuum.—The space between the upper end of the tube and the upper surface of the mercury is called a **vacuum,** meaning that it is an entirely empty space, and does not contain any substance, solid, liquid, or gaseous. If there was a gas of some kind there, no matter how small the quantity might be, it would expand, filling the space, and its tension would cause the column of mercury to fall and become shorter, according to the amount of gas or air present. The space is then called a **partial vacuum.** If the mercury fell 1 inch, so that the column was only 29 inches high, we should say, in ordinary language, that there were *29 inches of vacuum.* If it fell 8 inches, we would say that there were 22 inches of vacuum; if it fell 16 inches, we would say that there were 14 inches of vacuum, etc. Hence, when the vacuum gauge of a condensing engine shows 26 inches of vacuum, there is enough air in the condenser to produce a pressure of $\frac{30 - 26}{30} \times 14.7 = \frac{4}{30} \times 14.7 = 1.96$ pounds per square inch. In all cases where the mercurial column is used to measure a vacuum, the height of the column in inches gives the number of inches of vacuum. Thus, if the column were 5 inches high, or the vacuum gauge showed 5 inches, the vacuum would be 5 inches.

If the tube had been filled with water instead of mercury, the height of the column of water to balance the pressure of the atmosphere would have been $30 \times 13.6 = 408$ inches = 34 feet. This means that if a tube were filled with water, inverted and placed in a dish of water in a manner similar to the experiment made with the mercury, that the resulting height of the column of water would be 34 feet.

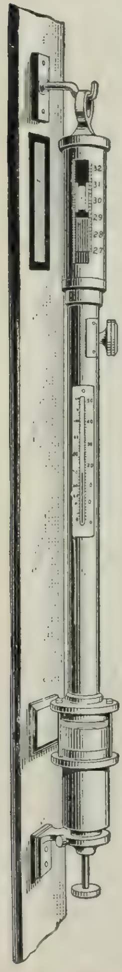


Fig. 190.

1044. The **barometer** is an instrument used for measuring the pressure of the atmosphere. There are two kinds in general use—the mercurial barometer and the aneroid barometer. The **mercurial barometer** is shown in Fig. 190. The principle is the same as in the case of the inverted tube shown in Fig. 189. The tube and cup at the bottom are protected by a brass or iron casing. At the top of the tube is a graduated scale which can be read to $\frac{1}{1000}$ of an inch, by means of a vernier. Attached to the casing is an accurate thermometer for determining the temperature of the outside air at the time the barometric observation is taken. This is necessary, since mercury expands when the temperature is increased, and contracts when the temperature falls; for this reason a standard temperature is assumed, and all barometer readings are reduced to this temperature. This standard temperature is usually taken at 32° F., at which temperature the height of the mercurial column is 30 inches. Another correction is made for the altitude of the place above sea level, and a third correction for the effects of capillary attraction.

1045. In Fig. 191 is shown a cut of an **aneroid barometer.** These instruments are made in various sizes, from the size of a large watch up to an 8 or 10 inch face. They consist of a cylindrical box of metal, with a top of thin, elastic, corrugated metal. The air is removed from the box. When the atmospheric pressure increases, the top is pressed inwards, and when it is diminished, the top is pressed outwards by its own elasticity, aided by a spring beneath. These movements of the cover are transmitted and multiplied by a combination of delicate levers which act upon an index hand and cause it to move either to the right or left over a graduated scale. These barometers are self-correcting (compensated) for variations in temperature. They are very portable, occupying but a small space, and are so delicate that they are said to show a difference in the atmospheric pressure when transferred from the table to the floor. They must be handled with care, as they are easily injured. The mercurial barometer is the standard.

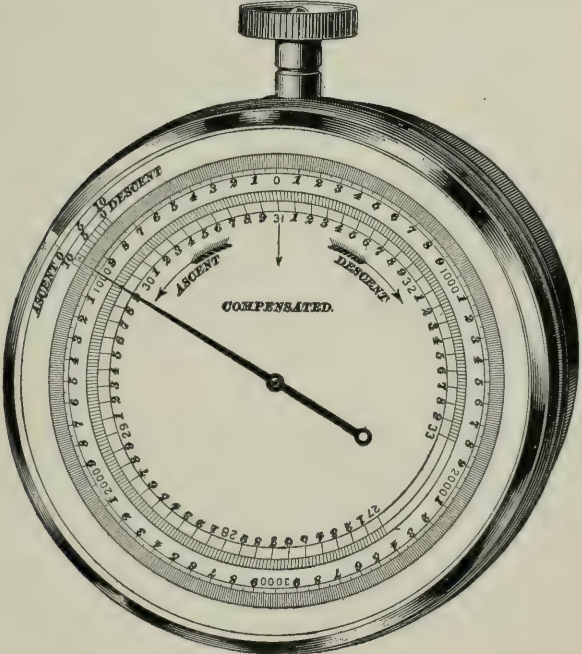


FIG. 191.

1046. With air, as with water, the lower we get, the greater the pressure, and the higher we get, the less the pressure. At the level of the sea, the height of the mercurial column is about 30 inches; at 5,000 feet above the sea, it is 24.7 inches; at 10,000 feet above the sea, it is 20.5 inches; at 15,000 feet above the sea, it is 16.9 inches; at 3 miles, it is 16.4 inches, and at 6 miles above the sea level, it is 8.9 inches.

The density also varies with the altitude; that is, a cubic foot of air at an elevation of 5,000 feet above the sea level will not weigh as much as a cubic foot at sea level. This is proved conclusively by the fact that at a height of 3½ miles the mercurial column measures but 15 inches, indicating that half the weight of the entire atmosphere is below that. It is known that the height of the earth's atmosphere is at least 50 miles; hence, the air just before reaching the limit must be in an exceedingly rarefied state. It is by means of barometers that great heights are measured. The aneroid barometer has the heights marked on the dial, so that it can be read directly. With the mercurial barometer, the heights must be calculated from the reading.

1047. The atmospheric pressure is everywhere present, and presses all objects in all directions with equal force. If a book is laid upon the table, the air presses upon it in every direction with an equal average force of 14.7 pounds per square inch. It would seem as though it would take considerable force to raise a book from the table, since, if the size of the book were 8 inches by 5 inches, the pressure upon it is $8 \times 5 \times 14.7 = 588$ pounds; but there is an equal pressure beneath the book to counteract the pressure on the top. It would now seem as though it would require a great force to open the book, since there are two pressures of 588 pounds each, acting in opposite directions, and tending to crush the book; so it would but for the fact that there is a layer of air between each leaf acting upwards and downwards with a pressure of 14.7 pounds per square inch. If two metal plates be made as perfectly smooth and flat as it is possible to get them, and the edge of one be laid upon the edge of the other, so that one may be slid upon the other, and the air thus excluded, it will take an immense force, compared with the weight of the plates, to separate them. This is because the full pressure of 14.7 pounds per square inch is then exerted upon each plate with no counteracting equal pressure between them.

If a piece of flat glass be laid upon a flat surface that has been previously moistened with water, it will require considerable force to separate them; this is because the water helps to fill up the pores in the flat surface and glass, and thus creates a partial vacuum between the glass and the surface, thereby reducing the counter pressure beneath the glass.

1048. Tension of Gases.—In Fig. 189 the space above the column of mercury was said to be a vacuum, and that if any gas or air was present, it would expand, its tension forcing the column of mercury downwards. If enough gas is admitted to cause the mercury to stand at 15 inches, the tension of the gas is evidently $\frac{14.7}{2} = 7.35$ pounds per square inch, since the pressure of the outside air of 14.7 pounds per square inch only balances 15 inches, instead of 30 inches, of mercury; that is, it balances only half as much as it would if there were no gas in the tube; therefore, the pressure (tension) of the gas in the tube is 7.35 pounds. If more gas is admitted until the top of the mercurial column is just level with the mercury in the cup, the gas in the tube has then a tension equal to the outside pressure of the atmosphere. Suppose that the bottom of the tube is fitted with a piston, and that the total length of the inside of the tube is 36 inches. If the piston be shoved upwards so that the space occupied by the gas is 18 inches long, instead of 36 inches, the temperature remaining the same as before, it will be found that the tension of the gas within the tube is 29.4 pounds per square inch. It will be noticed that the volume occupied by the gas is only half that in the tube before the piston was moved, while the pressure is twice as great, since $14.7 \times 2 = 29.4$ pounds. If the piston be shoved up, so that the space occupied by the gas is only 9 inches, instead of 18 inches, the temperature still remaining the same, the pressure will be found to be 58.8 pounds per square inch. The volume has again been reduced one-half, and the pressure increased 2 times, since $29.4 \times 2 = 58.8$ pounds. The space now occupied by the gas is 9 inches long, whereas, before the piston was moved it was 36 inches long; as the tube was assumed to be of uniform diameter throughout its length, the volume is now $\frac{9}{36} = \frac{1}{4}$ of its original volume, and its pressure is $\frac{58.8}{14.7} = 4$ times its original pressure. Moreover, if the temperature of the confined gas remains the same, the pressure and volume will always vary in a similar way. The law which states these effects is called *Mariotte's Law*, and is as follows:

1049. Mariotte's Law.—*The temperature remaining the same, the volume of a given quantity of gas varies inversely as the pressure.*

The meaning of this is: If the volume of the gas is diminished to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, etc., of its former volume, the tension will be increased 2, 3, 5, etc., times, or if the outside pressure be increased 2, 3, 5, etc., times, the volume of the gas will be diminished to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, etc., of its original volume, the temperature remaining constant. It also means that if a gas is under a certain pressure, and the pressure is diminished to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{10}$, etc., of its original pressure, that the volume of the confined gas will be increased 2, 3, 10, etc., times—its tension decreasing at the same rate.

Suppose 3 cubic feet of air to be under a pressure of 60 pounds per square inch in a cylinder fitted with a movable piston; then, the product of the volume and pressure is $3 \times 60 = 180$. Let the volume be increased to 6 cubic feet, then the pressure will be 30 pounds per square inch, and $30 \times 6 = 180$, as before. Let the volume be increased to 24 cubic feet, it is then $\frac{24}{3} = 8$ times its original volume, and the pressure is $\frac{1}{8}$ of its original pressure, or $60 \times \frac{1}{8} = 7\frac{1}{2}$ pounds, and $24 \times 7\frac{1}{2} = 180$, as in the two preceding cases. It will now be noticed that if a gas be enclosed within a confined space, and allowed to expand without losing any heat, *the product of the pressure, and the corresponding volume for one position of the piston, is the same as for any other position of the piston.* If the piston were to compress the air, the same result would be obtained.

Let p = pressure for one position of the piston;
p_1 = pressure for any other position of the piston;
v = volume corresponding to the pressure p;
v_1 = volume corresponding to the pressure p_1.

$$\text{Then, } p\,v = p_1\,v_1. \qquad \textbf{(53.)}$$

1050. Knowing the volume and the pressure for any position of the piston, and the volume for any other position, the pressure may be calculated, or, if the pressure is known for any other position, the volume may be calculated.

Example.—If 1.875 cubic feet of air be under a pressure of 72 pounds per square inch (*a*) what will be the pressure when the volume is increased to 2 cubic feet? (*b*) to 3 cubic feet? (*c*) to 9 cubic feet?

Solution.—Solving formula **53,** for p_1, the unknown pressure,

(*a*) $p_1 = \frac{p\,v}{v_1} = \frac{72 \times 1.875}{2} = 67\frac{1}{2}$ lb. per sq. in. Ans.

(*b*) $p_1 = \frac{72 \times 1.875}{3} = 45$ lb. per sq. in. Ans.

(*c*) $p_1 = \frac{72 \times 1.875}{9} = 15$ lb. per sq. in. Ans.

Example.—Ten cubic feet of air have a tension of 5.6 pounds per square inch; (*a*) what is the volume when the tension is 4 pounds? (*b*) 8 pounds? (*c*) 25 pounds? (*d*) 100 pounds?

Solution.—Solving formula **53,** for v_1,

(*a*) $v_1 = \frac{p\,v}{p_1} = \frac{5.6 \times 10}{4} = 14$ cu. ft. Ans.

(*b*) $v_1 = \frac{5.6 \times 10}{8} = 7$ cu. ft. Ans.

(*c*) $v_1 = \frac{5.6 \times 10}{25} = 2.24$ cu. ft. Ans.

(*d*) $v_1 = \frac{5.6 \times 10}{100} = .56$ cu. ft. Ans.

1051. NOTE.—There are two ways of measuring the pressure of a gas: by means of an instrument called a **manometer,** and by means of a **gauge.** The manometer generally used is practically the same as a mercurial barometer, except that the tube is much longer, so that pressures equal to several atmospheres may be measured, and is enlarged and bent into a U shape at the lower end; both lower and upper ends are open, the lower end being connected to the vessel containing the gas whose pressure it is desired to measure. The gauge is so common that no description of it will be given here. With both the manometer described above and the gauge, the pressures recorded are the amounts by which they exceed the atmospheric pressure, and are called the **gauge pressures.** To find the real pressure, called the **absolute pressure,** the atmospheric pressure must be added to the gauge pressure. In all formulas in which the pressure of a gas or steam is used, the absolute pressure must be used, unless the gauge pressure is distinctly specified as being the proper pressure to use. For convenience, all pressures given in Arts. **1039** to **1088,** inclusive, and in the questions referring to these articles, will be absolute pressures, and the word "absolute" will be omitted to avoid its constant repetition.

1052. As a necessary consequence of Mariotte's law, it may be stated that *the density of a gas varies directly as the pressure, and inversely as the volume; that is, the density increases as the pressure increases, and decreases as the volume increases.*

This is evident, since if a gas has a tension of 2 atmospheres, or 14.7 × 2 = 29.4 pounds per square inch, it will weigh twice as much as the same volume would if the tension was 1 atmosphere, or 14.7 pounds per square inch. For, let the volume be increased until it is twice as great as the original volume, the tension will then be 1 atmosphere. The total weight of the gas has not been changed, but there are now 2 cubic feet for every 1 cubic foot of the original volume, and the weight of 1 cubic foot now is only half as great as before. Thus, the density decreases as the volume increases, and as an increase of pressure causes a decrease of volume, the density increases as the pressure increases.

Let D be the density corresponding to the pressure p and volume v, and D_1 be the density corresponding to the pressure p_1 and volume v_1; then,

$$p : D = p_1 : D_1, \text{ or } p\,D_1 = p_1\,D, \qquad \textbf{(54.)}$$

and

$$v : D_1 = v_1 : D, \text{ or } v\,D = v_1\,D_1. \qquad \textbf{(55.)}$$

Since the weight is proportional to the density, the weights may be used in place of the densities in formulas

54 and **55.** Thus, let W be the weight of a cubic foot of air or other gas, whose volume is v and pressure is p; let W_1 be the weight of a cubic foot when the volume is v_1 and pressure is p_1; then,

$$p W_1 = p_1 W. \qquad \textbf{(56.)}$$

$$v W = v_1 W_1. \qquad \textbf{(57.)}$$

EXAMPLE.—The weight of 1 cubic foot of air at a temperature of 60° F., and under a pressure of one atmosphere (14.7 pounds per square inch), is .0763 pound; what would be the weight per cubic foot if the volume were compressed until the tension was 5 atmospheres, the temperature still being 60° F.?

SOLUTION.—Applying formula **56,** $p W_1 = p_1 W$, or $1 \times W_1 = 5 \times .0763$. Hence, $W_1 = .3815$ lb. per cu. ft. Ans.

EXAMPLE.—If in the last example the air had expanded until the tension was 5 pounds per square inch, what would have been its weight per cubic foot?

SOLUTION.—Applying formula **56,** $p W_1 = p_1 W$. Here $p = 14.7$, $p_1 = 5$ and $W = .0763$. Hence, $14.7 \times W_1 = 5 \times .0763$, or $W_1 = \frac{.3815}{14.7} =$ .02595 lb. per cu. ft. Ans.

EXAMPLE.—If 6.75 cubic feet of air, at a temperature of 60° F., and a pressure of one atmosphere, are compressed to 2.25 cubic feet (the temperature still remaining 60° F.), what is the weight of a cubic foot of the compressed air?

SOLUTION.—Applying formula **57,** $v W = v_1 W_1$, or $6.75 \times .0763 = 2.25 \times W_1$; hence, $W_1 = \frac{6.75 \times .0763}{2.25} = .2289$ lb. per cu. ft. Ans.

1053. In all that has been said before, it has been stated that the temperature was constant; the reason for this will now be explained. Suppose five cubic feet of air to be confined in a cylinder placed in a vacuum, so that there will be no pressure due to the atmosphere, and suppose the cylinder to be fitted with a piston weighing say 100 pounds, and having an area of 10 square inches. The tension of the gas will be $\frac{100}{10} = 10$ pounds per square inch. Suppose that the temperature of the air is 32° F., and that it is heated until the temperature is 33° F., i. e., the temperature is 1°, it will be found that the piston has risen a certain amount, and, consequently, the volume has increased, while the pressure is the same as before, or 10 pounds per square inch. If more heat is applied until the temperature of the gas is 34° F., it will be found that the piston has again risen, and the volume again increased, while the pressure still remains the same. It will be found that for every increase of temperature there will be a corresponding increase of volume. The law which expresses this change, is called *Gay-Lussac's Law*, and is as follows:

1054. Gay-Lussac's Law.—*If the pressure remains constant, every increase of temperature of 1° F. produces in a given quantity of gas an expansion of* $\frac{1}{492}$ *of its volume at 32° F.*

If the pressure remains constant, it will also be found that every decrease of temperature of 1° F., will cause a decrease of $\frac{1}{492}$ of the volume at 32° F.

Let v = original volume of gas;
v_1 = final volume of gas;
t = temperature corresponding to volume v;
t_1 = temperature corresponding to volume v_1.

Then,
$$v_1 = v\left(\frac{460 + t_1}{460 + t}\right). \qquad \textbf{(58.)}$$

That is, *the volume of gas after heating (or cooling) equals the original volume, multiplied by 460, plus the final temperature, divided by 460, plus the original temperature.*

EXAMPLE.— 5 cubic feet of air at a temperature of 45°, are heated under constant pressure up to 177°; what is its volume?

SOLUTION.—Applying formula **58**,
$$v_1 = v\left(\frac{460 + t_1}{460 + t}\right) = 5\left(\frac{460 + 177}{460 + 45}\right) = 6.307 \text{ cu. ft.} \quad \text{Ans.}$$

1055. Suppose that a certain volume of gas is confined in a vessel so that it cannot expand; in other words, suppose that the piston of the cylinder before mentioned to be fastened so that it cannot move. Let a gauge be placed on the cylinder so that the tension of the confined gas can be registered. If the gas is heated, it will be found that for every increase of temperature of 1° F., there will be a corresponding increase of $\frac{1}{492}$ of the tension. That is, the volume remaining constant, the tension increases $\frac{1}{492}$ of the original tension for every degree rise of temperature.

Let p = the original tension;
t = the corresponding temperature;
p_1 = final tension;
t_1 = final temperature.

Then, $$p_1 = p\left(\frac{460 + t_1}{460 + t}\right). \qquad \textbf{(59.)}$$

That is, *if a certain quantity of gas be heated (or cooled) from $t°$ to $t_1°$, the volume remaining constant, the resulting tension p_1 will be equal to the original tension, multiplied by 460, plus the final temperature, divided by 460, plus the original temperature.*

Example.—If a certain quantity of air is heated under constant volume from 45° to 177°, what is the resulting tension, the original tension being 14.7 pounds per square inch?

Solution.—Applying formula **59,**

$$p_1 = p\left(\frac{460 + t_1}{460 + t}\right) = 14.7\left(\frac{460 + 177}{460 + 45}\right) = 18.542 \text{ lb. per sq. in.} \quad \text{Ans.}$$

1056. According to the modern and now generally accepted theory of heat, the atoms and molecules of all bodies are in an incessant state of vibration. The vibratory movement in the liquids is faster than in the solids, and in the gases, faster than in either of the other two. Any increase of heat increases the vibrations, and a decrease of heat decreases them. From experiments and calculations based upon higher mathematics, it has been concluded that at 460° below zero, on the Fahrenheit scale, all these vibrations cease. This point is called the **absolute zero,** and all temperatures reckoned from this point are called the **absolute temperatures.** The point of absolute zero has never been reached, the lowest recorded temperature being about 393° F. below zero, but, nevertheless, it has a meaning, and is used in many formulas, being nearly always denoted by T. The ordinary temperatures are denoted by t. When the word temperature alone is used, the meaning is the same as ordinarily used, but when absolute temperature is specified, 460° F. must be added to the temperature.

The absolute temperature corresponding to 212° F. is $460 + 212 = 672°$ F. If the absolute temperature is given, the ordinary temperature may be found by subtracting 460 from the absolute temperature. Thus, the absolute temperature being 520° F., what is the temperature?

$$520° - 460° = 60°.$$

Let p = pressure in pounds per square inch;
V = volume of air in cubic feet;
T = absolute temperature;
W = weight in pounds.

Then, $$pV = .37052\ T. \qquad \textbf{(60.)}$$

That is, *the pressure in pounds per square inch, multiplied by the volume of the air in cubic feet, equals .37052 times the absolute temperature corresponding to the pressure p and volume V.*

In this formula, the weight of the air is 1 pound.

Example.—The pressure upon 9 cubic feet of air weighing 1 pound is 20 pounds per square inch; what is the temperature?

Solution.—Applying formula **60,** $p\,V = .37052\ T$, or $20 \times 9 = .37052\ T$; hence, $T = \frac{180}{.37052} = 485.8°$, nearly. $485.8° - 460 = 25.8°$, the temperature. Ans.

Example.—What is the volume of 1 pound of air whose temperature is 60° F. under a pressure of one atmosphere?

Solution.—Applying formula **60,** $p\,V = .37052\ T$. Substituting, $14.7 \times V = .37052 \times (460 + 60) = .37052 \times 520$, or $V = \frac{.37052 \times 520}{14.7} =$ 13.107 cubic feet. Ans.

1057. If the weight of the air be greater or less than 1 pound, the following formula must be used:

$$pV = .37052\ W\ T. \qquad \textbf{(61.)}$$

That is, *the pressure in pounds per square inch, multiplied by the volume in cubic feet, equals .37052 times the weight in pounds multiplied by the absolute temperature.*

Example.— 3 cubic feet of air weighing .35 pound, are under a pressure of 48 pounds per square inch; what is the temperature of the air?

Solution.—Applying formula **61,** $p\,V = .37052\,WT$. Substituting, $48 \times 3 = .37052 \times .35 \times T$, or $T = \frac{48 \times 3}{.37052 \times .35} = 1{,}110.4°$. Then, $1{,}110.4° - 460° = 650.4°$. Ans.

EXAMPLE.—What is the weight of 1 cubic foot of air at a temperature of 32°, and under a pressure of one atmosphere?

SOLUTION.—Applying formula **61**, $pV = .37052\,WT$. Substituting, $14.7 \times 1 = .37052 \times (460 + 32) \times W$, or

$$W = \frac{14.7}{.37052 \times 492} = .0806382 \text{ lb.} \quad \text{Ans.}$$

If the pressure be taken as 14.69856 pounds per square inch, and the absolute zero as 459.4°, instead of 460° below zero, and if .370514 be used, instead of .37052, more exact values, the weight of 1 cubic foot would be $\frac{14.69856}{.370514 \times 491.4} = .08073$ lb., nearly.

EXAMPLE.—What is the exact volume of 1 pound of air at a temperature of 32°, and at a pressure of one atmosphere? Take absolute zero at 459.4, and the pressure as 14.69856 pounds per square inch.

SOLUTION.— $pV = .370514\,WT$, or $14.69856 \times V = .370514 \times 1 \times (459.4 + 32)$. $V = \frac{.370514 \times 491.4}{14.69856} = 12.387$ cu. ft. Ans.

1058. If in the formula $pV = .37052\,WT$, both sides of the equation be divided by T (which, of course, does not alter the equality), there results the expression $\frac{pV}{T} = .37052\,W$. Let p_1, V_1 and T_1 represent the pressure, volume and temperature of the same weight of air in another state; then, $p_1V_1 = .37052\,WT_1$. Dividing both sides by T_1, $\frac{p_1V_1}{T_1} = .37052\,W$. Therefore, since $\frac{pV}{T}$ and $\frac{p_1V_1}{T_1}$ are equal to the same thing (i. e., $.37052\,W$), they are equal to each other, and

$$\frac{pV}{T} = \frac{p_1V_1}{T_1} \qquad \textbf{(62.)}$$

This very important formula is the complete expression of Gay-Lussac's law, and is true for any of the so-called permanent gases. It was from this formula that formulas **58** and **59** were derived. Thus, let the pressure be constant; then, $p = p_1$, and $\frac{pV}{T} = \frac{pV_1}{T_1}$, or $V_1 = \frac{VT_1}{T} = V\left(\frac{460 + t_1}{460 + t}\right)$. Similarly, letting the volume be constant, $V = V_1$, and $\frac{pV}{T}$

$= \frac{p_1 V}{T_1}$, or $p_1 = \frac{p T_1}{T} = p\left(\frac{460 + t_1}{460 + t}\right)$. So, also, by letting the temperature be constant, $T = T_1$ and $\frac{pV}{T} = \frac{p_1 V_1}{T}$, or $pV = p_1 V_1$, which is the same as formula **53.**

1059. In formulas **53, 62, 63,** and **64,** it matters not with what units the pressures and volumes are measured, except that they must be the same throughout the same example, and the pressures must always be *absolute pressures.*

EXAMPLES FOR PRACTICE.

1. A vessel contains 25 cubic feet of gas at a pressure of 18 pounds per square inch; if 125 cubic feet of gas having the same pressure are forced into the vessel, what will be the resulting pressure? Ans. 108 lb. per sq. in.

2. A pound of air has a temperature of 126°, and a pressure of 1 atmosphere; what volume does it occupy? Ans. 14.77 cu. ft.

3. The volume of steam in the cylinder of a steam engine at cut-off is 1.35 cubic foot, and the pressure is 85 pounds per square inch; if the pressure at the end of the stroke is 25 pounds per square inch, what is the new volume? Ans. 4.59 cu. ft.

4. A certain quantity of air has a volume of 26.7 cubic feet, a pressure of 19.3 pounds per square inch, and a temperature of 42°; what is the weight? Ans. 2.77 lb.

5. A receiver contains 180 cubic feet of gas at a pressure of 20 pounds per square inch; if a vessel holding 12 cubic feet, to be filled from the receiver until its pressure is 20 pounds per square inch, what will be the pressure in the receiver? Ans. $18\frac{2}{3}$ lb. per sq. in.

6. 10 cubic feet of air having a pressure of 22 pounds per square inch, and a temperature of 75°, are heated until the temperature is 300°; the volume remaining the same, what is the new pressure? Ans. 31.25 lb. per sq. in.

7. If a spherical shell whose outside diameter is 18 inches, has a part of the air within it removed until the pressure is 5 pounds per square inch, what is the total pressure due to the atmosphere tending to crush the shell? Ans. 9,873.42 lb.

THE MIXING OF GASES.

1060. If two liquids which do not act chemically upon each other are mixed together and allowed to stand, it will be found that after a time the two liquids have separated, and that the heavier has fallen to the bottom. If two equal vessels, containing gases of different densities, be put in communication with each other, they will be found to have mixed in equal proportions after a short time. If one vessel be higher than the other, and the heavier gas be in the lower vessel, the same result will occur. The greater the difference of the densities of the two gases, the quicker they will mix. It is assumed that no chemical action takes place between the two gases. When the two gases have the same temperature and pressure, the pressure of the mixture will be the same; this is evident, since the total volume has not been changed, and unless the volume or temperature changes, the pressure cannot change. This property of the mixing of gases is a very valuable one, since, if they acted like liquids, carbonic acid gas (the result of combustion), which is $2\frac{1}{2}$ times as heavy as air, would remain next to the earth, instead of dispersing into the atmosphere, the result being that no animal life could exist.

1061. Mixtures of Equal Volumes of Gases Having Unequal Pressures.—*If two gases having equal volumes and temperatures, but different pressures, be mixed in a vessel whose volume equals one of the equal volumes of the gas, the pressure of the mixture will be equal to the sum of the two pressures, provided that the temperature remains the same as before.*

Example.—Two vessels containing 3 cubic feet of gas, each at a temperature of 60°, and subjected to pressures of 40 pounds and 25 pounds per square inch, respectively, are placed in communication with each other, and all the gas is compressed into one vessel. If the temperature of the mixture is also 60°, what is the pressure?

Solution.—According to the rule just given, the pressure will be $40 + 25 = 65$ pounds per square inch. This may be proven by applications of Mariotte's law; thus, compress the gas whose pressure is 25 pounds per square inch until its pressure is 40 pounds; its volume may be found thus: $p v = p_1 v_1$, or $25 \times 3 = 40 \times v$; whence, $v = 1.875$ cubic feet. Let communication be established between the two vessels, the pressure will evidently be 40 pounds and the total volume $3 + 1.875 = 4.875$ cubic feet. If this be compressed until the volume is 3 cubic feet, the temperature remaining at 60° throughout the whole operation, the final pressure may be found by formula **53,** $p v = p_1 v_1$.

Thus, $40 \times 4.875 = p_1 \times 3$, and $p_1 = \frac{40 \times 4.875}{3} = 65$ pounds per square inch, as before.

1062. Mixture of Two Gases Having Unequal Volumes and Pressures.

Let v and p be the volume and pressure, respectively, of one of the gases.

Let v_1 and p_1 be the volume and pressure, respectively, of the other gas.

Let V and P be the volume and pressure, respectively, of the mixture. Then, if the temperature remains the same,

$$V P = v p + v_1 p_1. \qquad \textbf{(63.)}$$

That is, *if the temperature is constant, the volume after mixture, multiplied by the resulting pressure, equals the volume of one gas before mixture multiplied by its pressure, plus the volume of the other gas multiplied by its pressure.*

Example.—Two gases of the same temperature, having volumes of 7 cubic feet and 4½ cubic feet, and whose pressures are 27 pounds and 18 pounds per square inch, respectively, are mixed together in a vessel whose volume is 10 cubic feet. The temperature of the two gases and of the mixture being 60° F., what is the resulting pressure?

Solution.—Applying formula **63,** $P V = p v + p_1 v_1$, or $P \times 10 = 27 \times 7 + 4\frac{1}{2} \times 18$. Hence, $P = \frac{189 + 81}{10} = 27$ lb. per sq. in. Ans.

1063. Mixture of Two Volumes of Air Having Unequal Pressures, Volumes, and Temperatures.

If a body of air having a temperature t_1, a pressure p_1, and a volume v_1 be mixed with another volume of air having a temperature t_2, a pressure p_2, and a volume v_2, to form a volume V having a pressure P and a temperature t, then, either the new temperature t, the new volume V, or the new pressure P may be found, if the other two quantities are known, by the following formula, in which T_1, T_2, and T are the absolute temperatures corresponding to t_1, t_2, and t:

$$PV = \left[\frac{p_1 v_1}{T_1} + \frac{p_2 v_2}{T_2}\right] T. \qquad \textbf{(64.)}$$

EXAMPLE.—Five cubic feet of air having a tension of 30 pounds per square inch, and a temperature of 80° F., are required to be compressed together with 11 cubic feet of air having a tension of 21 pounds per square inch, and a temperature of 45° F., in a vessel whose cubical contents are 8 cubic feet. The new pressure is required to be 45 pounds per square inch. What is the temperature of the mixture?

SOLUTION.—Substituting in formula **64**,

$$45 \times 8 = \left[\frac{30 \times 5}{540} + \frac{21 \times 11}{505}\right] \times T, \text{ or } 360 = .7352\ T. \text{ Hence, } T = \frac{360}{.7352} = 489.66°, \text{ nearly, and } t = 29.66°. \text{ Ans.}$$

EXAMPLES FOR PRACTICE.

1. Two vessels contain air at pressures of 60 and 83 pounds per square inch. The volume of each vessel is 8.47 cubic feet. If all of the air in both vessels is removed to another vessel, and the new pressure is 100 pounds per square inch, what is the volume of the vessel, the temperature being the same throughout?

Ans. 12.11 cu. ft.

2. A vessel contains 11.83 cubic feet of air at a pressure of 33.3 pounds per square inch. It is desired to increase the pressure to 40 pounds per square inch by supplying air from a second vessel which contains 19.6 cubic feet of air at a pressure of 60 pounds per square inch. What will be the pressure in the second vessel after the pressure in the first has been raised to 40 pounds per square inch?

Ans. 55.96 lb. per sq. in.

3. If 4.8 cubic feet of air having a tension of 52 pounds per square inch and a temperature of 170° are mixed with 13 cubic feet having a tension of 78 pounds per square inch and a temperature of 265°, what must be the volume of the vessel containing the mixture in order that the tension of the mixture may be 30 pounds per square inch and the temperature 80°? Ans. 32.31 cu. ft.

PNEUMATIC MACHINES.

THE AIR PUMP.

1064. The **air pump** is an instrument for removing air from an enclosed space. A section of the principal parts is shown in Fig. 192, and the complete instrument in Fig. 193. The closed vessel R is called the **receiver,** and the space which it encloses is that from which it is desired to remove the air. The receiver is usually made of glass, and the edges are ground so as to be perfectly air-tight. When made in the form shown, it is called a **bell jar receiver.**

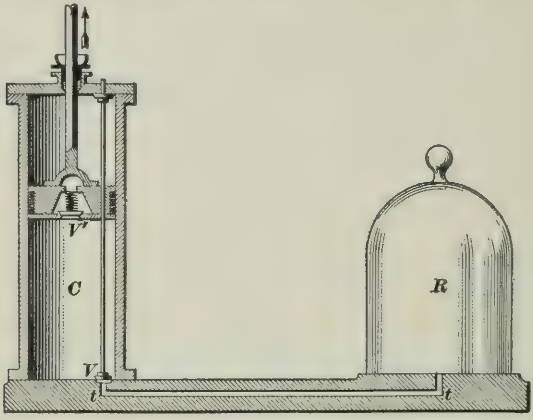


FIG. 192.

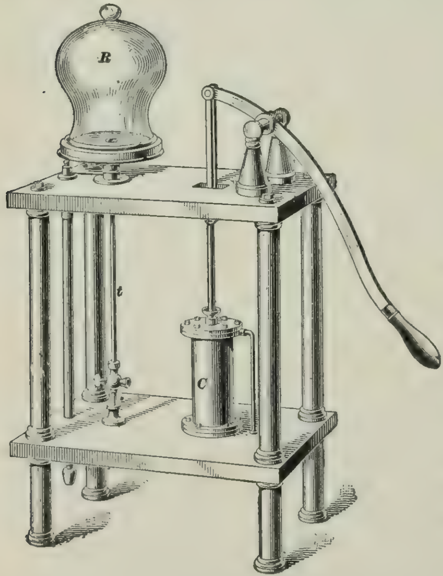


FIG. 193.

The receiver rests upon a horizontal plate in the center of which is an opening communicating with the pump cylinder *C* by means of a bent tube *t*. The pump piston fits the cylinder accurately, and has a valve *V'* opening upwards. At the junction of the tube with the cylinder is another valve *V* also opening upwards. When the piston is raised the valve *V'* closes, and, since no air can get into the cylinder from above, the piston leaves a vacuum behind it. The pressure on top of *V* being now removed, the tension of the air in the receiver *R* causes *V* to rise; the air in the receiver then expands and occupies the space displaced by the piston, the space in the tube t and in the receiver R. The piston is now pushed down, the valve V closes, the valve V' opens, and the air in C escapes. The lower valve V is sometimes supported, as shown in Fig. 192, by a metal rod passing through the piston and fitting it somewhat tightly. When the piston is raised or lowered, this rod moves with it. A button near the upper end of the rod confines its motion to within very narrow limits, the piston sliding upon the rod during the greater part of the journey.

1065. Degrees and Limits of Exhaustion.—Suppose that the volume of R and t together is four times that of C, and that there are, say, 200 grains of air in R and t, and 50 grains in C, when the piston is at the top of the cylinder. At the end of the first stroke, when the piston is again at the top, 50 grains of air in the cylinder C will have been removed, and the 200 grains in R and t will occupy the spaces R, t, and C. The ratio between the sum of the spaces R and t and the total space $R+t+C$ is $\frac{4}{5}$; hence, $200 \times \frac{4}{5} = 160$ grains = the weight of air in R and t after the first stroke. After the second stroke, the weight of the air in R and t would be $(200 \times \frac{4}{5} \times \frac{4}{5} = 200 \times (\frac{4}{5})^2 = 200 \times \frac{16}{25} = 128$ grains. At the end of the third stroke, the weight would be $[200 \times (\frac{4}{5})^2] \times \frac{4}{5} = 200 \times (\frac{4}{5})^3 = 200 \times \frac{64}{125} = 102.4$ grains. At the end of n strokes, the weight would be $200 \times (\frac{4}{5})^n$. It is evident that *it is impossible to remove all of the air that is contained in R and t by this method.* It requires an exceedingly good air pump to reduce the tension of the air in R to $\frac{1}{50}$ of an inch of mercury. When the air has become so rarefied as this, the valve V' will not lift, and, consequently, no more air can be exhausted.

1066. Sprengel's Air Pump.—In Fig. 194, $c\,d$ is a glass tube longer than 30 inches, open at both ends, and connected by means of India rubber tubing with a funnel A filled with mercury and supported by a stand. Mercury is allowed to fall into this tube at a rate regulated by a clamp at c. The lower end of the tube $c\,d$ fits in the flask B, which has a spout at the side a little higher than the lower end of $c\ d$; the upper part has a branch at x to which a receiver R can be tightly fixed. When the clamp at c is opened the first portions of the mercury which run out close the tube and prevent air from entering from below. These drops of mercury act like little pistons, carrying the air in front of them and forcing it out through the bottom of the tube. The air in R expands to fill the tube every time that a drop of mercury falls, thus creating a partial vacuum in R, which becomes more nearly complete as the process goes on. The escaping mercury falls into the dish H, from which it can be poured back into the funnel from time to time. As the exhaustion from R goes on, the mercury rises in the tube $c\ d$ until, when the exhaustion is complete, it forms a continuous column 30 inches high; in other words, it is a barometer, whose Torricellian vacuum is the receiver R. This instrument necessarily requires a great deal of time for its operation, but the results are very complete, a vacuum of $\frac{1}{43000}$ of an inch of mercury being sometimes obtained. By use of chemicals in addition to the above, a vacuum of $\frac{1}{650000}$ of an inch of mercury has been obtained.

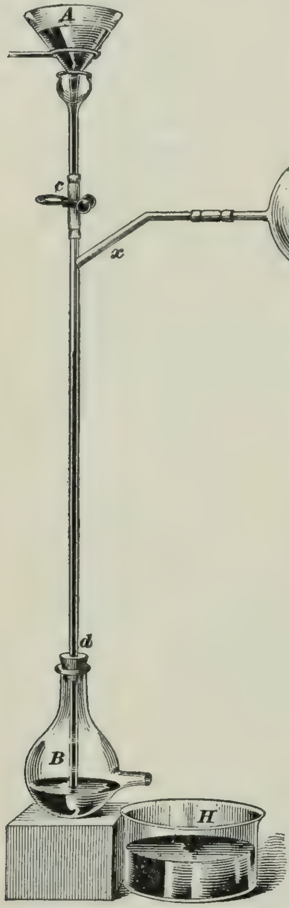


FIG. 194.

1067. NOTE.—A theoretically perfect vacuum is sometimes called a **Torricellian vacuum.**

1068. Magdeburg Hemispheres.—By means of the two hemispheres shown in Fig. 195, it can be proven that the atmosphere presses upon a body equally in all directions. They were invented by Otto Von Guericke, of Magdeburg, and are called the **Magdeburg hemispheres.** One of the hemispheres is provided with a stop-cock, by which it can be screwed on to an air pump. The edges fit accurately and are well greased, so as to be air-tight. As long as the hemispheres contain air, they can be separated with ease; but when the air in the interior is pumped out by means of an air pump, they can be separated only with great difficulty. The force required to separate them will be equal to the area of the largest circle of the hemisphere (projected area) in square inches, multiplied by 14.7 pounds.

This force will be the same in whatever position the hemisphere may be held, thus proving that the pressure of air upon it is the same in all directions.

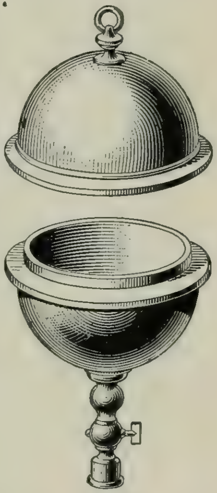


FIG. 195.

1069. The Weight Lifter.—The pressure of the atmosphere is very clearly shown by means of an apparatus like that illustrated in Fig. 196. Here, a cylinder fitted with a piston is held in suspension by a chain. At the top of the cylinder is a plug *A*, which can be taken out. This plug is removed, the piston pushed up (the force necessary being equal to the weight of the piston and rod *B*) until it touches the cylinder head. The plug is then screwed in, and the piston will remain at the top until a weight has been hung on the rod equal to the area of the piston, multiplied by 14.7 pounds, less the weight of the piston and rod. If a force was applied to the rod sufficiently great to force the piston downwards, it would raise any weight less than the above to the top of the cylinder. Suppose the weight to be removed, and the piston to be supported, say midway of the length of the cylinder. Let the plug be removed and air admitted above the piston, then screw the plug back into its place; if the piston be shoved upwards, the farther up it goes, the greater will be the force necessary to push it, on account of the compression of the air. If the piston is of large diameter, it will also require a great force to pull it out of the cylinder, as a little consideration will show. For example, let the diameter of the piston be 20 inches, the length of the cylinder 36 inches, plus the thickness of the piston, and the weight of the piston and rod 100 pounds. If the piston is in the middle of the cylinder, there will be 18 inches of space above it, and 18 inches of space below it. The area of the piston is $20^2 \times .7854 = 314.16$ square inches, and the atmospheric pressure upon it is $314.16 \times 14.7 = 4{,}618$ pounds, nearly. In order to shove the piston upwards 9 inches, the pressure upon it must be twice as great, or 9,236 pounds, and to this must be added the weight of the piston and rod, or $9{,}236 + 100 = 9{,}336$ pounds. The force necessary to cause the piston to move upwards 9 inches would then be $9{,}336 - 4{,}618 = 4{,}718$ pounds. Now, suppose the piston to be moved downwards until it is just on the point of being pulled out of the cylinder. The volume above it will then be twice as great as before, and the pressure one-half as great, or $4{,}618 \div 2 = 2{,}309$ pounds. The total upward pressure will be the pressure of the atmosphere less the weight of the piston and rod, or $4{,}618 - 100 = 4{,}518$ pounds, and the force necessary to pull it downwards to this point will be $4{,}518 - 2{,}309 = 2{,}209$ pounds.

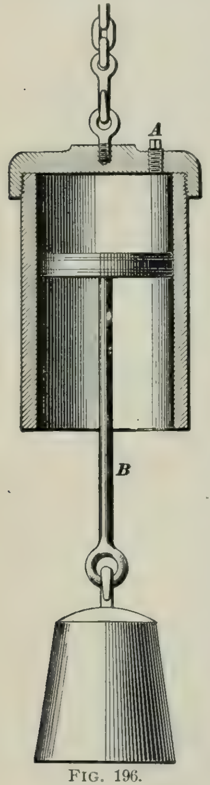


FIG. 196.

1070. The Baroscope.—The buoyant effect of air is very clearly shown by means of an instrument called the **baroscope,** shown in Fig. 197. It consists of a scale beam, from one extremity of which is suspended a small weight, and from the other a hollow copper sphere. In air they exactly balance each other; but when placed under the receiver of an air pump and the air exhausted, the sphere sinks, showing that it is really heavier than the small weight. Before the air is exhausted, each body is buoyed up by the weight of the air it displaces, and since the sphere displaces the most air, it loses more weight by reason of this displacement than the small weight. Suppose that the volume of the sphere exceeds that of the weight by 10 cubic inches; the weight of this volume of air is 3.1 grains. If this weight be added to the small weight, it will overbalance the sphere in air, but will exactly balance it in a vacuum.

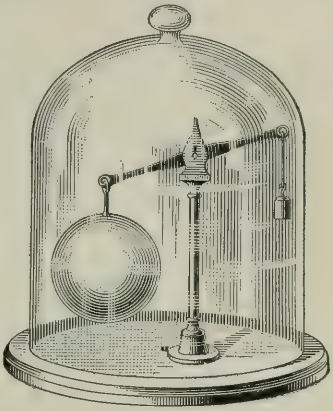


FIG. 197.

AIR COMPRESSORS.

1071. For many purposes compressed air is preferable to steam or other gas for use as a motive power. In such cases **air compressors** are used to compress the air. These are made in many forms, but the most common one is to place a cylinder, called the *air cylinder*, in front of the cross-head of a steam engine, so that the piston of the air cylinder can be driven by attaching its piston rod to the cross-head, in a manner similar to a steam pump. A cross-section of the air cylinder of a compressor of this kind is shown in Fig. 198, in which A is the piston and B is the piston rod, driven by the cross-head of a steam engine not shown in the figure. Both ends of the lower half of the cylinder are fitted with inlet valves D and D', which allow the air to enter the cylinder, and both ends of the upper half are fitted with discharge valves *F* and *F'*, which allow the air to escape from the cylinder after it has been compressed to the required pressure.

Suppose the piston *A* to be moving in the direction of the arrow; then the inlet valves *D* in the left-hand end of the cylinder from which the piston is moving will be forced inwards by the pressure of the atmosphere, which overcomes the resistance of the light spring *C*, thus allowing the air to flow in and fill the cylinder. On the other side of the piston, the air is being compressed, and, consequently, it acts with the springs *S* to force the inlet valves *D'* in the right-hand end of the cylinder to their seats. In the right-hand end of the cylinder, the discharge valves *F'* are opened when the pressure of the air in the cylinder is great enough to overcome the resistance of the light springs *E'*

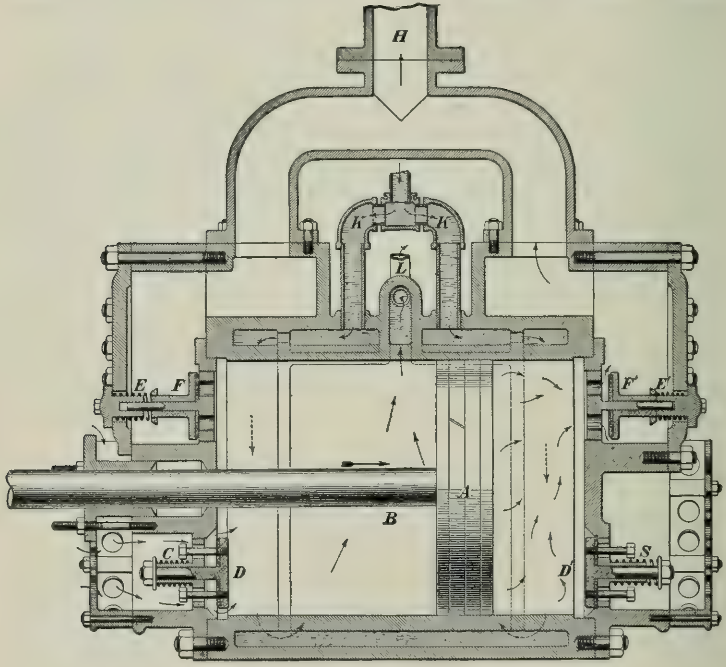


FIG. 198.

and the tension of the air in the passages leading to the discharge pipe *H*, and the discharge valves *F* are pressed against their seats by the springs *E* and the tension of the air in the passages. Suppose it is desired to compress the air to 59 pounds per square inch, and we wish to find at what point of the stroke the discharge valves will open. Now, 59 pounds per square inch equals a pressure of 4 atmospheres, very nearly; hence, when the pressure in the cylinder becomes great enough to force air out through the discharge valves, the volume must be one-quarter of the volume at atmospheric pressure, or the valves will open when the piston has traveled three-quarters of its stroke, provided the air be compressed at constant temperature.

The air, after being discharged from the cylinder, passes out through the delivery pipe *H*, and from thence is conveyed to its destination. It was shown in the early part of this paper that when air or any other gas was compressed its temperature was increased. For high pressures this increase of temperature becomes a serious consideration, for two reasons: 1st. When the air is discharged at a high temperature, the pressure falls considerably when it has cooled down to its normal temperature, and this represents a serious loss in the economical working of the machine. 2d. The alternate heating and cooling of the compressor cylinder by the hot and cold air is very destructive to it, and increases the wear to a great extent. To prevent the air from heating, cooling devices are resorted to, the most common one being the so-called **water jacket.** This is effected in the following manner: The cylinder walls are hollow, as shown in the cut; the cold water enters this hollow space in the cylinder wall through the pipe *K K*, and flows around the cylinder, finally passing out through the discharge pipe *L*. The water tends to keep the cylinder walls cold, and these cool the air as it is compressed.

1072. The Cartesian Diver. — The instrument shown in Fig. 199, called the **cartesian diver,** illustrates the elasticity of air and the transference of pressure in all directions in water. It consists of a glass jar filled with water, having a rubber bulb at the top filled with air. The image in the jar is made of glass and is hollow, the weight being less than an equal volume of water, so that it will float at the top of the jar. The tail of the image has a hole in it, the water being prevented from getting inside of the image by the tension of the air within it. If the bulb be squeezed, the air in it will be forced out, creating a pressure upon the water which, being transferred in all directions, causes the water to flow into the tail of the image, compressing the air inside, and thus causing it to fall to the bottom of the jar. When the bulb is released, the air flows back into it; the pressure upon the water is removed, the air within the image expands; the image, again becoming lighter than water, rises to the top of the jar.

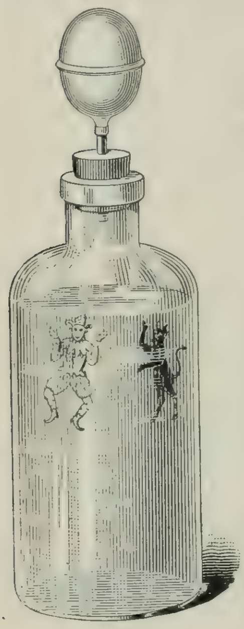


FIG. 199.

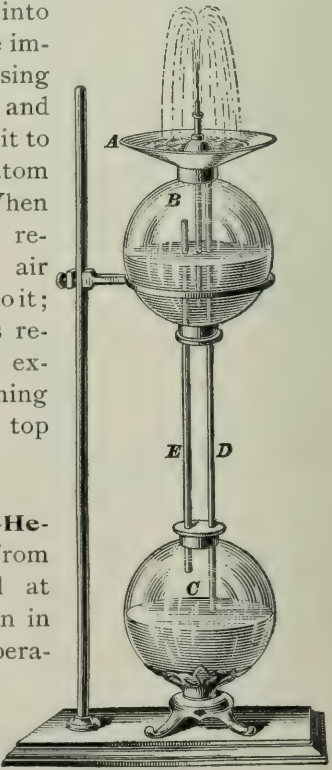


FIG. 200.

1073. Hero's Fountain.—**Hero's fountain** derives its name from its inventor, Hero, who lived at Alexandria 120 B. C. It is shown in Fig. 200. It depends for its operation upon the elastic properties of air. It consists of a brass dish *A*, and two glass globes *B* and *C*. The dish communicates with the lower part of the globe C by a long tube D, and another tube E connects the two globes. A third tube passes through the dish A to the lower part of the globe B. This last tube being taken out, the globe B is partially filled with water; the tube is then replaced and water is poured into the dish. The water flows through the tube D into the lower globe, and expels the air, which is forced into the upper globe. The air thus compressed acts upon the water and makes it jet out through the shortest tube, as represented in the figure. Were it not for the resistance of the atmosphere and friction, the water would rise to a height above the water in the dish equal to the difference of the level of the water in the two globes.

THE SIPHON.

1074. The action of the **siphon** illustrates the effect of atmospheric pressure. It is simply a bent tube of unequal branches, open at both ends, and is used to convey a liquid from a higher point to a lower, over an intermediate point higher than either. In Fig. 201, A and B are two vessels, B being lower than A, and $A\ C\ B$ is the bent tube or siphon. Suppose this tube to be filled with water and placed in the vessels, as shown, with the short branch $A\ C$ in the vessel A. The water will flow from the vessel A into B, so long as the level of the water in B is below the level of the water in A, and the level of the water in A is above the lower end of the tube $A\ C$. The atmospheric pressure upon the surfaces of A and B tends to force the water up the tubes $A\ C$ and $B\ C$. When the siphon is filled with water, each of these pressures is counteracted in part by

FIG. 201.

the pressure of the water in that branch of the siphon which is immersed in the water upon which the pressure is exerted. The atmospheric pressure opposed to the weight of the longer column of water will, therefore, be more resisted than that opposed to the weight of the shorter column; consequently, the pressure exerted upon the shorter column will be greater than that upon the longer column, and this excess pressure will produce motion.

Let $A =$ the area of the tube in square inches.

$h = D\,C =$ the vertical distance in inches between the surface of the water in B and the highest point of the center line of the tube.

$h_1 = E\,C =$ the distance in inches between the surface of the water in A and the highest point of the center line of the tube.

The weight of the water in the short column is $.03617\,A\,h_1$, and the resultant atmospheric pressure, tending to force the water up the short column, is $14.7 \times A - .03617\,A\,h_1$. The weight of the water in the long column is $.03617\,A\,h$, and the resultant atmospheric pressure, tending to force the water up the long column, is $14.7\,A - .03617\,A\,h$. The difference between these two is $(14.7\,A - .03617\,A\,h_1) - (14.7\,A - .03617\,A\,h) = .03617\,A(h - h_1)$. But $h - h_1 = E\,D =$ the difference between the levels of the water in the two vessels. To find the discharge from a siphon, use the difference $h - h_1$, reduced to feet, as the head, and the total length of the siphon between the two water levels, as the length of the pipe; the discharge may then be calculated by formula **50**, Art. **1032.**

It will be noticed that the short column must not be higher than 34 feet for water, or the siphon will not work, since the pressure of the atmosphere will not support a column of water that is higher than 34 feet; 28 feet is considered to be the greatest height for which a siphon will work well.

1075. Intermittent Springs.—Sometimes a spring is observed to flow for a time and then cease; then, after an interval, to flow again for a time. The generally accepted explanation of this is that there is an underground reservoir fed with water through fissures in the earth, as shown in Fig. 202. The outlet for the water is shaped like a siphon, as shown. When the water in the reservoir reaches the same height as the highest point of outlet, it flows out until the level of the water in the reservoir falls below the mouth of the siphon, the water flowing out of the reservoir faster than it is supplied to it. This flow then ceases until the water in the reservoir has again reached the level of the highest point of the siphon.

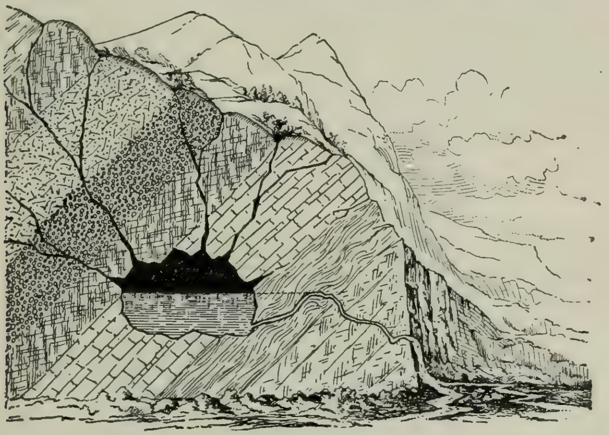


FIG. 202.

THE INJECTOR.

1076. A section of an injector is shown in Fig. 203. There are many different kinds of these instruments, but the principle is the same in all. When they are used for lifting water from a point below the discharge orifice and forcing it into the boiler of a steam engine or locomotive, they depend for their lifting action upon the creation of a partial vacuum by the action of steam. In the injector shown in Fig. 203, F is the connection for the steam pipe from the boiler, P is the connection for the pipe from the water supply, N is the connection to which the discharge pipe leading to the boiler is attached, and the waste water and steam are discharged through the **overflow** nozzle *O*.

The method of operation is as follows: The valve *B* is first opened by turning the wheel *W*; the **primer** valve *R* is then opened by the handle *J*, thus permitting steam to flow through the passage *E* and a connection, not shown in the figure, to the nozzle *u*. From *u* the jet of steam rushes out through *O*. A passage connects the chamber surrounding *u* with the space above the valve *L*. The jet of steam from *u* out through *O* carries with it the air in the chamber to which *O* is connected, thus forming a partial vacuum in the space above *L*; the air in the passages *D*, *C*, *G*, *H*, *K*, *T*, and in the water pipe connected at *P* is thus drawn out through the valve *L*, and a partial vacuum is formed, which permits the pressure of the atmosphere to force water through *P* until it finally fills the passages and flows out through *L* and the overflow nozzle *O*. As soon as water appears at *O*, the valve *R* is closed and the main steam valve *A* is opened by the wheel *S*, thus admitting steam to the passages *C*, *H*, *K*. This steam draws water from *G* through the opening surrounding *H* and discharges it through *K* with such a high velocity that it rushes past

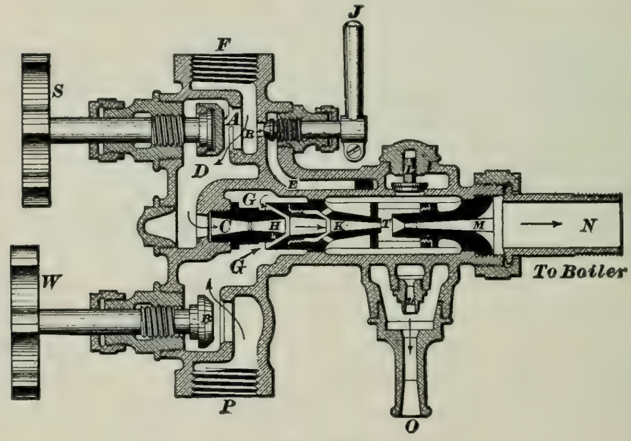


FIG. 203.

the opening *T* into the nozzle *M* and thence into the boiler.

THE LOCOMOTIVE BLAST.

1077. Fig. 204 shows the front end of a locomotive. *E* is the exhaust pipe, the center of which is directly in line with the center of the smokestack *S*. *T*, *T* are the tubes through which the hot furnace gases are discharged. The exhaust steam has a pressure of about two pounds above the atmosphere, and rushes through the exhaust pipe *E* and up the smokestack *S* with a very high velocity, taking the air out with it, and producing a partial vacuum in the space in front of the tubes. No air can get in this space except through the grates of the fire-box; consequently, this partial vacuum created in front of the tubes as described causes an influx of air through the grate, and produces the **forced draft,** or **blast.** The faster the engine runs, the greater the quantity of air drawn through the grate.

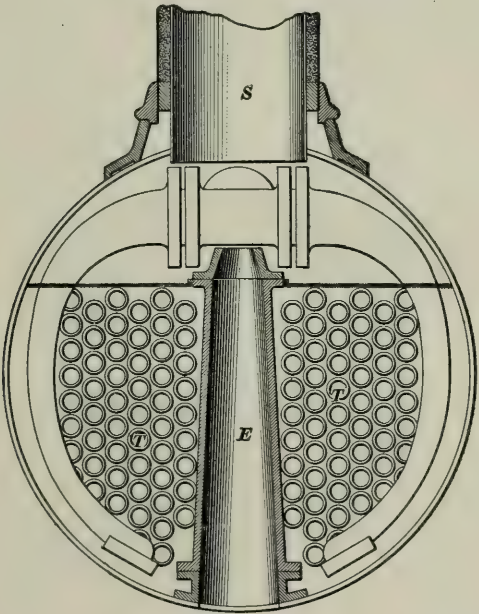


FIG. 204.

PUMPS.

1078. The Suction Pump.—A section of an ordinary suction pump is shown in Fig. 205. Suppose the piston to be at the bottom of the cylinder and to be just on the point of moving upwards in the direction of the arrow. As the piston rises it leaves a vacuum behind it, and the atmospheric pressure upon the surface of the water in the well causes it to rise in the pipe *P*, for the same reason that the mercury rises in the barometer tube. The water rushes up the pipe and lifts the valve *V*, filling the empty space in the cylinder *B* displaced by the piston. When the piston has reached the end of its stroke, the water entirely fills the space between the bottom of the piston and the bottom of the cylinder and also the pipe *P*. The instant that the piston begins its down stroke, the water in the chamber *B* tends to fall back into the well, and its weight forces the valve *V* to its seat, thus preventing any downward flow of the water. The piston now tends to compress the water in the chamber *B*, but this is prevented through the opening of the valves *u*, *u*

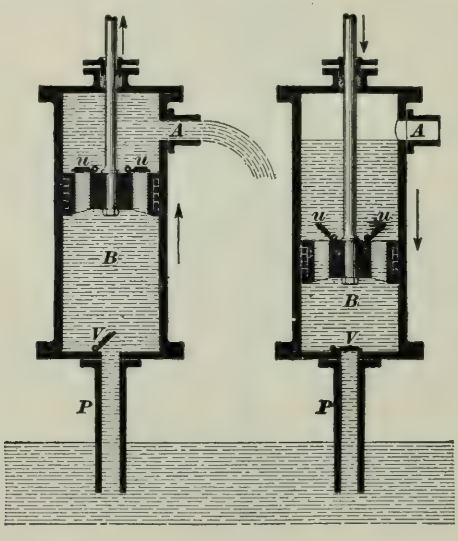


FIG. 205.

in the piston. When the piston has reached the end of its downward stroke, the weight of the water above closes the valves *u*, *u*. All the water resting on the top of the piston is then lifted with the piston on its upward stroke, and discharged through the spout *A*, the valve *V* again opening, and the water filling the space below the piston as before.

It is evident that the distance between the valve *V* and the surface of the water in the well must not exceed 34 feet, the highest column of water which the pressure of the atmosphere will sustain, since otherwise the water in the pipe would not reach to the height of the valve *V*. In practice this distance should not exceed 28 feet. This is due to the fact that there is a little air left between the bottom of the piston and the bottom of the cylinder, a little air leaks through the valves which are not perfectly air-tight, and a pressure is needed to raise the valve against its weight, which, of course, acts downwards. There are many varieties of the suction pump, differing principally in the valves and piston, but the principle is the same in all.

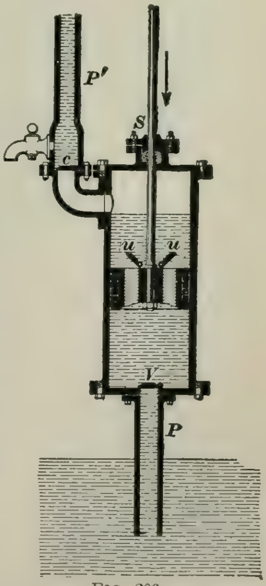


FIG. 206.

1079. The Lifting Pump.—A section of a **lifting pump** is shown in Fig. 206. These pumps are used when water is to be raised to greater heights than can be done with the ordinary suction pump. As will be perceived, it is essentially the same as the pump previously described, except that the spout is fitted with a cock and has a pipe attached to it, leading to the point of discharge. If it is desired to discharge the water at the spout, the cock may be opened; otherwise, the cock is closed, and the water is lifted by the piston up through the pipe *P′* to the point of discharge, the valve *c* preventing it from falling back into the pump, and the valve *V* preventing the water in the pump from falling back into the well. It is not necessary that there should be a second pipe P', as shown in the figure, for the pipe P may be continued straight upwards, as shown in Fig. 207.

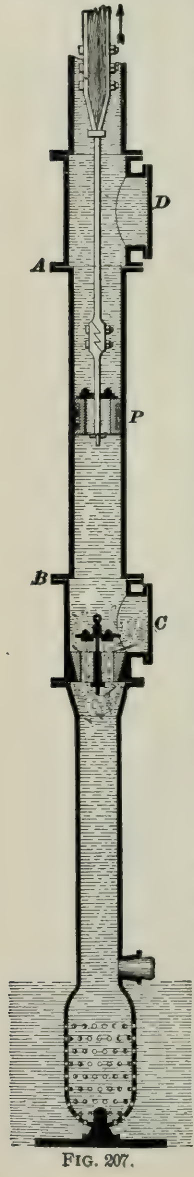


FIG. 207.

1080. In the figure is shown a section of a lifting pump for raising water from great depths, as from the bottom of mines to the surface. This pump consists of a series of pipes connected together, of which the lower end only is shown in the cut. That part of the pipe included between the letters A and B forms the pump cylinder in which the piston P works. That part of the pipe above the highest point of the piston travel, through which the water is discharged, is called the **delivery pipe,** and the part below the lowest point of the piston travel is called the **suction pipe.** The lower end of the suction pipe is expanded, and has a number of small holes in it, to keep out the solid matter. C is a plate covering an opening, and which may be removed to allow the suction valve to be repaired. D is a plate covering a similar opening through which the piston and piston valves may be repaired. The piston rod, or rather the piston stem, is made of wrought iron, inserted with wood, and connected with the piston. The only limit to the height to which a pump of this kind can raise water is the strength of the piston rod. Lifting pumps of this kind are used to raise water from great depths to the earth's surface; hence, a very long piston rod is necessary. In the lifting pump shown in Fig. 206 the water is raised from a point a few feet below the earth's surface to a point considerably higher. This requires the piston rod to move through a stuffing-box, as shown at *S*, and also necessitates the rod being round, in order that the water may not leak out.

1081. Force Pumps.—The **force pump** differs from the lifting pump in several important particulars, but chiefly in the fact that the piston is solid; that is, it has no valves. A section of a *suction and force pump* is shown in Fig. 208. The water is drawn up the suction pipe as before, when the piston rises; but when the piston reverses, the pressure on the water caused by the descent of the piston opens the valve *V'* and *forces* the water up the delivery pipe *P'*. When the piston again begins its upward movement, the valve *V'* is closed by the pressure of the water above it, and the valve *V* is opened by the pressure of the atmosphere on the water below it, as in the previous cases. For an arrangement of this kind, it is not necessary to have a stuffing-box. The water may be forced to almost any desired height. The force pump differs again from the lifting pump in respect to its piston rod, which should not be longer than is absolutely necessary in order to prevent it from *buckling*,

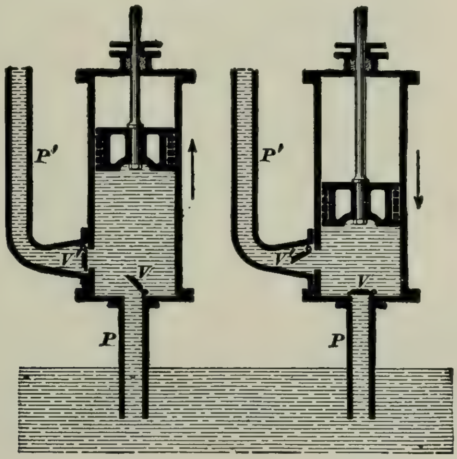


FIG. 208.

while in the lifting pump the length of the piston rod is a matter of indifference.

1082. Plunger Pumps.—When force pumps are used to convey water to great heights, the pressure of the water in the cylinder becomes so great that it becomes extremely difficult to keep the water from leaking past the piston, and the constant repairing of the piston packing becomes a nuisance. To obviate this difficulty the piston is made very long, as shown in Fig. 209, and is then called a **plunger.** The suction valve in this case consists of two clack valves inclined to each other and resting upon a square pin *A*; they are prevented from flying back too far during the up stroke of the plunger by the two uprights *I*, *I*. During the down stroke of the plunger the valves at *A* are closed and the delivery valve at *B* is open. A little air is always carried into the cylinder of a pump with the entering of the water. In force pumps this fact becomes a serious consideration, since, after repeated strokes, the air accumulates, and during the down stroke of the plunger it is compressed. After a time it would become sufficiently compressed to entirely prevent the water from entering through the suction valve, the pressure on the top of the valve being greater than that of the atmosphere below. In the pump shown in the figure, the plunger is a trifle smaller than the cylinder, and the air collects around the plunger below the stuffing-box. To remove this air a narrow passage *C*, shown by the dotted lines, that can be closed at its upper end by the cock *D*, connects the interior of the pump with the atmosphere when the cock is open. It is evident that this

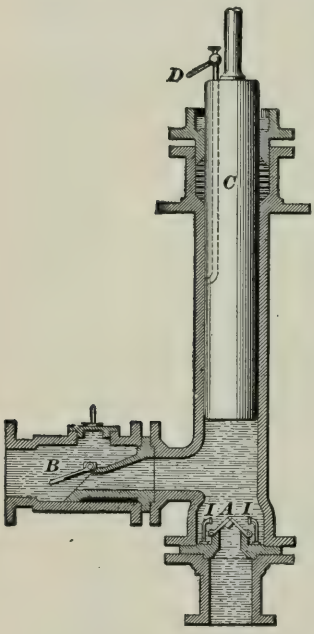


FIG. 209.

cock must not be opened, except during the down stroke of the plunger; for, if it were open during the up stroke, the pressure below the plunger being less than the pressure of the atmosphere above, the air would rush in instead of being expelled.

1083. Double-Acting Pumps.—In the pumps previously described, the discharge was intermittent; that is, the pump could only discharge when the piston was moving in one direction. In some cases it is necessary that there should be a continuous discharge; in all cases it takes more power to run the pump with an intermittent discharge, as a little consideration will show. If the height that the water is to be raised is considerable, its weight will be very great, and the entire mass must be put in motion during one stroke of the piston.

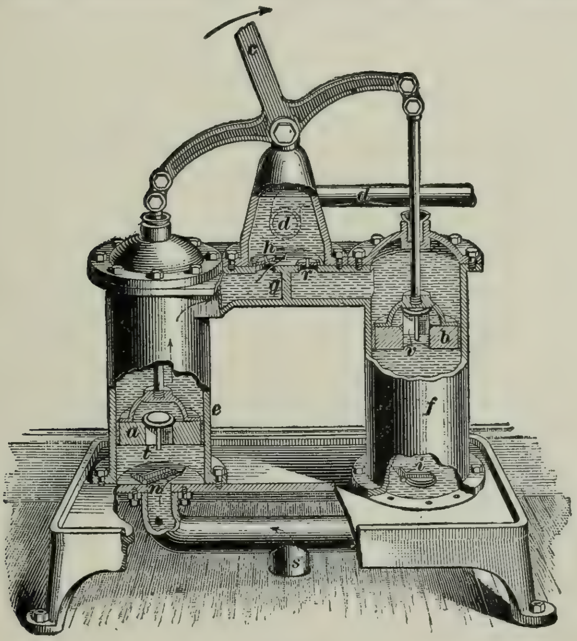


FIG. 210.

In order to obtain the advantage of a more continuous discharge, double-acting pumps are used. Fig. 210 shows a part sectional view of such a pump. Two pistons *a* and *b* are used, which are operated by one handle *c* in the manner shown. The pump has one suction pipe *s* and one discharge pipe *d*. The cylinders *e* and *f* are separated by a diaphragm *g*, so that they cannot communicate with each other above the pistons. In the figure, the handle *c* is moving to the right, the piston *a* upwards, and the piston *b* downwards. As the piston *a* moves upwards, it lifts the water above it and causes it to flow through the delivery valve *h* into the discharge pipe *d*. This upward movement of the piston creates a partial vacuum below it in the cylinder *e*, and causes the water to rush up the suction pipe *s* into the cylinder, as shown by the arrows. In the cylinder *f*, the downward movement of the piston *b* raises the piston valve *v*, and the weight of the water on the suction valve *i* keeps it closed. When the handle *c* has completed its movement to the right and begins its return, all of the valves on the right-hand side open except *v*, and those on the left-hand side close except *n*; water is then discharged into the delivery pipe by the cylinder *f*, and only at the instant of reversal is the flow into the delivery pipe *d* stopped.

1084. Air Chambers.—In order to obtain a continuous flow of water in the delivery pipe, with as nearly a uniform velocity as possible, an **air chamber** is usually placed on the delivery pipe of force pumps as near to the pump cylinder as the construction of the machine will allow. The air chambers are usually pear-shaped, with the small end connected to the pipe. They are filled with air which the water compresses during the discharge. During the suction, the air thus compressed expands and acts as an accelerating force upon the moving column of water, a force which diminishes with the expansion of the air, and helps to keep the velocity of the moving column more nearly uniform. An air chamber is sometimes placed upon the suction pipe. These air chambers not only tend to promote a uniform discharge, but they also equalize the stresses upon the pump, and prevent shocks due to the incompressibility of water.

They serve the same purpose in pumps that a fly-wheel does to the steam engine. Unless the pump moves very slowly, it is absolutely necessary to have an air chamber on the delivery pipe.

1085. Steam Pumps.—Steam pumps are force pumps operated by steam acting upon the piston of a steam engine, directly connected to the pump, and in many cases cast with the pump. A section of a double-acting steam pump showing the steam and water cylinders, with other details, is illustrated in Fig. 211. Here *G* is a steam piston, and *R* the piston rod, which is secured at its other end to the plunger *P*. *F* is a partition cast with the cylinder, which prevents the water in the left-hand half from communicating with that in the right-hand half of the cylinder. Suppose the piston to be moving in the direction of the arrow. The volume of the left-hand half of the pump cylinder will be increased by an amount equal to the area of the circumference of the plunger, multiplied by the length of the stroke, and the volume of the right-hand half of the cylinder will be diminished by a like amount. In consequence

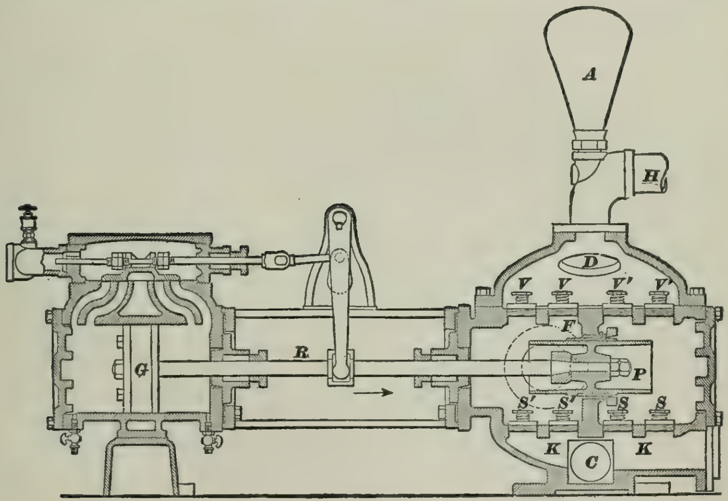


FIG. 211.

of this, a volume of water in the right-hand half of the cylinder equal to the volume displaced by the plunger in its forward movement will be forced through the valves *V'*, *V'* into the air chamber *A*, through the orifice *D*, and then discharged through the delivery pipe *H*. By reason of the partial vacuum in the left-hand half of the pump cylinder, owing to this movement of the plunger, the water will be drawn from the reservoir through the suction pipe *C* into the chamber *K*, *K*, lifting the valves *S'*, *S'*, and filling the space displaced by the plunger. During the return stroke the water will be drawn through the valves *S*, *S* into the right-hand half of the pump cylinder, and discharged through the valves *V*, *V* in the left-hand half. Each one of the four suction and four discharge valves is kept to its seat, when not working, by light springs, as shown.

There are many varieties and makes of steam pumps, the majority of which are double-acting. In many cases two steam pumps are placed side by side, having a common delivery pipe. This arrangement is called a **duplex pump.** It is usual to so set the steam pistons of duplex pumps that when one is completing the stroke the other is in the middle of its stroke. A double-acting duplex pump made to run in this manner, and having an air chamber of sufficient size, will deliver water with nearly a uniform velocity.

In mine pumps for forcing water to great heights, the plungers are made solid, and in most cases extended through the pump cylinder. In many steam pumps pistons are used instead of plungers, but when very heavy duty is required plungers are preferred.

1086. Centrifugal Pumps.—Next to the direct-acting steam pump, the **centrifugal pump** is the most valuable instrument for raising water to great heights that has yet been described. As the name denotes, the effects produced by centrifugal force are made use of. Fig. 212 represents one with half of the casing removed. The hub *S* is hollow, and is connected directly to the suction pipe. The curved arms *a*, called **vanes** or **wings,** are revolved with a high velocity in the direction of the arrow, and the air enclosed between them is driven out through the discharge passage and delivery pipe *D D*. This creates a partial vacuum in the casing and suction pipe, and causes the water to flow in through *S*. This water is also made to revolve with the vanes, and, of course, with the same velocity. The centrifugal force of the revolving water causes it to fly outwards towards the end of the vanes, and becomes greater the farther away it gets from the center. This causes it to leave the vanes, and finally to leave the pump by means of the discharge passage and delivery pipe *D D*. The height to which the water can be forced depends upon the velocity of the revolving vanes. In the construction of the centrifugal pump, particular care is required in giving the correct form to the vanes; the efficiency of the machine depends greatly upon this point being attended to. What is required is to raise the water, and the energy used to drive the pump should be devoted as far as possible to this one purpose. The water when it is raised should be delivered with as little velocity as possible, for any velocity which the water then possesses, has been produced at the expense of the energy used to drive the pump. The form of the vanes is such that the water is delivered at the desired height with the least expenditure of energy.

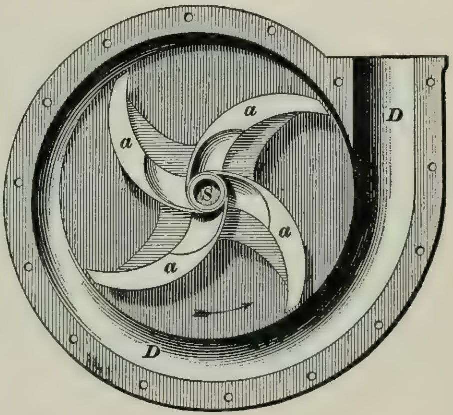


FIG. 212.

The number of vanes depends upon the size and capacity of the pump. It will be noticed that, in the pump shown in the figure, the vanes have sharp edges near the hub. The object of this is to provide for a free ingress of the water, and also to cut any foreign substance that may enter the pump and prevent it from working properly.

Almost any liquid can be raised with these pumps, but when used for pumping chemicals, the casing and vanes are made of materials that the chemicals will not affect.

1087. The Hydraulic Ram.—The construction of a hydraulic ram is shown in Fig. 213. This machine is used for raising water from a point below the level of the water in a spring or reservoir to a point considerably higher, with no power other than that afforded by the inertia of a moving column of water. In the figure, *a* is a pipe called the *drive pipe*, connecting the ram with the reservoir; the valve *b* slides freely in a guide, and is provided with lock-nuts to regulate the distance that the valve can fall below its seat. When the water is first turned on by opening the valve *n*, the valve *b* is already opened, and the water flows out through *c*, as shown. As the discharge continues, the velocity of the water in the drive pipe will increase until the upward pressure against the valve *b* is sufficient to force the valve to its seat. The actual closing of the valve takes place very suddenly, and the momentum of the column of water, which was moving with an increasing velocity through the drive pipe *a*, will very rapidly force some water through the valve *d* into the air-chamber *f*. Immediately after this, a rebound takes place, and for a short interval of time the water flows

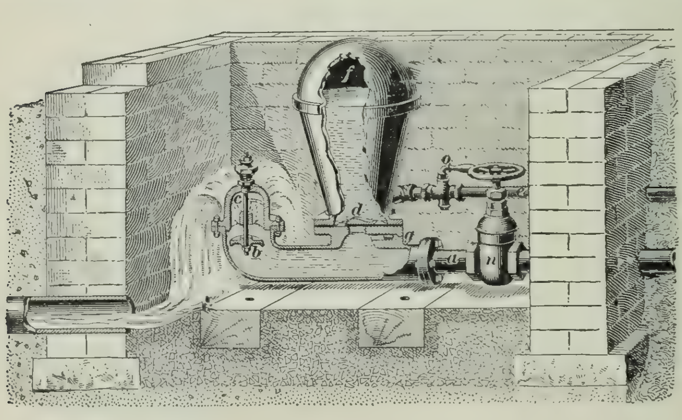


FIG. 213.

back up the drive pipe *a* and tends to form a vacuum under the air chamber valve *d*; this opens the snifter valve *g* and admits a little air, which accumulates under the valve *d* and is forced into the air chamber with the next shock. This air keeps the air chamber constantly charged; otherwise, the water, being under a greater pressure in the air chamber than it is in the reservoir, would soon absorb the air in the chamber and the ram would cease to work until the chamber was recharged with air. The rebound also takes the pressure off the under side of valve *b* and causes it to drop, and the above-described operations are repeated. The delivery pipe is shown at *c*; a steady flow of water is maintained through it by the pressure of the air in the chamber *f*; this air also acts as a cushion when valve *b* suddenly closes, and prevents undue shock to the parts of the ram.

The height to which water can be raised by the hydraulic ram depends upon the weight of the valve *b* and the velocity of the water in *a*.

1088. Power Necessary to Work a Pump:

Rule I.—*In all pumps, whether lifting, force, steam, single- or double-acting, or centrifugal, the number of foot-pounds of power needed to work the pump is equal to the weight of the water in pounds, multiplied by the vertical distance in feet between the level of the water in the well, or source, and the point of discharge, plus the work necessary to overcome the friction and other resistances.*

Rule II.—*The work done in one stroke of a pump is equal to the weight of a volume of water equal to the volume displaced by the piston during the stroke, multiplied by the total vertical distance in feet through which the water is to be raised, plus the work necessary to overcome the resistances.*

A little consideration will make Rule II evident. Suppose that the height of the suction is 25 feet; that the vertical distance between the suction valve and the point of discharge is 100 feet; that the stroke of the piston is 15 inches, and that its diameter is 10 inches. Let the diameters of the suction pipe and delivery pipe be 4 inches each. The volume displaced by the pump piston or plunger in one stroke equals $\frac{10^2 \times .7854 \times 15}{1,728} = .68177$ cubic foot. The weight of an equal volume of water $= .68177 \times 62.5 = 42.61063$ pounds. Now, in order to discharge this water, *all* of the water in the suction and delivery pipes had to be moved through a certain distance in feet equal to .68177 divided by the area of the pipes in square feet.

Four inches $= \frac{1}{3}$ of a foot. $(\frac{1}{3})^2 \times .7854 = \frac{.7854}{9} = .0872\frac{2}{3}$ square foot. $.68177 \div .0872\frac{2}{3} = 7.8125$ feet.

The weight of the water in the delivery pipe is $(\frac{1}{3})^2 \times .7854 \times 100 \times 62.5 = 545.42$ pounds.

The weight of the water in the suction pipe is $(\frac{1}{3})^2 \times .7854 \times 25 \times 62.5 = 136.35$ pounds.

$545.42 + 136.35 = 681.77$ pounds = the total weight of water moved in one stroke. The distance that it is moved in one stroke is 7.8125 feet. Hence, the number of foot-pounds necessary for one stroke is $681.77 \times 7.8125 = 5,326.33$ foot-pounds. Had this result been obtained by Rule II, the process would have been as follows: The weight of the water displaced by the piston in one stroke was found to be 42.61063 pounds. $42.61 \times 125 = 5,326.33$ pounds, which is exactly the same as the result obtained by the previous method, and is a great deal shorter.

EXAMPLE.—What must be the necessary horsepower of a double-acting steam pump if the vertical distance between the point of discharge and the point of suction is 96 feet? The diameter of the pump cylinder is 8 inches, the stroke is 10 inches, and the number of strokes per minute is 120. Allow 25% for friction, etc.

SOLUTION.—Since the pump is double-acting, it raises a quantity of water equal to the volume displaced by the plunger at every stroke. The weight of the volume of water displaced in one stroke $= (\frac{8}{12})^2 \times .7854 \times \frac{10}{12} \times 62.5 = 18.18$ pounds, nearly.

$18.18 \times 96 \times 120 = 209,433.6$ foot-pounds per minute.

Since 25% is to be allowed for friction, the actual number of foot-pounds per minute $= 209,433.6 \div .75 = 279,244.8$ foot-pounds per minute.

One horsepower $= 33,000$ foot-pounds per minute; hence, $\frac{279,244.8}{33,000} = 8.462$ H. P., nearly. Ans.

STRENGTH OF MATERIALS.

MATERIALS USED FOR CONSTRUCTION.

1089. The principal materials used in engineering construction are *timber*, *brick*, *stone*, *cast iron*, *wrought iron*, and *steel.* Table 19 gives their average weights per cubic foot.

TABLE 19.

Material.	Average Weight. Pounds per Cubic Foot.	Approximate Weight of Piece 1″ Square and 1′ Long in Lb.
Timber	40	.278
Brick	125	
Stone	160	
Cast Iron	450	3.125
Wrought Iron	480	3.333⅓
Steel	490	3.403

CAST IRON.

1090. **Cast iron** is a combination of pure iron with from 2% to 6% of carbon.

Pig iron is the result of the first smelting, and is obtained directly from the ore. Pig iron is rarely, if ever, used for anything except to be remelted and made into cast iron or wrought iron.

1091. Cast iron is of two kinds, *white cast iron* and *gray cast iron.* The first is a chemical compound of iron with from 2% to 6% of carbon, nearly all the carbon being chemically combined with the iron. The second, or gray cast iron, contains a part of the carbon in chemical combination, and the rest in the state of graphite mechanically mixed with the iron. When a piece of gray cast iron is broken, a large number of black specks are seen on the broken ends; these specks are pure carbon in the form of graphite.

1092. White cast iron contains hardly any free carbon. It is of two kinds, granular and crystalline. Both are very hard and brittle, and are only used for conversion into wrought iron or steel.

1093. Gray cast iron is divided into three classes, known as Nos. 1, 2, and 3.

No. 1 contains the largest amount of carbon in mechanical mixture, which makes it soft and fusible, though not as strong as Nos. 2 and 3. It is very suitable for making castings where precision in form is required, as, owing to its fluidity, it fills the mold well; but it is not suitable for castings requiring strength.

No. 2 is most suitable for use in constructions, as it is stronger than No. 1, and not so soft.

No. 3 contains the smallest amount of carbon in the graphitic (uncombined) form, and is, in consequence, harder and more brittle. It is fit only for the massive and heavy parts of machinery.

1094. Cast iron has certain advantages and disadvantages as a material for engineering construction. It is easy to give it any desired form. A pattern of the piece desired is made; a mold is made in the sand, the pattern is removed, and the melted iron poured in. Cast iron resists oxidation (rust) better than either wrought iron or steel. Its compressive (crushing) strength is very high, but its tensile (stretching) strength is comparatively low. It cannot be riveted, or welded by forging. It is brittle, breaking off without giving much warning, and stretching but little before giving away. It is liable to have hidden and small surface defects and air bubbles, and this makes its strength uncertain.

Another serious drawback in the use of cast iron is its liability to initial stresses from inequality in cooling after having been poured into the molds. Thus, if one part of the casting is very thin and another very thick, the thin part cools first, and, in cooling, contracts; the thick part, cooling afterwards, causes stresses in the thin part, which may be sufficient to break it, or, if not, there may be so great a stress in the thin part that a small additional force will break it.

Cast iron is not well adapted to a tensile stress, nor to resist shocks. It is used for columns and posts in buildings, on account of its high compressive strength. In machinery, it is used in all those parts where weight, mass, or form is of more importance than strength, as in frames and bed plates of machines, and for hangers, pulleys, gear wheels, etc. It is also used for water mains where the pressure to be resisted is not too great.

1095. Malleable cast iron is made by heating the casting in an annealing oven, in powdered hematite ore. It can be hammered into any desired shape when cold, but is very brittle when hot.

WROUGHT IRON.

1096. Wrought iron is the product resulting from the reduction of the carbon in cast iron. It is obtained by melting white cast iron, and passing an oxidizing flame over it. When the carbon is burned out, the mass of iron is left in a pasty condition, full of holes. It is then taken out, and hammered or rolled in order to unite it into one mass. The result of this first process is not suitable for use in any construction of importance, and has to be reheated and rerolled a number of times, in order to make it homogeneous, and to remove flaws from within the iron.

At best, therefore, wrought iron is a series of welds; if a piece is broken, the separate layers of which it is composed can be seen plainly. It cannot be melted and run into molds, as can be done with cast iron; but it can be easily welded by forging; that is, two pieces of wrought iron can be united by raising them to the proper temperature and hammering them together. Wrought iron is much more capable of bearing a tensile or transverse stress than cast iron; it is tougher, stretches more, and gives more warning before fracture.

It withstands shocks far better than cast iron. Two pieces may be punched or drilled and riveted together. The entire process weakens the iron, and cast iron would not withstand it. It has also to withstand flanging and the stresses due to changes of temperature.

Wrought iron cannot be hardened like steel by heating and then dipping in water, but may be *case-hardened* by rubbing the surface with potassium cyanide or potassium ferrocyanide while at a cherry-red heat and then dipping in water. The cyanide causes the iron to be carbonized to a slight depth; that is, through a depth of about $\frac{1}{32}$ of an inch the iron is converted into steel which can be hardened. Cast iron may be hardened in the same way.

The quality of wrought iron varies considerably, and the terms by which it is known in the market refer to the amount of working which the iron has received. We thus have common bar iron, best iron, double best, and triple best. These terms are only rough indications of quality. When wrought iron is rolled cold under great pressure, it has a smooth polished surface, and its strength is greatly increased.

When the word iron is used alone, wrought iron is meant.

STEEL.

1097. Steel is a chemical compound of iron and carbon; that is, it contains no carbon in a free state, as cast iron does. Its tensile strength is greater than that of wrought iron, and its compressive strength greater than that of cast iron. It is by far the strongest material used in the mechanic arts. Its strength varies greatly with its purity and the amount of carbon it contains. According to the amount of carbon in it, steel is divided into high grade, medium grade, and low grade, the high grades having the most carbon. Steel, unlike wrought iron, is fusible; unlike cast iron, it can be forged; and, with the exception of the higher grades, it can be welded by heating and hammering, although care must be exercised in so doing.

1098. The special characteristic of steel (the very lowest grades excepted) is that, when it is raised to a cherry red heat and suddenly cooled, it becomes brittle and exceedingly hard, and that by subsequent heating and slow cooling, the hardness may be reduced to any desired degree down to the point of least hardness that steel possessing that amount of carbon can have. The first process is called *hardening*, and the second *tempering*.

If the surface of a piece of steel that has been hardened is polished slightly so as to remove the dark scale or soot which covers it, and is then reheated, it will be found that, as the temperature increases, a series of colors appear. These colors are always the same for the same temperature, and, if the steel is suddenly cooled when one of the colors appears, it acquires a degree of hardness which is always the same for the same color and for the same quality of steel.

In Table 20 are given the colors with the corresponding temperatures that occur in tempering different kinds of tools.

TABLE 20.

Tools.	Color.	Temperature, Fahr.
Lancets	Pale yellow	430°
Razors and scalpels	Pale straw	450°
Penknives, chisels for cast iron, and screw taps	Yellow	470°
Scissors and chisels for wrought iron	Brown	490°
For carpenters' tools in general	Red	510°
Fine watch springs and table knives	Purple	530°
Swords and lock springs	Blue, bright	550°
Daggers, fine saws and needles	Blue, full	560°
Common saws and springs	Blue, dark	600°

Steel that has been hardened may be softened by heating it to a hardening temperature and then cooling it very slowly; this process is called **annealing.**

1099. Steel is made in one of the three following ways:

1. By adding carbon to wrought iron.
2. By removing carbon from cast iron.
3. By melting together cast and wrought iron in suitable proportions.

Several processes, varying with the quality of the product desired and the use for which it is intended, are used in making steel. The processes in general use are the following:

(*a*) **Cementation,** in which bars of very pure wrought iron are heated to a high temperature in contact with carbon. The product, known as **blister steel,** is used for cutlery, tools, etc.

(*b*) **Crucible steel,** also called **cast steel,** is made by melting pure wrought iron in a crucible with enough charcoal and cast iron to introduce the required amount of carbon. It is used for making springs, cutlery, tools, etc.

(*c*) **Bessemer steel** is made by decarbonizing cast iron by forcing a powerful blast of air through a melted mass of the iron. This removes the greater part of its carbon. A small quantity of very pure cast iron, rich in carbon, is then added, bringing up the percentage of carbon to the required amount.

(*d*) **Open-hearth steel** is made by fusing a charge consisting of the suitable proportions of cast iron with wrought iron scrap, or with Bessemer steel scrap.

1100. Bessemer and open-hearth steel contain more impurities than blister and crucible steel do; but they are much cheaper, and are just as suitable for many purposes. It is only in consequence of the introduction of these two cheap varieties that steel can be extensively used, as blister and crucible steels would, in the majority of cases, be too expensive.

Bessemer and open-hearth steels contain from .05% to 1½% of carbon. The proportion of carbon in the best kinds of tool and cutlery steels is as follows:

Razor steel,	1½%.	Very difficult to forge, and easily burnt.
Saw file,	1⅜%.	Bears heat not above cherry redness.
Tool steel,	1¼%.	Ordinary cutting tools. Welds with difficulty.
" "	1⅛%.	For mandrils and heavy cutting tools.
" "	1%.	For chisels, gravers, etc.

At the Imperial Works at Neuberg, Austria, the following percentages of carbon are present in the different grades of Bessemer steel:

1.58% to 1.38%.	Cannot be welded, and is rarely used.
1.38% " 1.12%.	Great care must be used in working.
1.12% " .88%.	Welds easily; used for bits, chisels, etc.
.88% " .62%.	Used for cutting tools, files, etc.
.62% " .38%.	Mild steel; for tires, etc.
.38% " .15%.	Tempers slightly; for boiler plates and axles.
.15% " .05%.	Does not temper; steel for pieces of machinery.

1101. It will be noticed from what precedes that the hardness of steel depends upon the amount of carbon it contains.

Some kinds of crucible cast steel can be hardened by heating to a low red heat and then allowing them to cool slowly in the air without dipping in water. They are called self-hardening steels, the best known being Mushet's special tool steel. This contains about 2% carbon with 7% to 12% tungsten in alloy with the iron. The same property is characteristic of Hadfield's manganese steel, which contains between .8% and 1.2% of carbon and 7% to 20% of manganese.

As a rule, when a piece of steel is broken across the grain, the finer the grain and the whiter and cleaner the fracture the more carbon it contains.

STRESSES AND STRAINS.

1102. The molecules of a solid or rigid body being held together by the force of cohesion, this force must be overcome to a greater or less degree in order to change the form and size of the body, or to break it into parts. The internal resistance which a body offers to any force tending to overcome the force of cohesion is called a **stress.** If a weight of 1,000 pounds is held in suspension by a rod, there will be a stress of 1,000 pounds in the rod. In this country and England, stresses are measured in pounds or tons; in nearly all other civilized countries, in kilograms. Whenever a body is subjected to a stress, the total stress induced by the acting force at any section of the body is the same as the total stress at any other section.

1103. The **unit stress** (called also the **intensity** of stress) is the stress per unit of area; or, it is the total stress divided by the area of the cross-section. In the above illustration, if the area had been 4 sq. in. the unit stress would have been $\frac{1,000}{4} = 250$ lb. per sq. in. Had the area been $\frac{1}{2}$ sq. in., the unit stress would have been $\frac{1,000}{.5} = 2,000$ lb. per sq. in.

Let $P =$ the total stress in pounds;
$A =$ area of cross-section in square inches;
$S =$ unit stress in pounds per square inch.

Then, $$S = \frac{P}{A}, \text{ or } P = A\,S. \qquad \textbf{(65.)}$$

That is, *the total stress equals the area of the section, multiplied by the unit stress.*

When a body is stretched, shortened, or in any way deformed through the action of a force, the amount of deformation is called a **strain.** Thus, if the rod before mentioned had been elongated $\frac{1}{10}''$ by the load of 1,000 pounds, the strain would have been $\frac{1}{10}''$. Within certain limits, to be given hereafter, strains are proportional to the stresses producing them.

1104. The **unit strain** is the strain per unit of length or of area, but is usually taken per unit of length and called the *elongation* per unit of length. In this paper, the unit of length will be considered as one inch. The unit strain, then, equals the total strain divided by the length of the body in inches.

Let l = length of body in inches;
e = elongation in inches;
s = unit strain.

Then, $$s = \frac{e}{l}, \text{ or } e = ls. \qquad \textbf{(66.)}$$

1105. Whenever a force, no matter how small, acts upon a body, it produces a stress and a corresponding strain.

According to the manner in which forces act upon a body, the stresses are divided into the following classes:

1. *Tension*, which produces a tensile or pulling stress.
2. *Compression*, which produces a compressive or crushing stress.
3. *Shear*, which produces a shearing or cutting stress.
4. *Torsion*, which produces a torsional or twisting stress.
5. *Flexure*, which produces a transverse or bending stress.

TENSION.

1106. When two forces act upon a body in opposite directions (away from each other) the body is said to be in tension. The two forces tend to elongate the body and thus produce a tensile stress and strain. A weight supported by a rope affords a good example. The weight acts downwards, and the reaction of the support to which the upper end of the rope is fastened acts upwards; the result being that the rope is stretched more or less, and a tensile stress is produced in it. Another familiar example is the connecting rod of a steam engine on the return stroke. The cross-head then exerts a pull on one end of the rod, which is resisted by the crank-pin on the other.

Example.—An iron rod, 2 inches in diameter, sustains a load of 90,000 pounds; what is the unit stress?

Solution.—Using formula **65**,

$$P = A\,S, \text{ or } S = \frac{P}{A} = \frac{90{,}000}{2^2 \times .7854} = 28{,}647.82 \text{ lb. per sq. in.} \quad \text{Ans.}$$

EXPERIMENTAL LAWS.

1107. The following laws have been established by experiment:

1. *When a body is subjected to a small stress, a small strain is produced, and when the stress is removed the body springs back to its original shape. This leads to the conclusion that, for small stresses, bodies are perfectly elastic.*

2. *Within certain limits, the change of shape (strain) is directly proportional to the applied force.*

3. *When the stress is sufficiently great, a strain is produced which is partly permanent; that is, the body does not spring back entirely to its original form when the stress is removed. This lasting part of the strain is called a* **set,** *and in such cases the strain is not proportional to the stress.*

4. *Under a still greater stress, the strain rapidly increases, and the body is finally ruptured or broken.*

5. *A force acting suddenly, as a shock, causes greater injury than a force gradually applied.*

According to the first law, the body will resume its original form when the force is removed, provided the stress is not too great. This property is called *elasticity*. According to the second law, the strain is proportional to the stress within certain limits. Thus, if a pull of 1,000 pounds elongates a body .1″, a pull of 2,000 pounds would elongate it .2″. This is true up to a certain limit, beyond which the body will not resume its original form upon the removal of the stress, but will be permanently strained more or less, according to the amount of stress. The stress at the point where the set begins is called the **elastic limit.** All strains produced by stresses within (less than) the elastic limit are directly proportional to the stresses.

1108. Stresses are equal but opposed to the external forces producing them, and are, therefore, measured and represented by these forces. Thus, as we have explained before, a force of 1,000 pounds produces a stress of 1,000 pounds. The external force is the force applied to a fixed body; the stress is the resistance offered by the body to a change of form; and when the body ceases to change (as when a rod ceases to elongate), the stress just balances the external force.

COEFFICIENT OF ELASTICITY.

1109. Amongst engineers, the term *elasticity* means the *resistance* which a body offers to a permanent change of form; and by *strength*, the resistance which a body offers to division or separation into parts.

It follows from this that those bodies which have the highest elastic limit are the most elastic.

1110. The **coefficient of elasticity** is the ratio of the unit stress to the unit strain, provided the elastic limit is not exceeded. Let S be the unit stress, s the unit strain, and E the coefficient of elasticity; then, by definition, $E = \frac{S}{s}$. Substituting the values of S and s obtained from formulas **65** and **66**,

$$E = \frac{S}{s} = \frac{P}{A} \div \frac{e}{l} = \frac{P\,l}{A\,e}. \qquad \textbf{(67.)}$$

1111. If in this formula we assume $e = l$, and $A = 1$ (1 square inch), then $E = P$. That is, the coefficient of elasticity is that force which, *if stress and strain continued proportional to each other*, would produce in a bar of unit area a strain equal to the original length of the bar ($l = e$). This, however, is never the case, as the elastic limit and the ultimate strength are reached before the applied force reaches the value E.

Example.—A wrought iron bar 2 inches square and 10 feet long is stretched .0528 inch by a stress of 44,000 pounds; what is the coefficient of elasticity?

Solution.—Using formula **67**,

$$E = \frac{P\,l}{A\,e} = \frac{44,000 \times 10 \times 12}{2^2 \times .0528} = 25,000,000 \text{ lb. per sq. in.} \quad \text{Ans.}$$

1112. The **ultimate strength** of any material is that unit stress which is just sufficient to break it.

1113. The **ultimate elongation** is the total elongation produced in a unit of length of the material having a unit of area, by a stress equal to the ultimate strength of the material.

1114. For the same size, quality, and kind of material, the ultimate strength, ultimate elongation, coefficient of elasticity, and elastic limit are the same for different pieces. Table 21 gives the *average* values of the coefficient of elasticity (E_1), elastic limit (L_1), ultimate strength (S_1), and ultimate elongation (s_1), of different materials, the quantities given being for *tension* only. As brick and stone are never used in tension, their values are not given.

TABLE 21.

Material.	Coefficient of Elasticity. E_1.	Elastic Limit. L_1.	Ultimate Tensile Strength. S_1.	Ultimate Elongation. s_1.
	Lb. per Sq. In.	Lb. per Sq. In.	Lb. per Sq. In.	In. per Linear Inch.
Timber	1,500,000	3,000	10,000	0.015
Cast Iron	15,000,000	6,000	20,000	0.005
Wrought Iron	25,000,000	25,000	55,000	0.20
Steel	30,000,000	50,000	100,000	0.10

The values in this table are subject to great variation, and cannot be depended upon in designing machine parts. Thus, the ultimate tensile strength of steel varies from less than 60,000 to more than 180,000 pounds per square inch, according to its purity and the amount of carbon it contains; that of cast iron from 12,000, or 13,000, to over 40,000; wrought iron varies from 40,000 to 72,000, according to quality, the latter value being for iron wire. Timber varies fully as much as, if not more than, any of the three preceding materials, its properties depending upon the kind of wood, its degree of dryness, the manner of drying, etc.

All the problems in this section will be solved by using the average values given in the preceding and following tables; the designer, however, should not use them, but either test the materials himself or state in the specifications what strength the material must have. For example, mild steel, for boiler shells, should have a tensile strength of not less than 60,000 or 65,000 pounds per square inch; Bessemer steel, for steel rails, not less than 110,000; open-hearth steel, for locomotive tires, not less than 125,000, and crucible cast steel, for tools, cutlery, etc., not less than 150,000. It is also customary to specify the amount of elongation. This is necessary because, as a rule, the elongation decreases as the tensile strength increases. Having tested the material about to be used, or having specified the lowest limits, the designer can ascertain the strength and stiffness of construction by means of the formulas and rules which are to follow.

EXAMPLE.—How much will a piece of steel 1 inch in diameter and 1 foot long elongate under a steady load of 15,000 pounds?

SOLUTION.—$E_1 = \frac{P\,l}{A\,e}$, or $e = \frac{P\,l}{A\,E_1}$.

From Table 21, $E_1 = 30,000,000$ for steel; hence,

$$e = \frac{15,000 \times 12}{1^2 \times .7854 \times 30,000,000} = .00764''. \quad \text{Ans.}$$

NOTE.—All lengths given in this treatise on Strength of Materials must be reduced to inches before substituting in the formulas.

EXAMPLE.—A piece of timber has a cross-section 2″ × 4″ and is 6 feet long. A certain stress produces an elongation of .144 inch; what is the value of the stress in pounds?

SOLUTION.—

$$E_1 = \frac{P\,l}{A\,e}, \text{ or } P = \frac{E_1\,A\,e}{l} = \frac{1,500,000 \times 2 \times 4 \times .144}{6 \times 12} = 24,000 \text{ lb.} \quad \text{Ans.}$$

COMPRESSION.

1115. If the length of the piece is not more than five times its least transverse dimension (its diameter, when round; its shorter side, when rectangular, etc.), the laws of compression are similar to those of tension. The strain is proportional to the stress until the elastic limit has been reached; after that, it increases more rapidly than the stress, as in the case of tension. The area of the cross-section is slightly enlarged under compression. In Table 22 are given the average compression values of E, L, and S for wood, brick, stone, cast iron, wrought iron, and steel. (See also Table 21.) E is not given for brick, nor L for cast iron, brick, or stone, because these values are not known. To distinguish between tension and compression when applying a formula, E_2, L_2, and S_2 will be used instead of E_1, L_1, and S_1.

TABLE 22.

Material.	Coefficient of Elasticity. E_2.	Elastic Limit. L_2.	Ultimate Compressive Strength. S_2.
	Lb. per Sq. In.	Lb. per Sq. In.	Lb. per Sq. In.
Timber...........	1,500,000	3,000	8,000
Brick	....	......	2,500
Stone.............	6,000,000	......	6,000
Cast Iron.........	15,000,000	......	90,000
Wrought Iron.....	25,000,000	25,000	55,000
Steel.............	30,000,000	50,000	150,000

1116. When the length of a piece subjected to compression is greater than ten times its least transverse dimension, it is called a *column*, and the material fails by a sidewise bending or flexure. The preceding table is to be used only for pieces whose length does not exceed *five* times the least dimension of the cross-section. (See Art. **1179***a*.)

EXAMPLE.—How much will a wrought iron bar 4 inches square and 15 inches long shorten under a load of 100,000 pounds?

SOLUTION.—$E_2 = \frac{P\,l}{A\,e}$, or $e = \frac{P\,l}{A\,E_2}$.

Hence, $e = \frac{100{,}000 \times 15}{16 \times 25{,}000{,}000} = .00375''$. Ans.

SHEAR.

1117. When two surfaces move in opposite directions very near together in such a manner as to cut a piece of material, or to pull part of a piece through the remainder, the piece is said to be *sheared.* A good example of a shearing stress is a punch; the two surfaces in this case are the bottom of the punch and the top of the die. Another example is a bolt with a thin head; if the pull on the bolt is great enough, it will be pulled through the head and leave a hole in it, instead of the bolt breaking by pulling apart, as would be the case with a thick head. In this case, the two surfaces are the under side of the head and the surface pressed against. Other examples are a knife cutting a piece of wood, and the ordinary shears from which this kind of stress takes its name.

TABLE 23.

Material.	Coefficient of Elasticity. E_3.	Ultimate Shearing Strength. S_3.
Timber (across the grain).....	....	3,000
Timber (with the grain)......	400,000	600
Cast Iron....................	6,000,000	20,000
Wrought Iron................	15,000,000	50,000
Steel........................	....	70,000

1118. Formula **65** applies in cases of shearing stress, but formulas **66** and **67** are never used for shearing. In the preceding table, E_3 and S_3 are used to represent, respectively, the coefficient of shearing elasticity and ultimate shearing strength.

Example.—What force is necessary to punch a one-inch hole in a wrought-iron plate $\frac{3}{8}$ of an inch thick?

Solution.—$1'' \times 3.1416 \times \frac{3}{8}'' = 1.1781$ sq. in. = area of punched surface = area of a cylinder 1″ in diameter and $\frac{3}{8}''$ high. Using formula **65**,

$$P = A\,S_3 = 1.1781 \times 50{,}000 = 58{,}905 \text{ lb.} \quad \text{Ans.}$$

EXAMPLE.—A wooden rod 4 inches in diameter and 2 feet long is turned down to 2 inches diameter in the middle so as to leave the enlarged ends each 6 inches long. Will a steady stress pull the rod apart in the middle, or shear the ends?

SOLUTION.—$P = A\ S_3 = 2 \times 3.1416 \times 6 \times 600 = 22{,}620$ lb. to shear off the ends.

The force required to rupture by tension is

$$P = A\ S_1 = 2^2 \times .7854 \times 10{,}000 = 31{,}416 \text{ lb.}$$

Since the former is only about $\frac{3}{4}$ of the latter, the piece will fail through the shearing off of the end. Ans.

Had a transverse stress been used, the force necessary to shear off a section of the end would have been

$$4^2 \times .7854 \times 3{,}000 = 37{,}700 \text{ lb.}$$

FACTORS OF SAFETY.

1119. It was previously stated that no stress should ever be applied to a machine part that would strain it beyond the elastic limit. The usual practice is to divide the ultimate strength of the material by some number depending upon the kind and quality of the material, and upon the nature of the stress; this number is called a **factor of safety.**

The factor of safety for any material is the ratio of its ultimate strength to the actual stress to which it is subjected, or for which it is intended.

In Table 23, 70,000 pounds per square inch is given as the ultimate shearing strength for steel. Now, suppose that the actual stress on a piece of steel is 10,000 pounds per square inch; then, the factor of safety for this piece would be $\frac{70{,}000}{10{,}000} = 7$.

1120. To find the proper allowable working strength of a material, divide the ultimate strength for tension, compression, or shearing, as the case may be, by the proper factor of safety.

Table 24 gives the factors of safety generally used in American practice. Factors of safety will always be denoted by the letter f in the formulas to follow.

TABLE 24.

Material.	For Steady Stress. (Buildings.)	For Varying Stress. (Bridges.)	For Shocks. (Machines.)
Timber	8	10	15
Brick and Stone	15	25	30
Cast Iron	6	10	15
Wrought Iron	4	6	10
Steel	5	7	10

1121. *Twice as much strain is caused by a suddenly applied stress as by one that is gradually applied.* For this reason a larger safety factor is used for shocks than for steady stresses. In general, the factor of safety for a given material must be chosen according to the nature of the stress.

The designer usually chooses his own factors of safety. If the material has been tested, or the specifications call for a certain strength, then the factor of safety can be chosen accordingly.

Example.—Assuming the mortar and brick to be of the same strength, how many tons could be safely laid upon a brick column 2 feet square and 8 feet high?

Solution.—$P = A\ S_2 = 2 \times 2 \times 144 \times 2{,}500 = 1{,}440{,}000$ lb. = 720 tons. The factor of safety for this case is 15 (see Art. **1120** and Table **24**); hence, $720 \div 15 = 48$ tons. Ans.

Example.—What must be the diameter of the journals of a wrought iron locomotive axle to resist shearing safely, the weight on the axle being 40,000 pounds?

Solution.—Let f be the factor of safety; then, $P = \frac{A\ S_3}{f}$, or $A = \frac{P f}{S_3}$. Since the axle has two journals, the stress on each journal is 20,000 lb. Owing to inequalities in the track, the load is not a steady one, but varies; for this reason, the factor of safety will be taken as 6. Then, $A = \frac{20{,}000 \times 6}{50{,}000} = 2.4$ sq. in. Therefore, $d = \sqrt{\frac{2.4}{.7854}} = 1\frac{3}{4}''$. Ans.

Example.—Considering the piston rod of a steam engine as if its length were less than ten times its diameter, what must be the diameter of a steel rod, if the piston is 18 inches in diameter and the steam pressure is 110 pounds per square inch ?

Solution.—Area of piston is $18^2 \times .7854 = 254.47$ sq. in. $254.47 \times 110 =$ 27,991.7 lb., or, say, 28,000 lb. = stress in the rod. $A = \frac{Pf}{S_2} = \frac{28,000 \times 10}{150,000}$ $= 1.87''$, nearly. Hence, diameter $= \sqrt{\frac{1.87}{.7854}} = 1.543''$, say $1\frac{9}{16}''$. Ans.

1122. When designing a machine, care should be taken (1) *to make every part strong enough to resist any stress likely to be applied to it; and* (2) *to make all parts of equal strength.* The reason for the first statement is obvious, and the second should be equally clear, since no machine can be stronger than its weakest part (proportioned, of course, for the stress it is to bear), and those parts of the machine which are stronger than others contain an excess of material which is wasted. In actual practice, however, this second rule is frequently modified. Some machines are intended to be massive and rigid, and need an excess of material to make them so; in others, there are difficulties in casting that modify the rule, etc., etc. In most cases, the designer must rely on his own judgment.

EXAMPLES FOR PRACTICE.

1. A cast iron bar is subjected to a steady tensile stress of 120,000 pounds. The cross-section is an ellipse whose axes are 6 and 4 inches. (*a*) What is the stress per square inch ? (*b*) What load will the bar carry with safety ?

Ans. { (*a*) 6,366 lb. per sq. in.
(*b*) 62,832 lb.

2. How much will a piece of steel 2 inches square and 10 inches long shorten under a load of 300,000 pounds ? Ans. .025".

3. A cylindrical wooden pin 1¾ inches in diameter is subjected to a double shearing stress. If the stress is suddenly applied, what total force is necessary to shear the pin ? Ans. 14,431 lb.

4. A wrought iron tie rod is ¾ inch diameter; how long must it be to lengthen ⅜ inch under a steady pull of 5,000 pounds ? Ans. 69 ft.

5. A steel bar having a cross-section of 5" × 4" and 14 feet long is lengthened .036 inch by a steady pull of 120,000 pounds; what is its coefficient of elasticity ? Ans. 28,000,000 lb. per sq. in.

6. Which is the stronger, weight for weight, a bar of chestnut wood whose tensile strength is 12,000 pounds per square inch and specific gravity .61, or a bar of steel whose tensile strength is 125,000 pounds per square inch?

7. What should be the diameter of a cast-iron pin subjected to a suddenly applied double shearing stress of 40,000 pounds to withstand the shocks with safety? Ans. 4⅜″, nearly.

8. What safe steady load may be placed upon a brick column 2 feet square and 9 feet high? Ans. 96,000 lb.

PIPES AND CYLINDERS.

1123. A pipe or cylinder subjected to a pressure of steam or water is strained equally in all its parts, and, when rupture occurs, it is in the direction of its length.

Let d = inside diameter of pipe in inches;
l = length of pipe in inches;
p = pressure in pounds per square inch;
P = total pressure.

Then, $P = p\,l\,d$.

This formula is derived from a principle of hydrostatics that the pressure of water in any direction is equal to the pressure on a plane perpendicular to that direction. In Fig. 214, suppose the direction of pressure to be as shown by the arrows; $A\ B$ would then be the plane perpendicular to this direction, the width of the plane being equal to the diameter, and the length equal to the length of the pipe. The area of the plane would then be $l \times d$, and the total pressure $P = p \times l \times d$, as above.

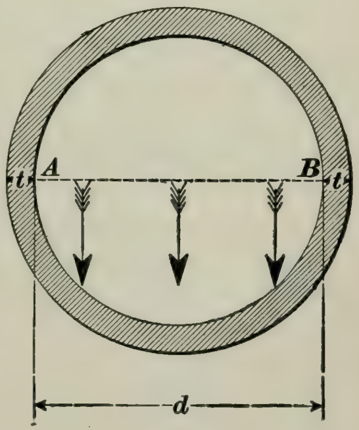


FIG. 214.

Suppose the pipe to have a thickness t, and let S be the working strength of the material; then, the resistance of the pipe on *each* side is $t\,l\,S$. Resistance must equal pressure; therefore, $p\,l\,d = 2\,t\,l\,S$, or

$$p\,d = 2\,t\,S. \qquad \textbf{(68.)}$$

Also, $p \frac{2tS}{d}$, which is the maximum pressure a pipe of a given material and of given dimensions can stand.

The pressure of water per square inch may be found by the formula, $p = .434\ h$, where h is the head in feet. In pipes where shocks are likely to occur, the factor of safety should be high. The thickness of a pipe to resist a given pressure varies directly as its diameter, the pressure remaining constant.

Example.—Find the factor of safety for a cast-iron water pipe 12 inches in diameter and $\frac{3}{4}$ inch thick, under a head of 350 feet.

Solution.—Here p, pressure per square inch, equals $.434\ h = .434 \times 350 = 151.9$ lb. Substituting, in formula **68**, the values given,

$$151.9 \times 12 = 2 \times \tfrac{3}{4} \times S, \text{ or } S = 1{,}215.2 \text{ lb. per sq. in.}$$

In Table 21, Art. **1114**, the ultimate tensile strength of cast iron is given as 20,000 pounds per square inch; then, the factor of safety is $f = \frac{20{,}000}{1{,}215.2} = 16 +$. Ans. The pipe would, therefore, be secure against shocks.

Example.—Find the proper thickness for a wrought-iron steam pipe 18 inches in diameter to resist a pressure of 140 pounds per square inch.

Solution.—Using a factor of safety of 10, the working strength $S = \frac{55{,}000}{10} = 5{,}500$ lb. per sq. in. From formula **68**, $t = \frac{p\,d}{2S} = \frac{140 \times 18}{2 \times 5{,}500} = 0.23$ in. In practice, however, the thickness is made somewhat greater than the formula requires.

CYLINDERS.

1124. The tendency of a cylinder subjected to internal pressure is to fail or rupture in the direction of its length, the same as a pipe.

Let π (pronounced *pi*) be the ratio of the circumference of a circle to its diameter $= 3.1416$; then, $\frac{1}{4}\pi = .7854 =$ ratio of area to the square of the diameter.

The total pressure on the cylinder head $= \frac{1}{4}\pi d^2 p$. Let $S =$ working unit stress $= \frac{S_1}{f}$, then $\pi\, d\, t\, S =$ the resistance to rupture caused by the pressure acting on the opposite cylinder heads and tending to elongate the cylinder. Since the resistance must equal the force, or pressure, $\frac{1}{4}\pi d^2 p = \pi d t S$, or

$$p d = 4 t S. \qquad \textbf{(69.)}$$

$$\text{Also, } p = \frac{4 t S}{d}.$$

Since, for longitudinal rupture, $p = \frac{2 t S}{d}$, it is seen that a cylinder is twice as strong against transverse rupture as against longitudinal rupture. Hence, other things being equal, the cylinder will always fail by longitudinal rupture.

1125. The foregoing formulas are for comparatively thin pipes and cylinders, in which the thickness is less than about $\frac{1}{10}$ inside radius. For pipes and cylinders whose thickness is greater than $\frac{1}{10}$ radius, use the following formula, in which r = the inner radius, and the other letters have the same meaning as before:

$$p = \frac{S t}{r + t}. \qquad \textbf{(70.)}$$

Substituting the values given in the example in Art. **1123,** in formula **70,** instead of formula **68,**

$$p = \frac{S t}{r + t}, \text{ or } S = \frac{p(r + t)}{t} = \frac{151.9 \times (6 + \frac{3}{4})}{\frac{3}{4}} =$$
$$151.9 \times 6\tfrac{3}{4} \times \tfrac{4}{3} = 1{,}367.1 \text{ lb.}$$

When formula **68** was used, $S = 1{,}215.2$ lb.; hence, formula **70** gives, for this case, a value 12½%, or ⅛ greater.

The formula for spheres is the same as that for transverse rupture of cylinders, or $p d = 4 t S$.

1126. A cylinder under external pressure is theoretically in a similar condition to one under internal pressure, so long as its cross-section remains a true circle. A uniform internal pressure tends to preserve the true circular form, but an external pressure tends to increase the slightest variation from the circle, and to render the cross-section elliptical. The distortion, when once begun, increases rapidly, and failure occurs by the collapsing of the tube rather than by the crushing of the material. The flues of a steam boiler are the most common instances of cylinders subjected to external pressure.

The letters having the same meaning as before, the following formula gives the collapsing pressure in pounds per square inch for wrought-iron pipe:

$$p = 9,600,000 \frac{t^{2.18}}{l\,d}. \qquad \textbf{(71.)}$$

Example.—What must be the thickness of a boiler tube 2 inches in diameter and 11 feet long, if the steam pressure is to be not over 160 pounds per square inch?

Solution.—Using formula **71**, with a factor of safety of 10, and solving for t,

$$t = \sqrt[2.18]{\frac{10\,p\,l\,d}{9,600,000}} = \sqrt[2.18]{\frac{10 \times 160 \times 11 \times 12 \times 2}{9,600,000}} = \sqrt[2.18]{\frac{11}{250}}.$$

Hence, $\log t = \frac{\log 11 - \log 250}{2.18} = \bar{1}.37773$, or $t = .2386''$, say $\frac{1}{4}''$.

EXAMPLES FOR PRACTICE.

1. What must be the thickness of a 16-inch cast-iron stand pipe which is subjected to a head of water of 250 feet? Assume that the stress is steady. Ans. .26″.

2. What should be the thickness of a wrought-iron boiler flue 15 feet long, 4 inches in diameter, and subjected to an external pressure of 200 pounds per square inch? Ans. .42″.

3. What pressure per square inch can be safely sustained by a cast-iron cylinder 12 inches in diameter and 3 inches thick? Ans. 1,111$\frac{1}{9}$ lb. per sq. in.

4. What external pressure per square inch can a wrought iron pipe 20 feet long, 3 inches in diameter, and $\frac{3}{8}$ inch thick, safely sustain and be secure against shocks? Ans. 157.2 lb. per sq. in.

5. A cast-iron cylinder 14 inches in diameter sustains a total pressure of 125 tons; what is the necessary thickness, assuming that the pressure is gradually applied, and that the cylinder is not subjected to shocks? Ans. 6.65″.

6. A cylindrical boiler shell 3 feet in diameter is subjected to a steady hydrostatic pressure of 180 pounds per square inch. What should its thickness be if made of steel having a tensile strength of 60,000 pounds per square inch? Ans. .27″.

ELEMENTARY GRAPHICAL STATICS.

Before taking up the subject of flexure, some fundamental principles of Graphical Statics not heretofore considered will be explained and applied to the case of beams.

FORCE DIAGRAM AND EQUILIBRIUM POLYGON.

1127. In Arts. **878** and **879,** the polygon of forces was used to find the resultant of several forces having a common point of application, or whose lines of action passed through a common point. A method of finding the resultant will now be given when the forces lie, or may be considered as lying, in the same plane, but their lines of action do not pass through a common point.

In Fig. 215, let F_1, F_2, and F_3 be three forces whose magnitudes are represented by the lengths of their respective lines, and their directions by the positions of the lines and by the arrow-heads. Construct the polygon of forces *O 1 2 3 O* as shown at (*d*), in the same manner as described in Art. **878,** *O 3* representing the direction and magnitude of the resultant. Everything is now known except the line on

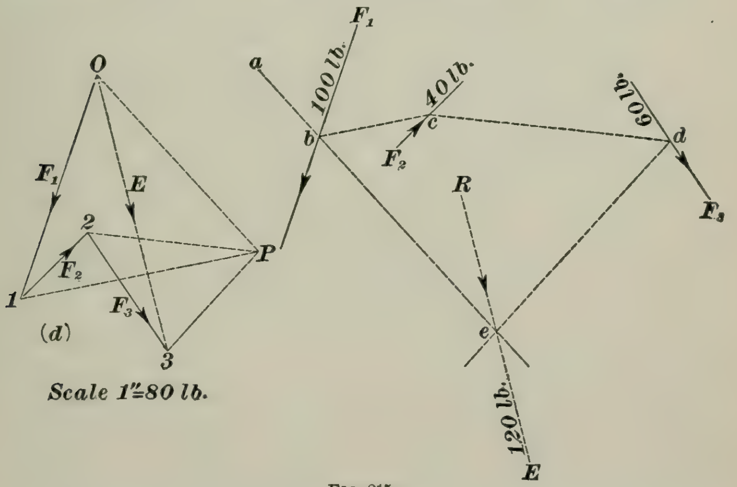


FIG. 215.

which the point of application of the resultant must lie. To find this, proceed as follows:

Choose *any* point P, and draw $P\,O$, $P\,1$, $P\,2$, and $P\,3$. Choose any point b, on the line of direction of one of the forces as F_1, and draw lines through b parallel to $P\,O$ and $P\,1$, the latter intersecting F_2, or F_2 prolonged, in c. Draw $c\,d$ parallel to $P\,2$, and intersecting F_3, or F_3 prolonged, in d. Draw $d\,e$ parallel to $P\,3$, intersecting the line $a\,b\,c$, parallel to $P\,O$, in e. The point e is a point on the line of direction of the resultant of the three forces. Hence, through e, draw R parallel and equal to $O\,3$ and acting in the same direction; it will be the resultant.

The method just described is applicable to any number of forces considered as acting in the same plane. The resultant can also be found when the forces act in different planes, but the method of finding it will not be described here.

The point P is called the **pole;** the lines $P\,O$, $P\,1$, $P\,2$, $P\,3$ joining the pole with the vertexes of the force polygon are called the **strings** or **rays;** the **force diagram** is the figure composed of the force polygon, $O\,1\,2\,3\,O$, the pole, and the strings. The polygon $b\,e\,d\,c\,b$ is called the **equilibrium polygon.**

Since the pole P may be taken anywhere, any number of force diagrams and equilibrium polygons may be drawn, all of which will give the same value for the resultant, and whose lines $d\,e$ and $a\,e$ will intersect on the resultant R. To test the accuracy of the work, take a new pole and proceed as before. If the work has been done correctly, $d\,e$ and $a\,e$ will intersect on R.

The equilibrium polygon gives an easy method of resolving a force into two components.

EXAMPLE.—In Fig. 216, let $F = 16$ pounds be the force, and let it be required to resolve it into two *parallel* components, A and B, at distances respectively of 5 feet and 15 feet from F. What will be the magnitudes of A and B?

SOLUTION.—Draw $O\,1$ to represent $F = 16$ lb. Choose any convenient pole P, and draw the rays $P\,O$ and $P\,1$. Take any point a on F, and draw *a b* parallel to *P O*, intersecting *A* in *b*, and *a c* parallel to *P 1*, intersecting *B* in *c*. Join *b* and *c* by the line *b c*. Through the pole *P*, draw *P d* parallel to *b c*, intersecting *O 1* in *d*. Then *O d* is the magnitude of *A*, measured to the scale to which *O 1* was drawn, and *d 1* is the magnitude of *B* to the same scale.

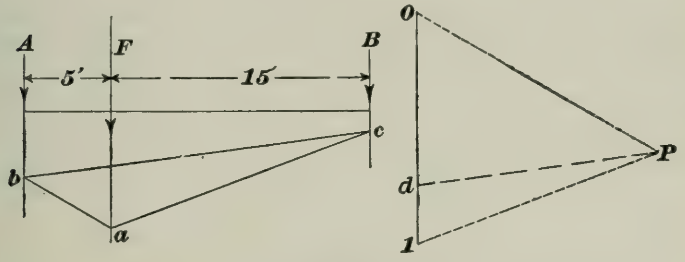


FIG. 216.

1128. If the components are not parallel to the given force, they must intersect its line of direction in a common point.

In Fig. 217, let $F = 16$ pounds be the force; it is required to resolve it into two components *A* and *B*, intersecting at *a*, as shown. Draw *O 1* to some convenient scale equal to 16 pounds; then draw *O P* and *1 P* parallel to *A* and *B*, and *O P* and *P 1* are the values of the components *A* and *B*, respectively, both in magnitude and direction.

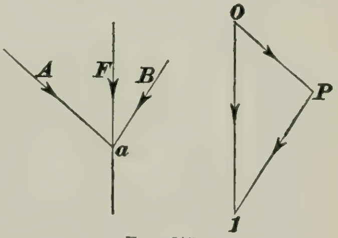


FIG. 217.

EXAMPLE.—Let F_1, F_2, F_3, F_4, and F_5, Fig. 218, be five forces whose magnitudes are 7, 10, 5, 12, and 15 pounds, respectively. It is required to find their resultant and to resolve this resultant into two components parallel to it and passing through the points *a* and *b*.

SOLUTION.—Choose any point *O*, Fig. 218, and draw *O 1* parallel and equal to F_1; *1–2* parallel and equal to F_2, etc.; *O 5* will be the value of the resultant, and its direction will be from *O* to *5*, opposed to the other forces acting around the polygon. Choose a pole *P*, and complete the force diagram. Choose a point *c* on F_1, and draw the equilibrium polygon *c d e f g h c*; the intersection of *c h*, parallel to *P O*, and *g h*, parallel to *P 5*, gives a point *h*, on the resultant *R*. Through *h*, draw *R* parallel to *O 5*, and it will be the position of the line of action of the resultant of the five forces. The components must pass through the points *a* and *b*, according to the conditions; hence, through *a* and *b*, draw V_1 and V_2 parallel to *R*. Since *O 5* represents the magnitude of *R*, draw *h k* and *h l* parallel to *P O* and *P 5*, respectively, as in Fig. 216 (they, of course, coincide with *c h* and *g h*, since the same pole *P* is used), intersecting V_1 and V_2 in *k* and *l*. Join *k* and *l*, and draw *P Q* parallel to *k l*. Then, $O\,Q = V_1$, and $Q\,5 = V_2$.

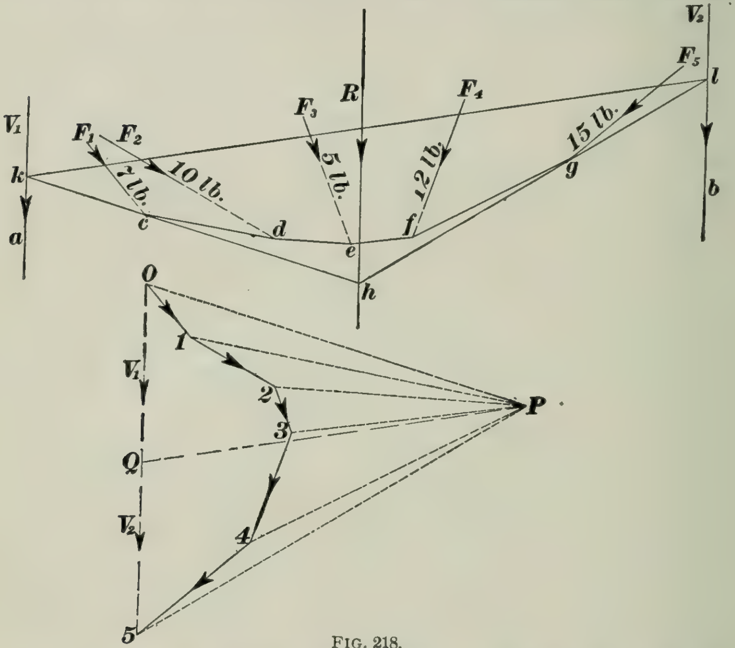


FIG. 218.

COMPOSITION OF MOMENTS.

1129. In Art. **906** it was stated that the moment of a force about a point is the product of the magnitude of the force by the perpendicular distance from the point to the line of action of the force. A force can act in two ways upon a body: it can either produce a motion of translation—that is, cause all the points of the body to move in straight parallel lines—or it can produce a motion of rotation—that is, make the body turn. A *moment* measures the capacity of a force to produce rotation about a given point. For example, suppose, in Fig. 219, that $A\ C$ is a lever 30 inches long, having a fulcrum at B 10 inches from A. If a weight is suspended from C, it will cause the bar to rotate about B in the direction of the arrow. A weight suspended from A will cause it to revolve in the opposite direction, as indicated by the arrow. Suppose, for simplicity, that the bar itself weighs nothing. If two weights of 12 pounds each are hung at A and C, it is evident that the bar will revolve in the direction of the arrow at C, on account of the arm $B\ C$ being longer than the arm $A\ B$. Let the weight at A be increased until it equals 24 pounds. The bar will then balance exactly, and any additional weight at A will cause the bar to rotate in the opposite direction, as shown by the arrow at that point. When the lever is balanced, it will be found that $24 \times 10 = 12 \times 20$, or, considering B as the center of moments, $24 \times$ perpendicular distance $A\ B = 12 \times$ perpendicular distance $B\ C$. In other words, the moment of W about B must equal the moment of P about B—that is, the two moments must be equal. Further, P tends to cause rotation in the direction in which the hands of a watch move, and will, for convenience, be considered positive, or $+$; W tends to cause rotation in a direction opposite to the hands of a watch, and will be considered negative, or $-$. Adding the two algebraically, $P \times B\ C + (-W \times A\ B) = P \times B\ C - W \times A\ B = 0$, since the two moments are equal. Hence the following general

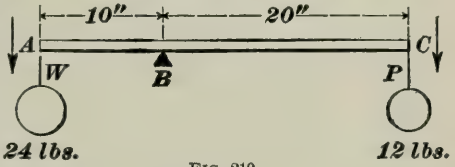


FIG. 219.

Rule.—*One of the necessary conditions of equilibrium is that the* **algebraic** *sum of the moments of all the forces about a given point should equal zero.*

Applications of this rule will occur farther on.

GRAPHICAL EXPRESSIONS FOR MOMENTS.

1130. The moment of a single force may be expressed graphically in the following manner: Let $F = 10$ pounds be the given force (see Fig. 220), and c, at a distance from

$F = f\,c = 7\frac{1}{2}$ feet, be the center of rotation (center of moments). Draw $O\,1$ parallel to F and equal to 10 pounds. Choose any point P as a pole, and draw the rays $P\,O$ and $P\,1$; also draw $P\,2$ perpendicular to $O\,1$. Through any point b on F, draw the sides $a\,b$ and $g\,b$ of the equilibrium polygon; they correspond to $b\,c$ and $d\,c$, Fig. 215, through the intersection of which the resultant must pass, the resultant F being given in the present case. Prolong $a\,b$, and draw $e\,d$ through c parallel to F, intersecting $b\,g$ and $a\,b$ in d and e. It can now be shown that the moment of F about $c = P\,2 \times d\,e$, when $P\,2$ is measured to the same scale as $O\,1$, and $e\,d$ is measured to the same scale as $c\,f$.

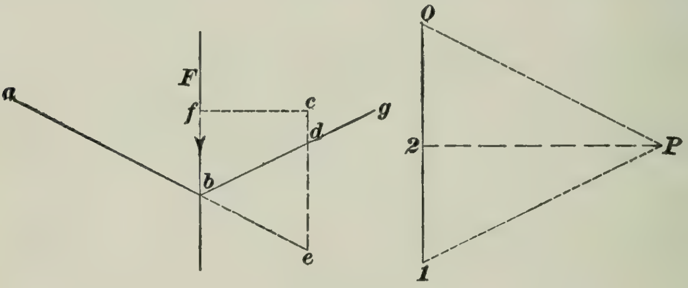


FIG. 220.

The line $P\,2$ is called the **pole distance** and will hereafter be always denoted by the letter H. The line $d\,e$ is called the **intercept.** Hence, the pole distance multiplied by the intercept equals the moment, or, denoting the intercept by y, moment $= H\,y$.

The statement made in the last sentence is one of the most important facts in Graphical Statics, and should be thoroughly understood. In the triangle $P\,O_1$, the lines $P\,O$ and P_1 represent the components of the force F in the directions $a\,b$ and $g\,b$, respectively, while the lines of action of those components are $a\,b$ and $g\,b$, meeting at b. As $d\,e$ is limited by $g\,b$ and $a\,b$ (produced), we may give the following definition: The *intercept* of a force whose moment about a point is to be found is the segment (or portion) which the two components (produced, if necessary) cut off from a line drawn through the center of moments parallel to the direction of the force.

1131. The pole distance and intercept for the moment of several forces about a given point may be determined in a similar manner, by first finding the magnitude and position of the resultant of all the forces; the moment of this resultant about the given point will equal the value of the resultant moment of all the forces which tend to produce rotation about that point.

EXAMPLE.—Let $F_1 = 20$ pounds, $F_2 = 25$ pounds, and $F_3 = 18$ pounds be three forces acting as shown in Fig. 221; find their resultant moment about the point *C*.

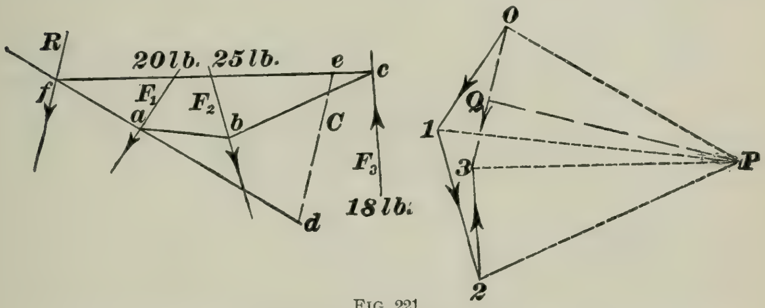


FIG. 221.

SOLUTION.—Draw the force diagram, equilibrium polygon, and resultant *R* as previously described. Draw *d C e* parallel to *R*. The intercept *d e*, multiplied by the pole distance $P\,Q =$ the resultant moment.

1132. If all the forces are parallel, the force polygon is a straight line. This is evident, since if a line of the force polygon be drawn parallel to one of the forces, and from one end of this line a second line be drawn parallel to another force, the second line will coincide with the first.

EXAMPLE.—Let $F_1 = 30$, $F_2 = 20$, and $F_3 = 20$, all in pounds, be three parallel forces acting downwards, as shown in Fig. 222. It is required to find their resultant moment (algebraic sum of the moments) and the moment of their resultant, all moments to be taken about the point *C*.

SOLUTION.—Lay off $O\,1 = 30$ lb. $= F_1$, $1\text{–}2 = 20$ lb. $= F_2$ and $2\text{–}3 = 20$ lb. $= F_3$, and $O\,3$ is the value of the resultant. Choose some point P as a pole and draw the rays. Take any point, as b, on any force, as F_1, and complete the equilibrium polygon $b\,c\,d\,e\,b$; then, the line of action of the resultant must pass through e. Through C draw $C\,i$ parallel to R and prolong $d\,e$. The moment of R about C equals the pole distance H, multiplied by the intercept $h\,i$, since $h\,i$ is that part of the line drawn through C parallel to R, and included between the lines $b\,e$ and $d\,e$ of the equilibrium polygon which meet upon R. Measuring $h\,i$ to the scale of distances (1 in. = 40 ft.), it equals 23 ft. Measuring H to the scale of forces, it equals 40 lb. Consequently, the moment of R about $C = 40 \times 23 = 920$ ft.-lb. Considering the force F_3, the intercept is $g\,i$, since F_3 is parallel to R, and, consequently, to $C\,i$; also, $g\,i$ is that part of the line $C\,i$ included between the sides $c\,d$ and $d\,e$, which meet on F_3. Measuring $g\,i$, it is found to equal 28 ft. Hence, the moment of F_3 about $C = 40 \times 28 = 1{,}120$ ft.-lb. The moment of F_2 about $C = H \times f\,g = 40 \times 13 = 520$ ft.-lb. The moment of $F_1 = H \times f\,h = 40 \times 18 = 720$ ft.-lb. Now F_3 and F_2 have positive moments, since they tend to cause rotation in the direction of the hands of a watch, while F_1 has a negative moment, since it tends to cause rotation in the opposite direction. Consequently, adding the moments algebraically, the resultant moment $= 1{,}120 + 520 - 720 = 920$ ft.-lb., the same as the moment of the resultant.

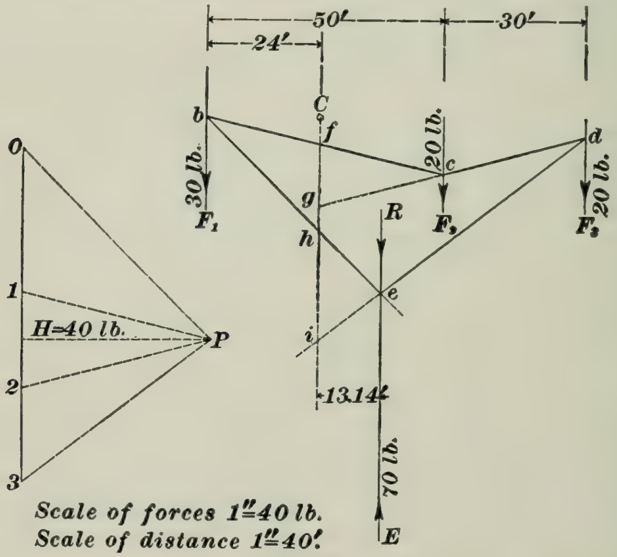


FIG. 222.

Having described the fundamental principles of Graphical Statics, the subject of Strength of Materials will now be continued, and the stresses due to flexure and torsion discussed.

BEAMS.

1133. Any bar resting upon supports in a horizontal position is called a **beam.**

1134. A **simple beam** is a beam resting upon two supports very near its ends.

1135. A **cantilever** is a beam resting upon one support in its middle, or which has one end fixed (as in a wall) and the other end free.

1136. A **restrained beam** is one which has both ends fixed (as a plate riveted to its supports at both ends).

1137. A **continuous beam** is one which rests upon more than two supports.

In this Course, the continuous beam will not be discussed, as the subject requires a knowledge of higher mathematics.

REACTIONS OF SUPPORTS.

1138. All forces which act upon beams will be considered as vertical, unless distinctly stated otherwise. According to the third law of motion, every action has an equal and opposite reaction. Hence, when a beam is acted upon by downward forces, the supports react upwards. It is required to find the value of the reaction at each support. If a simple beam is uniformly loaded or has a load in the middle, it is evident that the reaction of each support is one-half the load plus one-half the weight of the beam. If the load is not uniformly distributed over the beam, or if the load or loads are not in the middle, the reactions of the two supports will be different. The upward reactions are considered positive, and the downward forces negative. In order that a beam may be in equilibrium, three conditions must be fulfilled:

I. The algebraic sum of all vertical forces = 0.

II. The algebraic sum of all horizontal forces = 0.

III. The algebraic sum of the moments of all forces about any point = 0.

Since the loads act downwards and the reactions upwards, the first condition states that the sum of all the loads must equal the sum of the reactions of the supports.

EXAMPLE.—Let R_1 be the reaction of the left support, R_2 the reaction of the right support, and the distance between the two supports 14 feet. Suppose that loads of 50, 80, 100, 70, and 30 pounds are placed on the beam at distances from the left support equal to 2, 5, 8, 10, and 12½ feet, respectively. Required the reactions of the supports, neglecting the weight of the beam. See Fig. 223.

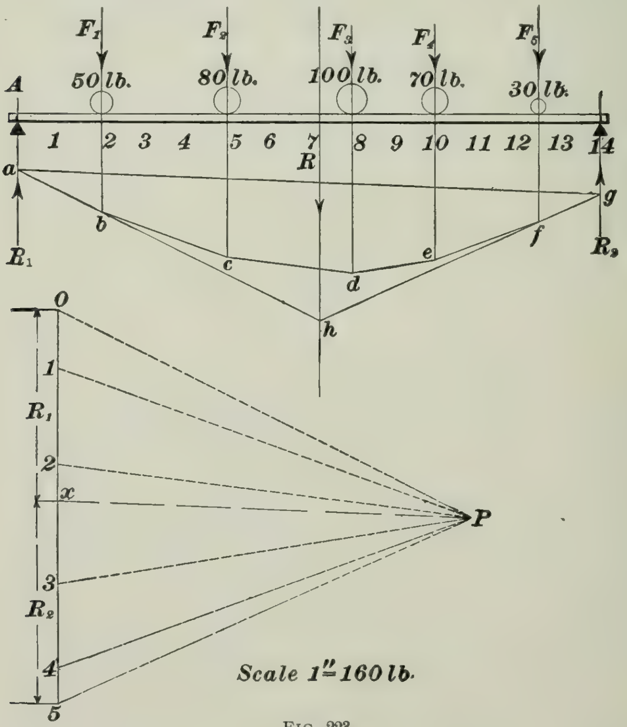


FIG. 223.

SOLUTION.—The reactions may be found graphically by resolving the resultant of the weights (which, in this case, acts vertically downwards) into two parallel components, passing through the points of support. The sum of the reactions is equal to the sum of the components, but the two sums have different signs. Draw the beam to some convenient scale, and locate the loads as shown in the figure. Draw the force diagram, making *O 1* = 50 lb., *1-2* = 80 lb., etc., *O 5* representing the force polygon.

Choose a point *b* on the line of action of the force F_1, and draw the equilibrium polygon *a b c d e f g a; a b* and *f g* intersect at *h*, the point through which the resultant *R* must pass. Draw *P x* parallel to *a g*, and *O x* will be the reaction (= component) R_1, and *x 5* the reaction R_2. Measuring *O x* and *x 5* to the same scale used to draw *O 5*, we find $R_1 = 161$ lb., and $R_2 = 169$ lb. By calculation, $R_1 = 161.4$ lb., and $R_2 = 168.6$ lb. This shows that the graphical method is accurate enough for all practical purposes. The larger the scale used, the more accurate will be the results.

The reactions and forces acting upwards will always be considered as positive, or +, and the downward weights as negative, or −. It is plain that the first condition of equilibrium is satisfied when the sum of the positive forces and reactions is equal to the sum of the negative forces.

THE VERTICAL SHEAR.

1139. In Fig. 223, imagine that part of the beam at a minute distance to the left of a vertical line passing through the point of support *A*, to be acted upon by the reaction R_1 = 160 pounds, and that part to the right of the line to be acted upon by the equal downward force due to the load. The two forces acting in opposite directions tend to shear the beam.

Suppose the line had been situated at the point marked *3* instead of at *A*; the reaction upwards would then be partly counterbalanced by the 50-pound weight, and the total reaction at this point would be 160 − 50 = 110 pounds. Since the upward reaction must equal the downward load at the same point, the downward force at *3* also equals 110 pounds, and the shear at this point is 110 pounds. At the point *6*, or any point between *5* and *8*, the downward force due to the weight at the left is 50 + 80 = 130 pounds, and the upward reaction is 160 pounds. The resultant shear is therefore, $160 - 130 = 30$ pounds. At any point between *8* and *10*, the shear is $160 - (50 + 80 + 100) = -70$ pounds. The negative sign means nothing more than that the weights exceed the reaction of the left-hand support.

The vertical shear equals the reaction of the left-hand support, minus all the loads on the beam to the left of the point considered.

For a simple beam, the greatest positive shear is at the left-hand support, and the greatest negative shear is at the right-hand support, and both shears are equal to the reactions at those points.

1140. The vertical shear may be represented graphically, as shown in Fig. 224, which is Fig. 223 repeated. Draw the force diagram, continue the lines of action of R_1 and

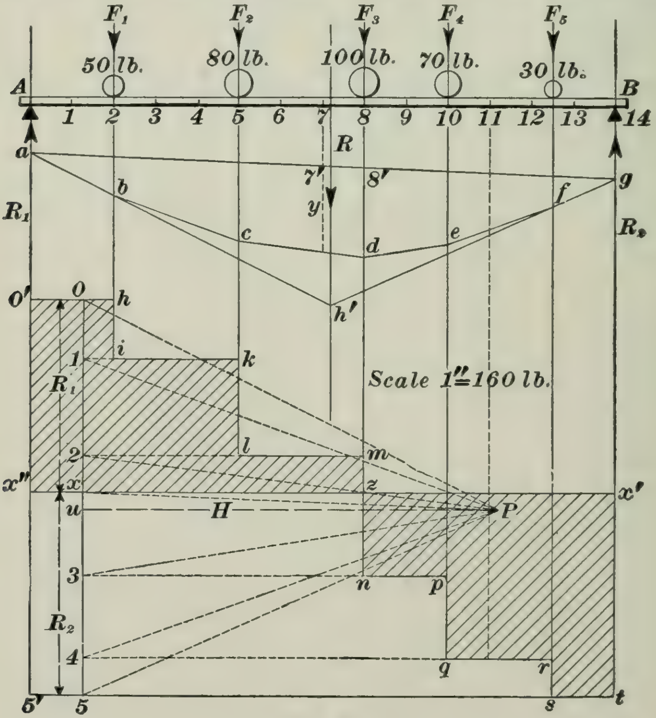


FIG. 224.

R_2 downwards and make $O'\,5' = O\,5$. Through x draw the horizontal line $x\,x'$, called the **shear axis.** The vertical shear is the same for any point between A and 2 and $= O\,x = O'\,x'' = 160$; hence, draw $O'\,h$ parallel to $x\,x'$, and any ordinate measured from $x\,x'$ to $O'\,h$, between O' and $h =$ 160 pounds = the vertical shear at any point between O' and h when $O'\,h = A\,2$, Through 1, draw $1\,k$ and project the points 2 and 5 upon it, in i and k. Then, the length of the ordinate between $x\,x'$ and $i\,k =$ the vertical shear between 2 and 5. In the same way, find the remaining points l, m, etc. The broken line $O'\,h\,i$..............t is called the **shear line;** and the figure $O'\,h\,i$______$r\,s\,t\,x'\,x''$ is called the **shear diagram.** To find the shear at any point, as 11, project the point upon the shear axis and measure the ordinate to the shear line, drawn through the projected point. If the ordinate is measured from the shear axis upwards, the vertical shear is positive; if downwards, it is negative. For the point 11, the vertical shear $= -140$ lb. The maximum negative vertical shear is -170 lb. $= x'\,t = x\;5$. The greatest shear, whether positive or negative, is the one which the beam must be designed to withstand.

1141. A beam seldom fails through shearing, but generally breaks by reason of the load bending and breaking it; that is, through flexure. In order to design a beam to resist flexure, the greatest (maximum) bending moment must be known.

The ***bending moment*** *at any point of a beam is the algebraic sum of the moments of all of the forces (the reaction included) acting upon the beam, on either side of that point, the point being considered as the center of moments.*

The expression "algebraic sum" refers to the fact that, when considering the forces acting at the left of the point taken, the moments of all the forces acting upwards are considered positive, and the moments of all the forces acting downwards are considered negative. Hence, the algebraic sum is the moment of the left reaction about the given point, minus the sum of the moments about the same point of all the downward forces between the reaction and the given point. Should there be any force or forces acting upwards, their moments must be *added*, since they are positive. If the forces on the right of the point are considered, all lever arms are negative, distances to the left of the point being +, to the right, − (see Art. **1129**). Hence, the downward forces give positive moments, and the right reaction gives a negative moment. This is as it should be, for the downward forces on the right and the upward forces on the left tend to rotate the beam in the direction of the hands of a watch, while the downward forces on the left and the upward forces on the right tend to rotate the beam in the opposite direction.

1142. To find the bending moment for any point of a beam, as 7 in Fig. 224, by the graphical method, draw the vertical line 7-7′ through the point. Let $y =$ that part of the line included between $a\ g$ and $a\ c\ f\ g$ of the equilibrium polygon (= vertical intercept). Then, $H \times y =$ the bending moment. H, of course, = the pole distance $= P\ u$. For any other point on the beam, the bending moment is found in the same manner—i. e., by drawing a vertical line through the point and measuring that part of it included between the upper and lower lines of the equilibrium polygon. The scale to which the intercept y is measured is the same as that used in drawing the length $A\ B$ of the beam. The pole distance H is measured to the same scale as $O\ 5$. In the present case, $y = 2.05$ feet, and $H = 349$ pounds; hence, the bending moment for the point 7 is $H\ y = 349 \times 2.05 = 715.45$ foot-pounds.

NOTE.—The expression "foot-pounds," used in stating the value of a moment, must not be confounded with foot-pounds of work. The former means simply that a force has been multiplied by a distance, while the latter means that a resistance has been overcome through a distance. In expressing the value of a moment, the force is usually measured in pounds or tons, and the distance in inches or feet; hence, the moment may be inch-pounds or inch-tons and foot-pounds or foot-tons. Unless otherwise stated, the bending moment will always be expressed in inch-pounds, the length of the beam being always measured in inches, and, consequently, also the length of the intercept y.

1143. If expressed in inch-pounds, the value of the moment just found is $715.45 \times 12 = 8,585.4$ inch-pounds.

It will be noticed that after the force diagram and equilibrium polygon have been drawn the value of the bending moment depends solely upon the value of y, since the length $P u = H$ is fixed. At the points a and g, directly under the points of support of the beam, $y = 0$; hence, for these two points, bending moment $= H y = H \times 0 = 0$; that is, for any simple beam, the bending moment at either support is zero. The greatest value for the bending moment will evidently be at the point *8*, since $d\,8'$ is the longest vertical line which can be included between $a\,g$ and $a\,c\,f\,g$.

The figure $a\,c\,e\,g\,a$ is called the **diagram of bending moments.**

1144. Consider now the case of a simple beam uniformly loaded. Let the distance between the supports in Fig. 225 be 12 feet, and let the total load uniformly distributed over the beam be 216 pounds. Divide the load into a convenient number of equal parts, the more the better, say 12, in this case. The load which each part represents is $216 \div 12 = 18$ pounds. For convenience, lay off $O\,C$ on the vertical through the left-hand support, equal to 216 pounds to the scale chosen, and divide it into 12 equal parts, $O\,a$, $a\,b$, etc.; each part will represent 18 pounds to the same scale. Choose a pole P, and draw the rays $P\,O$, $P\,a$, $P\,b$, etc. Through the points d, e, etc., the centers of gravity of the equal subdivisions of the load, draw the verticals $d\,1$, $e\,3$, $f\,5$, etc., intersecting the horizontals through O, a, b, etc., in *1*, *3*, *5*, etc. Draw $O\,1$, *1-2*, *2-3*, *3-4*, *4-5*, etc., and the broken line thus found will be the shear line. In drawing the shear line for a uniform load in this manner, it is assumed that each part of the total load is concentrated at its center of gravity, or, in other words, that a force equal to each small load (18 lb.) acts upon the beam at each of the points d, e, f, etc.

Construct the diagram of bending moments in the ordinary manner by drawing $g\,i$ parallel to $P\,O$, $i\,k$ parallel to $P\,a$, etc. Draw $P\,M$ parallel to $g\,h$, and $M\,q$ horizontal; $M\,q$ is the shear axis. When the load is uniform and the work has been done correctly, *O M* should equal *M C*—that is, the reactions of the two supports are equal.

1145. The shear line is not a broken line in reality, as shown, since the load is distributed evenly over the entire beam, and not divided into small loads concentrated at *d*, *e*, *f*, etc., as was assumed. The points *1*, *3*, etc., are evidently

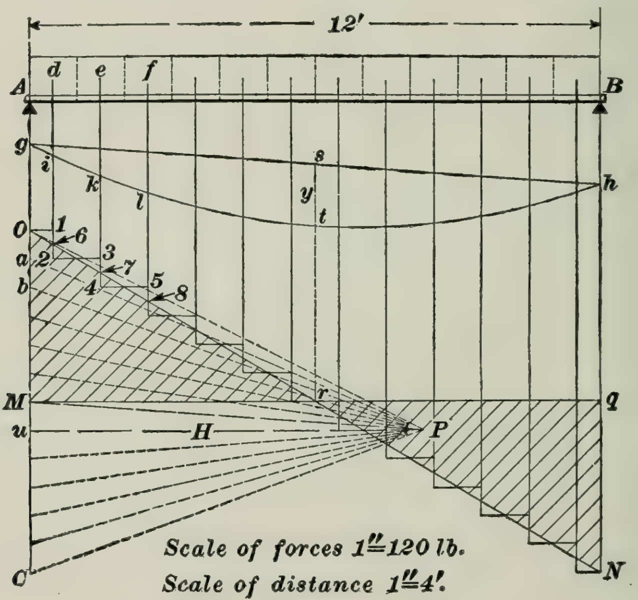


FIG. 225.

too high, and the points *2*, *4*, etc., too low. To find the real shear line, bisect the lines *1-2*, *3-4*, etc., locating the points *6*, *7*, *8*, etc. Trace a line through *O*, *6*, *7*, *8*..........*N*, and it will be the real shear line. For all cases of a uniform load, the shear line will be straight and may be drawn from *O* to *N* directly.

The diagram of bending moments is also not quite exact, but may be corrected by tracing a curve through the points *g* and *h*, which shall be tangent to *g i*, *i k*, *k l*, etc., at their middle points, as shown in the figure.

1146. To find the maximum bending moment for any beam having two supports, draw a vertical line through r, where the shear line cuts the shear axis, and the intercept, $s\,t = y$, on the diagram of bending moments, will be the greatest value of y, and, consequently, the greatest bending moment $= H \times y$. In the present example, $H = P\,u = 247.5$ pounds and maximum $y = s\,t = 15.72$ inches; hence, the maximum bending moment $= H\,y = 247.5 \times 15.72 = 3{,}890.7$ inch-pounds. The above is true, no matter how the beam may be loaded. In Fig. 224, the shear line cuts the shear axis at z, and $d\,8'$, on the vertical through z, was shown previously to be the maximum intercept.

1147. If there is a uniform load on the beam and one or more concentrated loads, as in Fig. 226, the method of finding the moment diagram and shear line is similar to that used in the last case. In Fig. 226, let the length $A\,B$ of the beam be 15 feet; the uniformly distributed load 180 pounds, with two concentrated loads, one of 24 pounds, 5 feet from A, and the other of 30 pounds, 11 feet from A. Draw the beam and loads as shown, the length of the beam and the distances of the weights from A being drawn to scale. Divide the uniform load into a convenient number of equal parts, say 10 in this case; each part will then represent $\frac{180}{10} = 18$ pounds. Draw $A\,O\,C$, as usual, and lay off 3 of the 18-pound subdivisions from O downwards; then lay off 24 pounds, to represent the first weight. Lay off four more of the equal subdivisions, and then the 30-pound weight. Finally, lay off the remaining three equal subdivisions, the point C being the end of the last 18-pound subdivision. $O\,C$ should then equal $180 + 24 + 30 = 234$ pounds to the scale to which the weights were laid off. It will be noticed in the above that the equal subdivisions of the load were laid off on $O\,C$ until that one was reached on which the concentrated loads rested, and that the concentrated loads were laid off before the equal subdivision on which the concentrated load rested was laid off. Had one of the concentrated loads been to the right of the center of gravity of the subdivision on which it rests, the weight of the subdivision would have been laid off first. Locate the centers of gravity of the equal subdivisions and draw the verticals through them as in the previous case. Choose a pole *P*, draw the rays, the diagram of bending moments, and the shear axis *M q*, as previously described.

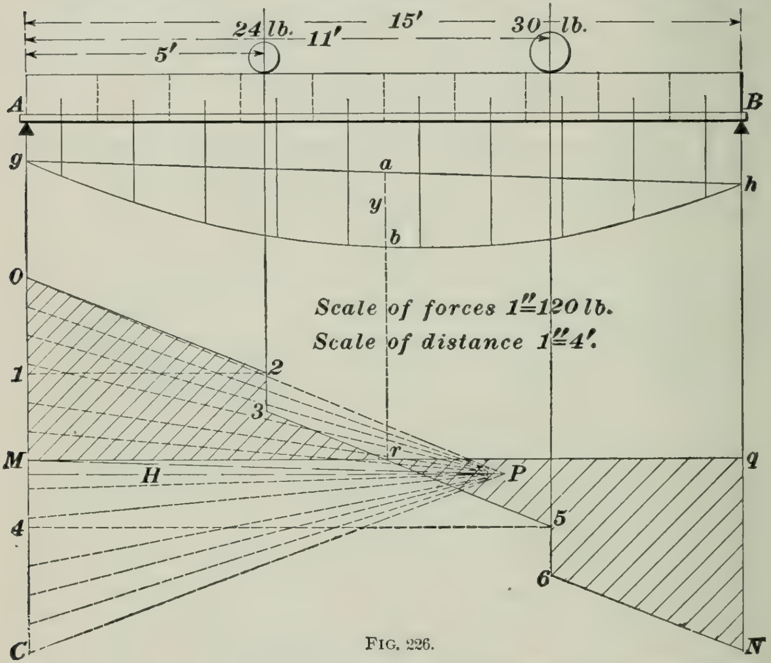


FIG. 226.

To find the maximum bending moment, the shear line must be drawn and its intersection with the shear axis determined.

The weight of the uniform load per foot of length is $\frac{180}{15} = 12$ lb. The weight of that part between *A* and the center of the 24-lb. weight is $12 \times 5 = 60$ lb. Lay off *O 1* = 60 lb. and draw *1-2* horizontal, cutting the vertical through the center of the 24-lb. weight in *2*. Draw *O 2* and it will be a part of the shear line. Lay off *2-3* vertically downwards, equal to 24 lb., locating the point *3*. Lay off *O 4* equal to $12 \times 11 + 24 = 156$ lb., and draw the horizontal *4-5*, cutting the vertical through the center of the 30-lb.

weight in *5*. Join *3* and *5* by the straight line *3-5*. Lay off *5-6* vertically downwards equal to 30 lb. Draw the horizontal $C\,N$, intersecting the vertical $B\,q\,N$ in N and join *6* and N by the straight line $6\,N$. The broken line O *2-3-5-6* N is the shear line and cuts the shear axis in the point r. Drawing the vertical $r\,b\,a$, through r, it intersects the moment diagram in a and b; hence, $a\,b$ is the maximum y. For this case, the maximum bending moment $= H \times y = 300 \times 18.36 = 5{,}508$ in.-lb.

It is better, in ordinary practice, to choose the pole P on a line perpendicular to $O\,C$ and at some distance from $O\,C$ easily measured with the scale used to lay off $O\,C$. Thus, suppose $O\,C$ to be laid off to a scale of $1'' = 60$ lb. At some convenient point, as M, on $O\,C$, draw a perpendicular line and choose a point on this line whose distance from $O\,C$ shall be easily measurable, say $3\frac{1}{2}''$. Then, H is known to be exactly $60 \times 3\frac{1}{2} = 210$ lb. and will not have to be measured when finding the bending moment $H\,y$.

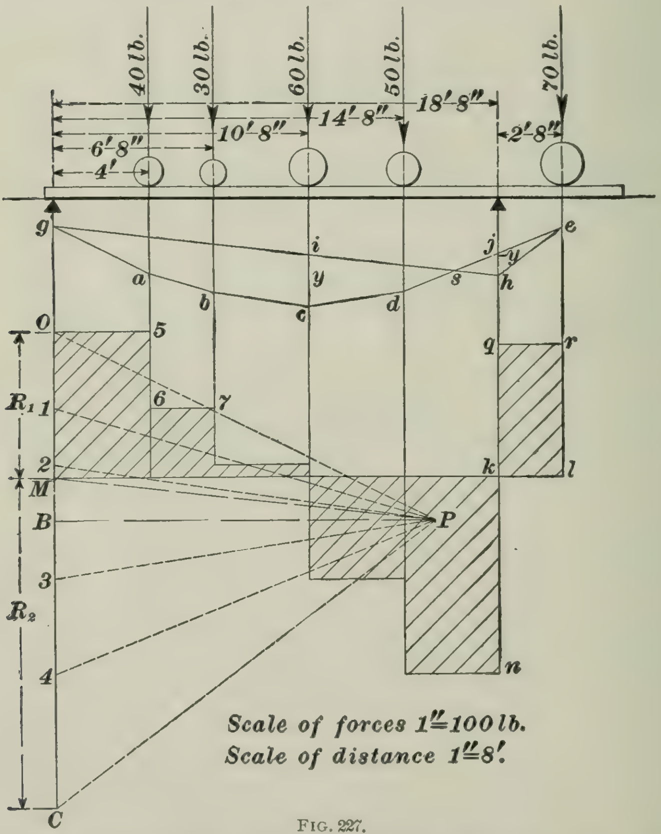
1148. If the student has familiarized himself with the method of constructing the shear and moment diagrams for concentrated loads, he will find no difficulty in understanding the preceding operations, which may be condensed into the following

Rule.—*Divide the beam into an even number of parts* (the greater the better), *and the uniform load into half as many. Consider the divisions of the load as concentrated loads applied, alternately, at the various points of division of the beam* (the ends included); *that is, the first point of division* (*the support*) *carries no load, the next one does, the following one does not*, etc. *Then proceed as in the ordinary case of concentrated loads.*

EXAMPLE.—Find the reactions of the supports, the maximum bending moments, and the maximum vertical shear of the beam shown in Fig. 227, which has one overhanging end.

SOLUTION.—Draw $O\,C$ and the force diagram in the usual manner. Construct the bottom curve of the moment diagram in the same manner as in the preceding cases. The side $d\,e$ is parallel to $P\,4$; $e\,h$ is parallel to $P\,C$, and cuts the vertical through the right reaction in h.

Join *h* and *g* by the straight line *g h*, and draw *P M* parallel to *g h*. Then, *O M* = 77 lb. = left reaction and *M C* = 173 lb. = right reaction. The shear line is drawn as in the previous cases until the point *n*, on the vertical *h n*, is reached; *k n* here denotes the vertical shear for any point between the 50-lb. weight and the right support, and this shear is negative. The point *k* denotes the intersection of the shear axis and the vertical through the right support. For any point to the right of *k*, between *k* and *l*, the vertical shear is positive, and is equal to 70 lb.; hence, lay off *k q* upwards equal to 70 lb., and draw *q r* horizontal. The line, *O 5-6-7* .... *n q r* is the shear line. Measuring *O M*, *k n*, and *k q*, it is found that *O M* = 77 lb., *k n* = − 103 lb., and *k q* = 70 lb.; therefore, *k n* = − 103 lb. = maximum vertical shear; *c i* is evidently the maximum *y*; hence, the maximum bending moment =



Scale of forces 1" = 100 lb.
Scale of distance 1" = 8'.

FIG. 227.

$Hy = PB \times ci = 200 \times 26'' = 5{,}200$ in.-lb. Any value of y measured in the polygon $g\,a\,b\,c\,d\,s\,g$ is positive, and any value measured in the triangle $e\,h\,s$ is negative. Consequently, the bending moment for any point between s and the vertical, through the center of the 70-lb. weight, is negative, since $H \times (-y) = -Hy$. In all cases, when designing beams with overhanging studs, the maximum bending moment, whether positive or negative, should be used. In the present case, the maximum negative y, or $h\,j$, is less than the maximum positive y, or $c\,i$; therefore, the maximum negative bending moment is also less than the maximum positive moment.

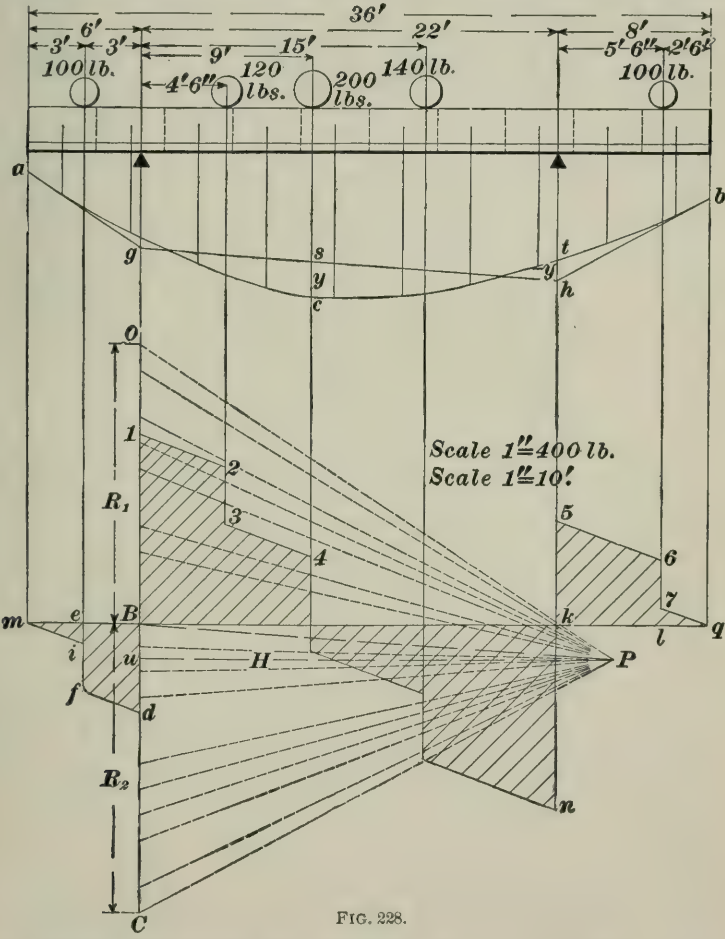


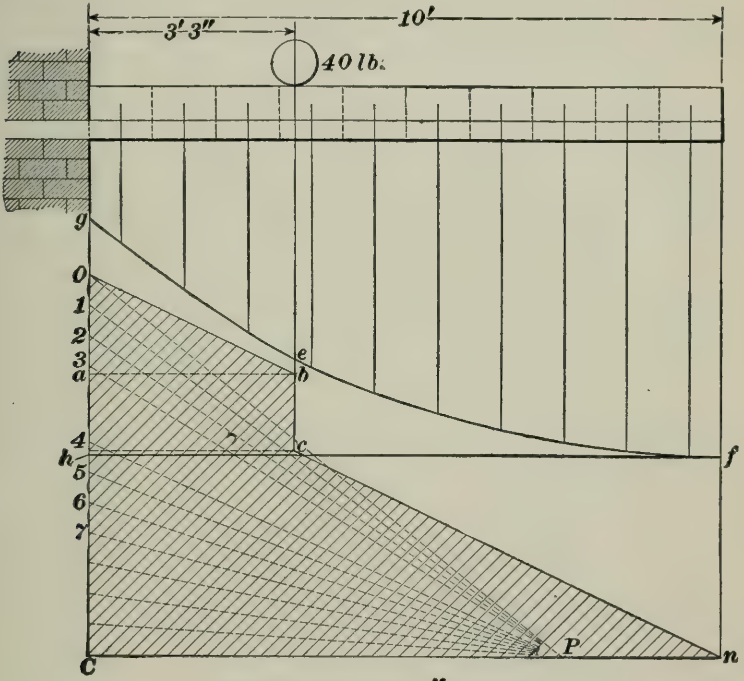
FIG. 228.

Example.—Fig. 228 shows a beam overhanging both supports, which carries a uniform load of 15 pounds per foot of length, and has five concentrated loads at distances from the supports as marked in the figure. Required the reactions of the supports, the maximum positive and negative bending moments, and the maximum vertical shear.

Solution.—Construct the force diagram and equilibrium polygon in the ordinary manner, continuing the latter to b and a, points on the verticals passing through the ends of the beam. Draw $b\,h$ and $a\,g$ parallel to $P\,C$ and $P\,O$ respectively, intersecting the verticals through the points of support in h and g. Join g and h, and draw $P\,B$ parallel to $g\,h$. Then, $O\,B = 588$ lb. $= R_1$, and $B\,C = 612$ lb. $= R_2$. Through B, draw the shear axis $m\,q$. To draw the shear line, proceed as follows: The shear for any point to the left of the left support is negative, and for any point to the right of the right support is positive; between the two supports it is positive or negative, according to the manner of loading, and the point considered. The negative shear at the left support $= 15 \times 6 + 100 = 190$ lb.; hence, lay off $B\,d$ downwards equal to 190 lb. For a point to a minute distance to the right of e, the shear is $15 \times 3 + 100 = 145$ lb. $= e\,f$, and for a minute distance to the left, it is $15 \times 3 = 45$ lb. $= e\,i$; at m, it is 0. Consequently, $m\,i\,f\,d$ is the shear line between the end of the beam and the left support. Lay off $O\,1 = B\,d = 190$ lb. and draw the shear line *1-2-3-4*——————n in the usual manner. Draw the shear line *5-6-7* q, laying off $k\,5 = 15 \times 8 + 100 = 220$ lb.; $l\,6 = 100 + 15 \times 2\tfrac{1}{2} = 137\tfrac{1}{2}$ lb., and *6-7* $= 100$ lb. At q, the vertical shear is again 0. The broken line $m\,i\,f\,d$ *1-2-3*——————n *5-6-7* q is the shear line. The maximum positive bending moment is $H \times y = P\,u \times s\,c = 1{,}000 \times 22.5 = 22{,}500$ in.-lb. The greatest maximum negative moment is $H \times (-y) = P\,u \times -t\,h = 1{,}000 \times -11.6 = -11{,}600$ in.-lb. It will be noticed that there are two negative and one positive maximum bending moments.

1149. The student should now be able to find the bending moment for any beam having but two supports, whatever the character of the loading. The bending moment plays a very important part in the flexure of beams, which is the next subject to be considered. In all cases of loading heretofore considered, no other forces than the loads themselves have been considered. Should forces act upon the beam which are not vertical, the force polygon will be no longer a straight line, but a broken line somewhat similar in character to O *1-2-3-4-5* in Fig. 218.

In Fig. 229 is shown a cantilever beam projecting 10 ft. from the wall. It carries a uniform load of 16 pounds per foot of length, and a concentrated load of 40 pounds at a distance of 3$\frac{1}{4}$ feet from the wall. The maximum bending moment is required. The method is similar to the last, except that, as there is but one support, there can be but one reaction. Since the beam is 10 feet long, the total weight of the uniform load is 16 × 10 = 160 pounds. Hence, the reaction = 160 + 40 = 200 pounds.



Scale of forces 1" = 100 lb.
Scale of distance 1" = 3'.

FIG. 229.

Draw $O\ C$ equal to 200 lb., to some convenient scale. Draw $P\ C$ perpendicular to $O\ C$ at C and choose the pole P at a convenient distance from $O\ C$. For convenience, divide the uniform load into 10 equal parts, as shown; then, each part will represent 16 lb. Lay off *O 1*, *1-2*, *2-3*, each equal to 16 lb., and *3-4* equal to 40 lb. Also, *4-5*, *5-6*, *6-7*, etc., equal to 16 lb. each. 3 ft. 3 in. $= 3\frac{1}{4}$ ft., and $16 \times 3\frac{1}{4} = 52$ lb. Lay off $O\,a = 52$ lb., and draw *a b*, meeting the vertical through the center of the weight in *b*. Draw *O b*; lay off *b c* equal to 40 lb. and draw *C n*. *O b c n* is the shear line. The perpendicular through the point *C* coincides with *P C*; hence, *n P C* is the shear axis. Draw the line *g e f* of the moment diagram as in the previous cases. In Fig. 226, and in the preceding figures, the line *g h* was drawn connecting the extreme ends of the bottom line; in other words, it joined the points where the equilibrium polygon cut the lines of direction of the reactions of the supports. This cannot be done in this case, because there is no right reaction; therefore, *g h* must be drawn by means of some other property of the polygon. In the previous cases, the shear axis was drawn perpendicular to *O C* at the point where a line through the pole *P* parallel to *g h* cut *O C*, or, in other words, the shear axis was drawn through the point which marked the end *M* of the left reaction *O M*. In the present case, Fig. 229, the point *C* is the end of the left reaction; hence, *f h* is parallel to *P C*, *h g* is the maximum *y*, and the bending moment $= H\,y = P\,C \times h\,g = 250 \times 45 =$ 11,250 inch-pounds.

It will also be noticed that the distance $y = h\,g$ is measured from the line *h f* upwards, while, for all points between the supports in the previous examples, this distance was measured downwards. The same observation is true for any point between *h* and *f*; hence, for a cantilever, all bending moments are negative.

1150. All the cases of beams heretofore given might have been solved by analytical methods—that is, by algebraic processes and formulas; but the graphical method is to be preferred, as it is usually shorter, very nearly as accurate, and less liable to error. Moreover, when the diagram has once been drawn, both the bending moment and the shear for any point can be instantly determined. Except in special cases, the graphical method will be employed to determine the maximum bending moment, but in some particular cases formulas will be used, by which the result can be obtained more quickly than by the graphical method.

EXAMPLES FOR PRACTICE.

1. A simple beam 24 feet long carries 4 concentrated loads of 160, 180, 240, and 120 pounds at distances from the left support of 4, 10, 16, and 21 feet, respectively. (*a*) What are the values of the reactions? (*b*) What is the maximum bending moment in inch-pounds?

$$\text{Ans.}\left\{\begin{array}{l}(a)\ R_1 = 333\tfrac{1}{3}\text{ lb.};\ R_2 = 366\tfrac{2}{3}\text{ lb.}\\(b)\ 28{,}480\text{ in.-lb.}\end{array}\right.$$

2. A simple beam carries a uniform load of 40 pounds per foot, and supports two concentrated loads of 500 and 400 pounds at distances from the left support of 5 and 12 feet, respectively. The length of the beam is 18 feet. What are (*a*) the reactions? (*b*) The maximum bending moment in inch-pounds?

$$\text{Ans.}\left\{\begin{array}{l}(a)\ R_1 = 854\tfrac{4}{9}\text{ lb.};\ R_2 = 765\tfrac{5}{9}\text{ lb.}\\(b)\ 48{,}846\text{ in.-lb.}\end{array}\right.$$

3. A cantilever projects 10 feet from a wall and carries a uniform load of 60 pounds per foot; it also supports three concentrated loads of 100, 300, and 500 pounds at distances from the wall of 2, 5, and 9 feet, respectively. Required, (*a*) the maximum vertical shear, and (*b*) the maximum bending moment in inch-pounds.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ -1{,}500\text{ lb.}\\(b)\ -110{,}400\text{ in.-lb.}\end{array}\right.$$

4. A beam which overhangs one support sustains six concentrated loads of 160 lb. each at distances from the left support of 4 ft. 9 in., 7 ft., 9 ft. 6 in., 12 ft., 15 ft., and 18 ft. 3 in., respectively, the distance between the supports being 16 ft. What are (*a*) the reactions? (*b*) The maximum bending moment?

$$\text{Ans.}\left\{\begin{array}{l}(a)\ R_1 = 295\text{ lb.};\ R_2 = 665\text{ lb.}\\(b)\ 20{,}460\text{ in.-lb.}\end{array}\right.$$

5. A beam which overhangs both supports equally carries a uniform load of 80 pounds per foot, and has a load of 1,000 pounds in the middle, the length of the beam being 15 feet, and the distance between the supports 8 feet. What is (*a*) the vertical shear? (*b*) The maximum bending moment?

$$\text{Ans.}\left\{\begin{array}{l}(a)\ 820\text{ lb.}\\(b)\ 25{,}800\text{ in.-lb.}\end{array}\right.$$

NOTE.—The student may not obtain the exact answers given above, but if his results do not differ by more than 1%, he will know that his method is right.

NEUTRAL AXIS.

1151. In Fig. 230, let $A\ B\ C\ D$ represent a cantilever. Suppose that a force F acts upon it at its extremity A. The beam will then be bent into the shape shown by $A'\ B\ C\ D'$.

It is evident from the cut that the upper part $A'B$ is now longer than it was before the force was applied; i. e., $A'B$ is longer than AB. It is also evident that $D'C$ is shorter than DC. Hence, the effect of the force F in bending the beam is to lengthen the upper fibers and to shorten the lower ones. In other words, when a cantilever is bent through the action of a load, the upper fibers are in tension and the lower fibers in compression. The reverse is the case in a simple beam in which the upper fibers are in compression and the lower fibers in tension. Further consideration will show that there must be a fiber, $S\,S''$, which is neither lengthened nor shortened when the beam is bent, i. e., $S'\,S'' = S\,S''$. When the beam is straight the fiber $S\,S''$, which is neither lengthened nor shortened when the beam is bent, is called the **neutral line.** There may be

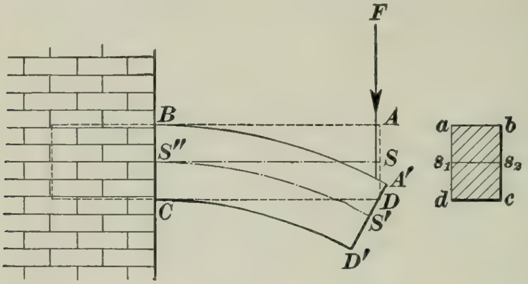


FIG. 230.

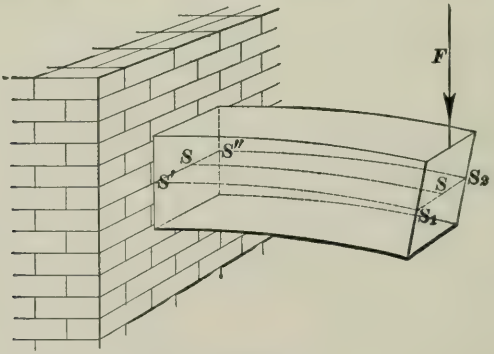


FIG. 231.

any number of neutral lines dependent only on the width of the beam. For, let $b\,a\,d\,c$, Fig. 230, be a cross-section of the beam. Project s upon it in s_1. Make $b\,s_2 = a\,s_1$ and draw $s_1\,s_2$; then, any line in the beam which touches $s_1\,s_2$ and is parallel to $S\,S''$ is a neutral line. Thus, in Fig. 231, $S_1\,S'$, $S\,S$, $S_2\,S''$, etc., are all neutral lines. The line $S_1\,S_2$ is called the **neutral axis,** and the surface $S_1\,S'\,S''\,S_2$ is called the **neutral surface**. The neutral axis, then, is the line of intersection of a cross-section with the neutral surface. It is shown in works on mechanics that *the neutral axis always passes through the center of gravity of the cross-section of the beam.*

1152. Experimental Law.—*When a beam is bent, the horizontal elongation (or compression) of any fiber is directly proportional to its distance from the neutral surface, and, since the strains are directly proportional to the horizontal stresses in each fiber, they are also directly proportional to their distances from the neutral surface, provided the elastic limit is not exceeded.*

1153. Suppose the beam to be a rectangular prism, then every cross-section will be a rectangle, and the neutral axis will pass through the center o. See Fig. 232.

Let the perpendicular distance from the neutral axis $M\,N$ to the *outermost fiber* be denoted by c, and the horizontal unit stress (stress per square inch) at the distance c from the axis by S. If a is the area of a fiber, the stress on the outermost fibers will be $a\,S$. The stress on a fiber at the distance unity (1 inch) from $M\,N$ is $\frac{a\,S}{c}$; and the stress on a fiber at the distance r_1 is $\frac{a\,S}{c} \times r_1 = \frac{a\,S\,r_1}{c}$. The moment of this stress about the axis $M\,N$ is $\frac{a\,S\,r_1}{c} \times r_1 = \frac{a\,S\,r_1^2}{c} = \frac{S}{c}\,a\,r_1^2$. The moment of the

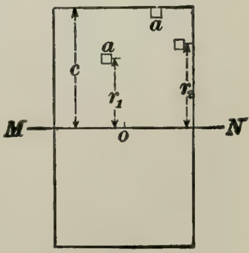


FIG. 232.

stress on any other fiber at a distance r_2 from MN is evidently $\frac{S}{c} a r_2^2$, and for a distance r_3, $\frac{S}{c} a r_3^2$, etc. If n is the number of fibers, the sum M of the moments of the horizontal stresses on all the fibers is

$$M = \frac{S}{c} a r_1^2 + \frac{S}{c} a r_2^2 + \frac{S}{c} a r_3^2 +, \text{ etc.}, =$$

$$\frac{S}{c}(a r_1^2 + a r_2^2 + a r_3^2 + \ldots\ldots a r_n^2) = \frac{S}{c} a (r_1^2 + r_2^2 + r_3^2 + \ldots\ldots r_n^2).$$

Now, let r be a quantity whose square equals the mean of the squares of $r_1, r_2, r_3, \ldots\ldots r_n$. Then, $r^2 = \frac{r_1^2 + r_2^2 + \ldots\ldots + r_n^2}{n}$; and, therefore, $r_1^2 + r_2^2 + r_3^2 + \ldots\ldots + r_n^2 = n r^2$. Substituting above, we get $M = \frac{S}{c} n a r^2$. But, since a is the area of one fiber, $n a$ is the area of all the fibers—that is, the area A of the cross-section; hence, the sum of the moments of all the horizontal stresses $= \frac{S}{c} A r^2$.

1154. The expression $A r^2$, which is found by dividing a section into a large number of minute areas (a, a, etc.), multiplying each area by the square of its distance from an axis (r_1^2, r_2^2, r_3^2, etc.), and then adding the products thus obtained, is called the **moment of inertia** of the section with respect to that axis, and is usually denoted by the letter I. Hence

$$I = A r^2. \qquad \textbf{(72.)}$$

1155. The quantity r, whose square is the mean of the squares of all the distances of the minute areas from the axis, is called the **radius of gyration.**

1156. The sum of the moments of all the horizontal stresses may then be written as $\frac{S}{c} A r^2 = \frac{S}{c} I$, or $S \frac{I}{c}$. This expression is called the **resisting moment,** since it is the measure of the resistance of the beam to bending (and, consequently, to breaking) when loaded. The resisting moment must equal the bending moment when the beam is in equilibrium; hence, denoting the bending moment by M,

$$M = S_4 \frac{I}{c}. \qquad \textbf{(73.)}$$

1157. The values of I and c depend wholly upon the size and form of the cross-section of the beam, and S_4 is the ultimate strength of flexure of the material.

In Table 25, the average ultimate strength of flexure S_4 is given for a number of different materials.

TABLE 25.

Material.	Ultimate Strength of Flexure in Lb. per Sq. In. S_4.
Cast Iron	38,000
Wrought Iron	45,000
Steel	120,000
Brass	17,000
Ash	14,000
Brick	1,000
Stone	2,000
Hemlock	7,200
Oak, white	12,500
Pine, white	9,000
Pine, yellow	11,000
Hickory	16,000

1158. Exact values of I for most cross-sections can only be determined by the aid of the calculus. The least value of I occurs when the axis passes through the center of gravity of the cross-section—that is, when I is found with reference to the neutral axis.

The least moments of inertia for a number of different sections are given in the Table of Moments of Inertia; also, the area of the sections and the values of c. The dotted line indicates the position of the neutral axis, about which the moment of inertia is taken.

In the table of Moments of Inertia, A is the area of the section, and π = ratio of the circumference of a circle to its diameter = 3.1416. It will be noticed that d is always taken vertically.

1159. To use formula **73,** find the bending moment in inch-pounds by the graphical method previously described, or calculate it by means of the table of Bending Moments. If it is desired to find the size of a beam that will safely resist a given bending moment, take S_4 from Table 25, Art. **1157,** and divide it by the proper factor of safety taken from Table 24. Then, formula **73** becomes

$$M = \frac{S_4 I}{f c}. \qquad \textbf{(74.)}$$

From this $\frac{I}{c} = \frac{M f}{S_4}$. Substituting the values of M, f, and S_4, the value of $\frac{I}{c}$ is found. The kind and shape of beam having been decided upon, the size can be so proportioned that $\frac{I}{c}$ for the section shall not be less than the value calculated above. An example will make this clear.

EXAMPLE.—What should be the size of an ash girder to resist safely a bending moment of 28,000 inch-pounds, the cross-section to be rectangular and the load steady?

SOLUTION.—$M = \frac{S_4 I}{f c}$, or $\frac{I}{c} = \frac{M f}{S_4}$.

$M = 28{,}000$; from Table 25, Art. **1157,** $S_4 = 14{,}000$; from Table 24, Art. **1120,** $f = 8$.

Then, $\frac{I}{c} = \frac{28{,}000 \times 8}{14{,}000} = 16$.

From the Table of Moments of Inertia, $I = \frac{b d^3}{12}$ and $c = \frac{d}{2}$ for a rectangle; hence, $\frac{I}{c} = \frac{b d^3}{12} \times \frac{2}{d} = \frac{b d^2}{6} = 16$, or $b d^2 = 96$. Any number of values of b and d can be found that will satisfy this equation. If b is taken as 6 inches, $d^2 = \frac{96}{6} = 16$ and $d = \sqrt{16} = 4$. Hence, the beam may be a 6 × 4, with the short side vertical. When possible, it is always better to have the longer side vertical. If b is taken as 2 inches, $d^2 = 48$ and $d = \sqrt{48} = 7$ inches, nearly; hence, a 2 × 7 will also answer the purpose. The advantage of using a 2 × 7 instead of a 6 × 4 is evident, since the 6 × 4 contains nearly twice as much material as the 2 × 7. Thus, the area of the cross-section of a 6 × 4 is 24 square inches, and of a 2 × 7, 14 square inches. Moreover, the 2 × 7, with its long side vertical, is slightly stronger than the 6 × 4, with its short side vertical, since $\frac{I}{c} = \frac{2 \times 7^2}{6} = 16\frac{1}{3}$ for the former, and $\frac{6 \times 4^2}{6} = 16$ for the latter. If the 6 × 4 had its longer side vertical, thus making it a 4 × 6, $\frac{I}{c}$ would then equal $\frac{4 \times 6^2}{6} = 24$, and the safe bending moment could be increased to $M = \frac{S_4 I}{f c} = \frac{14{,}000 \times 24}{8} = 42{,}000$ in.-lb.

1160. If the breaking bending moment, form, and size of the cross-section of the beam are known, the ultimate strength of flexure S_4 can be readily found from formula **73,** by substituting the values of M, I, and c, and solving for S_4.

Example.—A cast iron bar, 2 inches square, breaks when the maximum bending moment = 63,360 inch-pounds; what is its ultimate strength of flexure?

Solution.—$M = S_4 \frac{I}{c}$, or $S_4 = \frac{M c}{I}$. $c = \frac{d}{2} = 1''$.

$$I = \frac{d^4}{12} = \frac{2^4}{12}; \text{ therefore, } S_4 = \frac{63{,}360 \times 1}{\frac{1}{12} \times 2^4} = 47{,}520 \text{ lb. per sq. in.}$$

1161. In order to save time in calculating, the bending moments for cases of simple loading are given in the Table of Bending Moments. W denotes a concentrated load, and w the uniform load per inch of length. All dimensions are to be taken in inches when using the formulas.

For any other manner of loading than is described in the Table of Bending Moments, the maximum bending moment must be found by the graphical method.

Example.—A wrought iron cantilever, 6 feet long, carries a uniform load of 50 pounds per inch. The cross-section of the beam is an equilateral triangle, with the vertex downwards; what should be the length of a side?

Solution.—$M = \frac{w l^2}{2}$, from the Table of Bending Moments, $= \frac{50 \times (6 \times 12)^2}{2} = 129{,}600$ in.-lb. $I = \frac{b d^3}{36}$ and $c = \frac{2}{3} d$, from the Table of Moments of Inertia; hence, $\frac{I}{c} = \frac{b\,d^2}{24}$. $S_4 = 45{,}000$, from Table 25, Art. **1157**, and $f = 4$, from Table 24, Art. **1120**. Therefore, 129,600 $= \frac{45{,}000}{4} \times \frac{b\,d^2}{24}$, or $b\,d^2 = \frac{129{,}600 \times 4 \times 24}{45{,}000} = 276.48$. Since an equilateral triangle has been specified, b can not be given *any* convenient value in order to find d. For an equilateral triangle, $d = b \sin 60° = .866\ b$. Hence, $b\,d^2 = b\ (.866\ b)^2 = .75\ b^3$. Therefore, $b\,d^2 = .75\ b^3 = 276.48$ or $b = \sqrt[3]{\frac{276.48}{.75}}$ $\log b = \frac{\log 276.48 - \log .75}{3} = .85553$, or $b = 7.17''$, nearly.

Example.—What weight would be required to break a round steel bar 4 inches in diameter, 16 feet long, fixed at both ends and loaded in the middle?

Solution.—Use formula **73**; $M = \frac{S_4\,I}{c}$. Here $M = \frac{W\,l}{8}$, from the table of Bending Moments; $S_4 = 120{,}000$; $\frac{I}{c} = \frac{\frac{1}{64}\pi\,d^4}{\frac{1}{2}d} = \frac{\pi\,d^3}{32}$. Hence, $\frac{W\,l}{8} = \frac{W \times (16 \times 12)}{8} = \frac{120{,}000 \times 3.1416 \times 4^3}{32}$, or $W = \frac{120{,}000 \times 3.1416 \times 64 \times 8}{16 \times 12 \times 32} = 31{,}416$ lb.

DEFLECTION OF BEAMS.

1162. The deflection, or amount of bending, produced in a beam by one or more loads is given by certain general formulas, whose derivation is too complicated to be given here. We shall give the formulas only, illustrating their application by examples.

In the third column of the Table of Bending Moments are given expressions for the value of the greatest deflection of a beam when loaded as shown in the first column. From this it is seen that the deflection s equals a constant (depending upon the manner of loading the beam and upon the condition of the ends—whether fixed or free), multiplied by $\frac{W\,l^3}{E\,I}$. Let a represent the constant and s the deflection; then,

$$s = a\,\frac{W\,l^3}{E\,I}. \qquad \textbf{(75.)}$$

In the above formula, E = coefficient of elasticity, and is to be taken from Table 21, Art. **1114,** l = length in inches, W = total weight supported in pounds, and I is the moment of inertia about the neutral axis; a has values varying from $\frac{1}{384}$ to $\frac{1}{3}$.

It will be noticed that the deflection is given for only nine cases; for any other manner of loading a beam than those here given, it is necessary to use the calculus to obtain the deflection.

Example.—What will be the maximum deflection of a simple wooden beam 9 feet long, whose cross-section is an ellipse, having axes of 6 inches and 4 inches (short axis vertical), under a concentrated load of 1,000 pounds?

Solution.—Use formula **75,** $s = a\frac{Wl^3}{EI}$. From the Table of Bending Moments,

$$a = \frac{1}{48},\ W = 1{,}000 \text{ lb.},\ l = 9 \times 12 = 108 \text{ in.},$$

$$E = 1{,}500{,}000 \text{ and } I = \frac{\pi b d^3}{64} = \frac{\pi \times 6 \times 4^3}{64}.$$

$$\text{Hence, } s = \frac{1{,}000 \times 108^3 \times 64}{48 \times 1{,}500{,}000 \times \pi \times 6 \times 64} = .9282''.$$

1163. The principal use of the formula for deflection is to determine by its means the stiffness of a beam or shaft. In designing machinery, it frequently occurs that a piece may be strong enough to sustain the load with perfect safety, but the deflection may be more than circumstances will permit; in this case, the piece must be made larger than is really necessary for mere strength. An example of this occurs in the case of locomotive guides, and the upper guides of a steam engine when the engine runs under. It is obvious that they must be very stiff. In such cases it is usual to allow a certain deflection, and then proportion the piece so that the deflection shall not exceed the amount decided upon.

Example.—The breadth of the guides of a certain locomotive is not to exceed $2\frac{1}{4}$ inches. Regarding the guides as fixed at both ends, (*a*) what must be their depth to resist a load of 10,000 pounds at the middle? The guides are made of cast iron and are 38 inches long between the points of support. (*b*) What weight would these guides be able to support with safety? The deflection must not exceed $\frac{1}{200}$ of an inch. The cross-section is, of course, rectangular.

Solution.—Since the load comes on two guides, each piece must support $10{,}000 \div 2 = 5{,}000$ lb. In the formula,

$$s = \frac{1}{200} = a\,\frac{W\,l^3}{E\,I},\ a = \frac{1}{192} \text{ for this case, } W = 5{,}000,\ l = 38,$$

$$E = 15{,}000{,}000, \text{ and } I = \frac{b\,d^3}{12} = \frac{2\tfrac{1}{4}\,d^3}{12} = \frac{3}{16}d^3. \quad \text{Substituting,}$$

$$(a) \quad s = \frac{5{,}000 \times 38^3 \times 16}{192 \times 15{,}000{,}000 \times 3d^3} = \frac{1}{200},$$

or, $$d = \sqrt[3]{\frac{5{,}000 \times 38^3 \times 16 \times 200}{192 \times 15{,}000{,}000 \times 3}} = 4.67'', \text{ nearly, say } 4\tfrac{11}{16}''. \quad \text{Ans.}$$

(*b*) To find the weight which these guides could support with safety, use formula **74**, $M = \frac{S_4\,I}{f\,c}$, in which $M = \frac{W\,l}{8}$, $S_4 = 38{,}000$, $f = 10$, $\frac{I}{c} = \frac{b\,d^2}{6} = \frac{2\tfrac{1}{4} \times (4\tfrac{11}{16})^2}{6} = \frac{50{,}625}{6{,}144}$. Substituting,

$$W = \frac{38{,}000 \times 50{,}625 \times 8}{10 \times 6{,}144 \times 38} = 6{,}592 \text{ lb.}$$

Hence, the beam is over 30% stronger than necessary, the extra depth being required for stiffness.

EXAMPLES FOR PRACTICE.

1. How much will a simple wooden beam 16 ft. long, 2 in. wide and 4 in. deep deflect under a load in the middle of 120 lb.? Ans. 1.106″.

2. What should be the size of a rectangular yellow pine girder 20 ft. long, to sustain a uniformly distributed load of 1,800 lb.? Assume a factor of safety for a varying stress. Ans. 5″ × 8″.

3. A hollow cylindrical beam, fixed at both ends, has diameters of 8 in. and 10 in. If the beam is 30 ft. long and is made of cast iron, (*a*) what steady load will it safely support at 15 ft. from one of the supports? (*b*) What force will be required to rupture the beam if applied at this point?

Ans. { (*a*) 8,158 lb.
(*b*) 48,946 lb.

4. A simple cylindrical wrought-iron beam, resting upon supports 24 ft. apart, sustains three concentrated loads of 350 lb. each, at distances from one of the supports of 5, 12, and 19 ft.; what should be the diameter of the beam to withstand shocks safely? Ans. 4.71″, say 4¾″.

5. Find the value of $\frac{I}{c}$ for a hollow rectangle whose outside dimensions are 10 in. and 13 in., and inside dimensions are 8 in. and 10 in.; (*a*) when the long side is vertical; (*b*) when the short side is vertical.

Ans. { (*a*) 179.103.
(*b*) 131⅓.

6. What is the deflection of a steel bar 1 in. square and 6 ft. long, which supports a load of 100 lb. at the center? Ans. .31104″.

7. Which will be the stronger, a beam whose cross-section is an equilateral triangle, one side measuring 15 in., or one whose cross-section is a square, one side measuring 9 in.? Both beams are of the same length. Ans. The one having the square cross-section.

8. A wooden beam of rectangular cross-section sustains a uniform load of 50 lb. per foot. If the beam is 8″ × 14″ and 16 ft. long, how much more will it deflect when the short side is vertical than when the long side is vertical? Ans. .055417″.

COMPARISON OF STRENGTH AND STIFFNESS OF BEAMS.

1164. Consider two rectangular beams, loaded in the same manner, having the same lengths and bending moments, but different breadths and depths. Then, $M = S\frac{I}{c} = S\frac{b\,d^2}{6}$ (1) and $M = S\frac{b_1 d_1^2}{6}$ (2). Dividing (1) by (2), $\frac{M}{M} = \frac{6\,S\,b\,d^2}{6\,S\,b_1\,d_1^2} = \frac{b\,d^2}{b_1\,d_1^2} = 1$, or $b\,d^2 = b_1\,d_1^2$ (3).

Equation 3 shows that, if both beams have the same depth, their strengths will vary directly as their breadths, i. e., if the breadths are increased 2, 3, 4, etc., times, their strengths will also be increased 2, 3, 4, etc., times. It also shows that, if the breadths are the same and the depths are increased, the strengths will vary as the square of the depth, i. e., if the depths are increased 2, 3, 4, etc., times, the strengths will be increased 4, 9, 16, etc., times. Hence, it is always best, when possible, to have the long side of a beam vertical. If the bending moments are the same, but the weights and lengths are different, $M = g\,W\,l$ (1) and $M = g\,W_1\,l_1$ (2), when g denotes the fraction $\frac{1}{2}$, $\frac{1}{4}$, etc., according to the manner in which the ends are secured, and the manner of loading. Dividing (1) by (2) $\frac{M}{M} = \frac{g\,W\,l}{g\,W_1\,l_1}$, or $W\,l = W_1\,l_1$ (3).

Equation 3 shows that if the load W or W_1 be increased, the length l or l_1 must be decreased; consequently, the strength of a beam loaded with a given weight varies inversely as its length, i. e., if the load be increased 2, 3, 4, etc., times, the length must be shortened 2, 3, 4, etc., times, the breadth and depth remaining the same.

Example.—If a simple beam, loaded in the middle, has its breadth and depth reduced one-half, what proportion of the original load could it carry?

Solution.—In the preceding paragraphs, it was shown that the strength varied as the product of the breadth and the square of the depth, or $b_1 d_1^2 = \frac{1}{2} \times (\frac{1}{2})^2 = \frac{1}{8}$. Consequently, the beam can support only $\frac{1}{8}$ of the original load. Had the breadth remained the same, $(\frac{1}{2})^2 = \frac{1}{4}$ of the original load could have been supported. Had the depth remained the same, $\frac{1}{2}$ of the original load could have been supported.

Example.—A beam 10 ft. long, loaded in the middle, has a breadth of 4 in. and a depth of 6 in. The length is increased to 12 ft., the breadth to 6 in., and the depth to 8 in.; how many times the original load can it now support?

Solution.—The strength varies directly as the product of the breadth and square of the depth and inversely as the length, or as $\frac{b\,d^2}{l}$. If b, d, and l denote the original sizes, the strength of the two beams will be to each other as $\frac{b_1 d_1^2}{l} : \frac{b\,d^2}{l}$, or as $\frac{6 \times 8^2}{12} : \frac{4 \times 6^2}{10}$; $\frac{6 \times 8^2}{12} = 32$ and $\frac{4 \times 6^2}{10} = 14.4$. $\frac{32}{14.4} = 2\frac{2}{9}$. Consequently, the beam will support a load $2\frac{2}{9}$ times as great as the original beam.

1165. By a process of reasoning similar to that employed above, it can be shown that the maximum deflection of a beam varies inversely as the cube of the depth and directly as the cube of the length. In other words, if the depth be increased 2, 3, 4, etc., times, the deflection will be decreased 8, 27, 64, etc., times; and, if the length be increased 2, 3, 4, etc., times, the deflection will be increased 8, 27, 64, etc., times. Hence, if a beam is required to be very stiff, the length should be made as short and the depth as great as circumstances will permit.

COLUMNS.

1166. When a piece ten or more times as long as its least diameter or side (in general, its least transverse dimension) is subjected to compression, it is called a **column** or **pillar.**

The ordinary rules for compression do not apply to columns, for the reason that when a long piece is loaded beyond a certain amount, it buckles and tends to fail by flexure. This combination of flexure and compression causes the column to break under a load considerably less than that required to merely crush the material. It is likewise evident that the strength of a column is principally dependent upon its diameter, since that part having the least thickness is the part that buckles, or bends. A column free to turn in any direction, having a cross-section of 3″ × 8″, is not nearly so strong as one whose cross-section is 4″ × 6″. The strength of a very long column varies, practically, inversely as the square of the length; i. e., if a column *b* is twice as long as a column *a*, the strength of $b = (\frac{1}{2})^2 = \frac{1}{4}$ the strength of *a*, the cross-sections being equal.

1167. The conditions of the ends of a column play a very important part in determining their strength, and must always be taken into consideration. In Fig. 233, are shown three classes of columns. The column marked *a* is used in architecture, while the columns similar to *b* and *c* are used in bridge and machine construction.

According to theory, which is confirmed by experiment, a column having one end flat and the other rounded, like *b*, is $2\frac{1}{4}$ times as strong as a column having both ends rounded, like *c*.

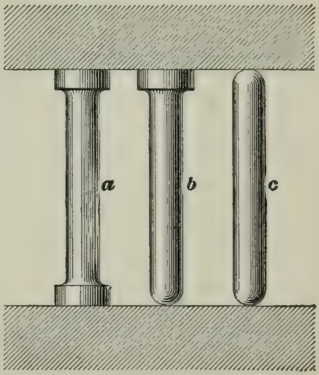


FIG. 233.

One having both ends flat, like *a*, is 4 times as strong as *c*, which has both ends rounded, the three columns being of the same length. If the length of *c* be taken as 1, the length of *b* may be $1\frac{1}{2}$, and that of *a* may be 2 for equal strength, the cross-sections all being the same; for, since the strengths vary inversely as the squares of the lengths, the strength of *c* is to that of *b* as $1 : \frac{1}{(1\frac{1}{2})^2}$ or as $1 : \frac{4}{9}$. But, since b is $2\frac{1}{4} = \frac{9}{4}$ times as strong as c, $\frac{4}{9} \times \frac{9}{4} = 1$, or the length of b being $1\frac{1}{2}$ times that of c, its strength is the same. Similarly, when a is twice as long as c its strength is the same.

1168. Columns like b and c do not actually occur in practice, an eye being formed at the end of the column and a pin inserted, forming what may be termed a *hinged end.* A steam engine connecting-rod is a good example of a column having two hinged ends, and a piston rod of a column having one end hinged and one end flat.

1169. There are numerous formulas for calculating the strength of columns, but the one that gives the most satisfactory results for columns of all lengths is the following:

$$W = \frac{S_2 A}{f\left(1 + \frac{A l^2}{g I}\right)}. \qquad \textbf{(76.)}$$

In this formula, $W =$ load, $S_2 =$ ultimate strength for compression, taken from Table 22, Art. **1115,** $A =$ area of section of a column in square inches, $f =$ factor of safety, $l =$ length in inches, $g =$ constant, to be taken from Table 26, and $I =$ least moment of inertia of the cross-section—that is, the moment of inertia about an axis passing through the center of gravity of the cross-section and parallel to the longest side. In other words, if the column has a rectangular cross-section whose longer side is b and shorter side d, the least moment of inertia is $\frac{b d^3}{12}$, the axis in this case being parallel to the long sides b. The values of g are given in the following table:

TABLE 26.

Material.	Both Ends Fixed.	One End Hinged.	Both Ends Hinged.
Timber	3,000	1,690	750
Cast Iron	5,000	2,810	1,250
Wrought Iron	36,000	20,250	9,000
Steel	25,000	14,060	6,250

EXAMPLE.—A column whose cross-section is an I (see Table of Moments of Inertia) is used as a column having both ends fixed, $d = 8''$, $d_1 = 6''$, $b = 6''$, and $b_1 = 3\frac{1}{2}''$. If made of wrought iron, what steady load will it sustain with safety, the length being 10 ft.?

SOLUTION.—From the Table of Moments of Inertia, least I $= \frac{1}{12}(d b^3 - d_1 b_1{}^3) = \frac{8 \times 6^3 - 6 \times 3.5^3}{12} = 122.5625$. $A = b d - b_1 d_1 = 6 \times 8 = 3.5 \times 6 = 27$ sq. in. $S_2 = 55,000$, $f = 4$, $l = 10 \times 12 = 120''$ and $g = 36,000$. Therefore, $W = \frac{55,000 \times 27}{4\left(1 + \frac{27 \times 120^2}{36,000 \times 122.5625}\right)} = \frac{1,485,000}{4.35247} = 341,186$ lb., nearly.

Had the column been less than $10 \times 6 = 60$ in. $= 5$ ft. long, the safe load would have been $\frac{55,000 \times 27}{4} = 371,250$ lb. Had it been twice as long, it would have supported a safe load of only

$$\frac{55,000 \times 27}{4\left(1 + \frac{27 \times 240^2}{36,000 \times 122.5625}\right)} = 274,498 \text{ lb.}$$

1170. In the actual designing of a column, the size of the cross-section is not known, but the form (square, round, etc.) is known, also the length, material, condition of ends and load it is to carry. To find the size of the cross-section, substitute $\frac{S_2}{f}$ in formula **65,** for S, and solve for A, obtaining $A = \frac{Pf}{S_2}$. Substituting in this equation the values of $P (= W)$, f, and S_2, this gives the value of A for a short piece less than 10 times the length of the shortest side, or diameter. Assume a value of A somewhat larger than that just found, and dimension a cross-section of the form chosen so that its area shall equal that assumed. Calculate the moment of inertia and substitute the values of W, A, I, l, and g in formula **76,** and solve for $\frac{S_2}{f}$. If the result last found equals the value of $\frac{S_2}{f}$ taken from Tables 22 and 24, the assumed dimensions are correct; if larger, the assumed dimensions must be increased; if smaller, they should be diminished, and in both cases the value of $\frac{S_2}{f}$ should be recalculated. An example will serve to illustrate the process.

Example.—What should be the diameter of a steel piston rod 5 feet long, the diameter of the piston being 18 inches and the greatest pressure 130 pounds per square inch?

Solution.—S_2 for this case $= 150{,}000$ lb. Since the piston rod is liable to shocks, a factor of safety of 10 should be used; hence, $\frac{S_2}{f} = \frac{150{,}000}{10} = 15{,}000$ lb. The load $W = 18^2 \times .7854 \times 130 = 33{,}081$ lb. $A = \frac{Pf}{S_2} = \frac{33{,}081}{15{,}000} = 2.2$ sq. in., nearly.

Assume that 3 sq. in. are needed. The diameter of a circle corresponding to an area of 3 sq. in. is $\sqrt{\frac{3}{.7854}} = 1.9544''$. Assume the diameter to be $1\frac{15}{16}'' = 1.9375$; the area will be $1.9375^2 \times .7854 = 2.9483$ sq. in. The value of $I = \frac{\pi d^4}{64} = \frac{3.1416 \times (1\frac{15}{16})^4}{64} = .69173$. $W = \frac{S_2 A}{f\left(1 + \frac{A\,l^2}{g\,I}\right)}$.

Consequently, $\frac{S_2}{f} = \frac{W}{A}\left(1 + \frac{A\,l^2}{g\,I}\right) = \frac{33{,}081}{2.9483}\left(1 + \frac{2.9483 \times (5 \times 12)^2}{14{,}060 \times .69173}\right) = 23{,}465$ lb.

As this value exceeds 15,000 lb., the diameter of the rod must be increased. Trying $2\frac{1}{8}''$ as the diameter, the area is 3.5466 sq. in., and $I = 1.00093$. Substituting these values as before, $\frac{S^2}{f} = \frac{33{,}081}{3.5466}\left(1 + \frac{3.5466 \times 60^2}{14{,}060 \times 1.00093}\right) = 17{,}790$ lb. This is still too large; hence, trying $2\frac{1}{4}''$, the area $= 3.976$ sq. in. $I = 1.258$ and

$$\frac{S_2}{f} = \frac{33{,}081}{3.976}\left(1 + \frac{3.976 \times 60^2}{14{,}060 \times 1.258}\right) = 15{,}052 \text{ lb.}$$

Consequently, the diameter should be $2\frac{1}{4}''$.

EXAMPLES FOR PRACTICE.

1. What safe steady load will a hollow cylindrical cast-iron column support, which is 14 feet long, outside diameter 10 inches, inside diameter 8 inches, and which has flat ends? Ans. 273,500 lb

2. A hollow wooden column having a square cross-section is to support a steady load of 15,155 pounds. If the thickness of the side is $1\frac{1}{2}$ inches, length of column 20 feet, and the ends flat, what should be the length of the sides of the cross-section, outside and inside?

Ans. Outside, 9″; inside, 6″.

3. Suppose a wrought-iron connecting-rod to have a rectangular cross-section of uniform size throughout its length. If the diameter of the steam cylinder is 40 inches, steam pressure 110 pounds per square inch, and the length of the rod is $12\frac{1}{2}$ feet, what should be the dimensions of the cross-section of the rod? Ans. $5\frac{1}{2}'' \times 9''$.

1171. The preceding method for determining the dimensions of the cross-section, when the load and length are given, is perfectly general, and can, therefore, be used in every case. It is, however, somewhat long and cumbersome. For the special cases of square, circular, and rectangular columns, the following formulas may be applied, if preferred. They seem complicated, but, when substitutions are made for the quantities given, the formulas will be found of relatively easy application.

For square columns, the side c of the square is given by the formula

$$c = \sqrt{\frac{Wf}{2\,S_2} + \sqrt{\frac{Wf}{S_2}\left(\frac{Wf}{4\,S_2} + \frac{12\,l^2}{g}\right)}}. \qquad \textbf{(77.)}$$

For circular columns, the diameter d of the circle is given by the formula

$$d = 1.4142\sqrt{\frac{.3183\,Wf}{S_2} + \sqrt{\frac{.3183\,Wf}{S_2}\left(\frac{.3183\,Wf}{S_2} + \frac{16\,l^2}{g}\right)}}. \qquad \textbf{(78.)}$$

For rectangular columns, assume the shorter dimension (depth $= d$). Then the longer dimension (breadth $= b$) is given by the formula

$$b = \frac{Wf\left(1 + \frac{12\,l^2}{d^2\,g}\right)}{d\,S_2}. \qquad \textbf{(79.)}$$

Should the dimensions given by the last formula be too much out of proportion, a new value may be assumed for d, and a new value found for b.

EXAMPLE.—Required the section of a square timber pillar to stand a steady load of 20 tons, the length of the column being 30 feet, and its ends both flat.

SOLUTION.—Here $S_2 = 8{,}000$ lb., $f = 8$, $g = 3{,}000$, $W = 40{,}000$ lb., $l = 30 \times 12 = 360$ in. These values, substituted in formula **77**, give

$$c = \sqrt{\frac{40{,}000 \times 8}{2 \times 8{,}000} + \sqrt{\frac{40{,}000 \times 8}{8{,}000}\left(\frac{40{,}000 \times 8}{4 \times 8{,}000} + \frac{12 \times 360^2}{3{,}000}\right)}}$$
$$= \sqrt{20 + \sqrt{40\left(10 + \frac{4 \times 36^2}{10}\right)}}$$
$$= \sqrt{20 + \sqrt{21{,}136}} = \sqrt{20 + 145.35}$$
$$= \sqrt{165.35} = 12.90 = 12\tfrac{7}{8}'', \text{ nearly, or say } 13''.$$

EXAMPLE.—Let it be required to solve the problem worked out by the general method in Art. **1170.**

SOLUTION.—Here $S_2 = 150{,}000$ lb., $f = 10$, $g = 14{,}060$, $W = 33{,}000$ lb., nearly, and $l = 5 \times 12 = 60$ in. From these data we have

$$\frac{.3183\ W f}{S_2} = \frac{.3183 \times 33{,}000 \times 10}{150{,}000} = \frac{.3183 \times 11}{5} = .7003,$$
$$\frac{16\, l^2}{g} = \frac{16 \times 60^2}{14{,}060} = \frac{8 \times 360}{703} = \frac{4.0967}{4.7970}.$$

Then, formula **78,**

$$d = 1.4142 \sqrt{.7003 + \sqrt{.7003 \times 4.80}}$$
$$= 1.4142 \sqrt{.7003 + \sqrt{3.3614}} = 1.4142 \sqrt{.7003 + 1.8334}$$
$$= 1.4142 \times 1.59 = 2.23 = 2\tfrac{1}{4}'', \text{ nearly,}$$

as found by the general or trial method.

The student may apply formula **79** to the solution of example 3 in the preceding article.

TORSION AND SHAFTS.

1172. When a force is applied to a beam in such a manner that it tends to twist it, the stress thus produced is termed **torsion.** In Fig. 234, $b\,c$ represents a beam fixed at one end; a load W is applied at the end of a lever arm $o\,n$, which twists the beam. If a straight line $c\,b$ is drawn parallel to the axis before the load is applied, it will be found, after the weight W has been hung from n, that the line $c\,b$ will take a position $c\,a$, forming a spiral. If the load does not strain the material beyond its elastic limit, *c a* will return to its original position *c b* when W is removed. It will also be found that the angles *a c b* and *a o b* are directly proportional to the loads.

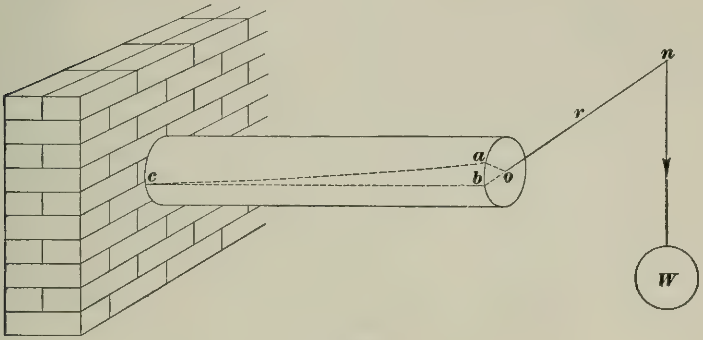


FIG. 234

Torsion manifests itself in the case of rotating shafts. Instead of one end being fixed, as in the previous case, the reisistance which the shaft has to overcome takes the place of the force which before was necessary for fixing one end. Should the shaft be too small, the resistance will overcome the strength of the material and rupture it.

1173. The angle *a o b*, which may be called the angle of twist, plays an important part in the designing of shafts. For all solid shafts below 11 inches in diameter, the following formula may be used:

$$d = c\sqrt[4]{Pr} = c_1\sqrt[4]{\frac{H}{N}}, \qquad \textbf{(80.)}$$

in which $d =$ diameter of round shaft or the side of a square shaft in inches; $c =$ constant from Table 27; $P =$ force or weight applied to the end of the lever arm, in pounds; $r =$ length of lever arm in inches, from center of shaft to point of application of P; $c_1 =$ constant from same table; $H =$ horsepower transmitted, and $N =$ number of revolutions per minute.

TABLE 27.

Material.	c.		c_1.	
	Round.	Square.	Round.	Square.
Wrought Iron......	.31	.272	4.92	4.31
Cast Iron	.353	.309	5.59	4.89
Steel	.297	.26	4.7	4.11

EXAMPLE.—What should be the diameter of a wrought-iron crank shaft for a 16″ × 20″ steam engine, if the greatest steam pressure is to be 90 lb. per sq. in.? (Assume that the entire steam pressure is transmitted through the crank-pin at some point of the stroke).

SOLUTION.—Total pressure on piston $= 16^2 \times .7854 \times 90 = 18{,}095.616$, say 18,000 lb. $= P$ in formula **80.** $r = \frac{20}{2} = 10''$. Therefore,

$$d = c\sqrt[4]{Pr} = .31\sqrt[4]{18{,}000 \times 10} = 6.385''.$$

A $6\frac{3}{8}''$ shaft would be sufficiently large.

EXAMPLE.—What horsepower could be safely transmitted by a 7-inch cast iron square shaft making 80 revolutions per minute?

SOLUTION.—Formula **80** gives, $d = c_1\sqrt[4]{\frac{H}{N}}$, or $H = \frac{Nd^4}{c_1{}^4} = \frac{80 \times 7^4}{4.89^4} =$ 335.93, say 336 horsepower.

1174. If the diameter of a wrought-iron shaft is greater than 12.4 inches, of a cast-iron shaft greater than 10.3 inches, or of a steel shaft greater than 13.6 inches, the following formula should be used:

$$d = k\sqrt[3]{Pr} = k_1\sqrt[3]{\frac{H}{N}}, \qquad (81.)$$

k and k_1 being taken from Table 28. If the shaft is hollow (round), either of the two following formulas may be used:

$$P = q\left(\frac{d_1{}^4 - d_2{}^4}{d_1 r}\right), \qquad (82.)$$

or

$$H = q_1 N\left(\frac{d_1{}^4 - d_2{}^4}{d_1}\right), \qquad (83.)$$

d_1 and d_2 being the outside and inside diameters, respectively, and q and q_1 constants to be taken from Table 28.

TABLE 28.

Material.	k	k_1	q	q_1
Wrought Iron	.0909	3.62	1,335	.0212
Cast Iron	.1145	4.56	669	.0106
Steel	.0828	3.3	1,767	.028

Example.—What horsepower can be safely transmitted by a hollow wrought-iron shaft making 60 revolutions per minute, and whose diameters are $9\frac{1}{2}$ and 12 inches?

Solution.—

$$H = q_1 N\left(\frac{d_1^4 - d_2^4}{d_1}\right) = .0212 \times 60\left(\frac{12^4 - 9.5^4}{12}\right) = 1{,}334.65 \text{ H.P.}$$

EXAMPLES FOR PRACTICE.

1. What should be the diameter of a steel shaft to transmit 500 horsepower at 200 revolutions per minute? Ans. 5.91″, say $5\frac{15}{16}$″.

2. How many horsepower will an 8″ round wrought-iron shaft transmit with safety, running at 150 R. P. M.? Ans. 1,048.5 H.P.

3. A hollow cast-iron shaft has an outside diameter of 10 inches and an inside diameter of 6 inches; at what speed should it be run to transmit 750 horsepower? Ans. 81.29 R. P. M.

4. A wrought-iron shaft 4 inches square runs at 110 revolutions per minute; what horsepower will it safely transmit? Ans. 81.6 H.P.

5. What should be the diameter of a wrought-iron shaft to transmit 6,000 horsepower at 100 revolutions per minute?

Ans. 14.172″, say $14\frac{3}{16}$″.

ROPES.

1175. The strength of hemp and manila ropes varies greatly, depending not so much upon the material and area of cross-section as upon the method of manufacture and the amount of twisting.

Hemp ropes are about 25% to 30% stronger than manila ropes or tarred hemp ropes. Ropes laid with tar wear better than those laid without tar, but their strength and flexibility are greatly reduced. For most purposes, the following formula may be used for the safe working load of any of the three ropes mentioned above:

$$P = 100\,C^2, \qquad \textbf{(84.)}$$

in which $P =$ working load in pounds and $C =$ circumference of rope in inches. This formula gives a factor of safety of from $7\frac{1}{2}$ for manila or tarred hemp rope to about 11 for best three-strand hemp rope. When excessive wear is likely to occur, it is better to make the circumference of the rope considerably larger than that given by the formula.

1176. Wire rope is made by twisting a number of wires (usually 19) together into a strand and then twisting several strands (usually 7) together to form the rope. It is very much stronger than hemp rope, and may be much smaller in size to carry the same load.

For iron wire rope of 7 strands, 19 wires to the strand, the following formula may be used, the letters having the same meaning as in formula **84**:

$$P = 600\,C^2. \qquad \textbf{(85.)}$$

Steel wire ropes should be made of the best quality of steel wire; when so made they are superior to the best iron wire ropes. If made from an inferior quality of steel wire, the ropes are not as good as the better class of iron wire ropes. When substituting steel for iron ropes, the object in view should be to gain an increase of wear rather than to reduce the size. The following formula may be used in computing the size or working strength of the best steel wire rope, 7 strands, 19 wires to the strand:

$$P = 1{,}000\,C^2. \qquad \textbf{(86.)}$$

Formulas **85** and **86** are based on a factor of safety of 6.

1177. When using ropes for the purpose of raising loads to a considerable height, the weight of the rope itself must also be considered and added to the load. The weight of the rope per running foot, for different sizes, may be obtained from the manufacturer's catalogue.

EXAMPLE.—What should be the allowable working load of an iron wire rope whose circumference is $6\frac{3}{4}$ inches? Weight of rope not to be considered.

SOLUTION.—Using formula **85**,

$$P = 600 \times (6\tfrac{3}{4})^2 = 27{,}337.5 \text{ lb.}$$

EXAMPLE.—The working load, including weight, of a hemp rope is to be 900 pounds; what should be its circumference?

SOLUTION.—Using formula **84**,

$$C = \sqrt{\frac{P}{100}} = \sqrt{\frac{900}{100}} = 3''.$$

1178. In measuring ropes, the circumference is used instead of the diameter, because the ropes are not round and the circumference is not equal to 3.1416 times the diameter. For three strands the circumference is about 2.86 d, for seven strands about 3 d, d being the diameter.

CHAINS.

1179. The size of a chain is always specified by giving the diameter of the iron from which the link is made. The two kinds of chain most generally used are the **open link** chain and the **stud link** chain. The former is shown by (*a*), Fig. 235, and the latter by (*b*). The stud prevents the two sides of a link from coming together when under a heavy pull, and thus strengthens the chain.

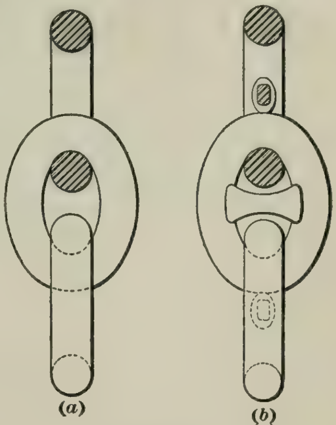


FIG. 235.

It is a good practice to anneal old chains which have become brittle by overstraining. This renders them less liable to snap from sudden jerks. The annealing process reduces their tensile strength, but increases their toughness and ductility, two qualities which are sometimes more important than mere strength.

Let P = safe load in pounds;
d = diameter of link in inches.

Then, for open link chains, made from a good quality of wrought iron,

$$P = 12{,}000\, d^2, \qquad \textbf{(87.)}$$

and, for stud link chains,

$$P = 18{,}000\, d^2. \qquad \textbf{(88.)}$$

EXAMPLE.—What load will be safely sustained by a $\frac{3}{4}$-inch open link chain?

SOLUTION.—Using formula **87**,

$$P = 12{,}000\ d^2 = 12{,}000 \times (\tfrac{3}{4})^2 = 6{,}750 \text{ lb.}$$

EXAMPLE.—What must be the diameter of a stud link chain to carry a load of 28,125 pounds?

SOLUTION.—Using formula **88**, $P = 18{,}000\ d^2$. Hence,

$$d = \sqrt{\frac{P}{18{,}000}} = \sqrt{\frac{28{,}125}{18{,}000}} = 1\tfrac{1}{4}''.$$

1179*a*. The statement in the last sentence of Art. **1116** may be modified in practice to include pieces whose lengths are not greater than *ten* times their least transverse dimensions, without material error. Although the stress is pure compression only for pieces whose lengths do not exceed *five* times their least transverse dimensions, the results obtained by formula **76** agree so closely with those obtained by formula **67**, when the length does not exceed ten times the least transverse dimension, that the latter may be used in all such cases. When the length is greater than ten times the least transverse dimension, the piece becomes a column, and formula **76** must be used.

ARITHMETIC.

(ARTS. 1–181.)

(1) What is arithmetic?

(2) What is a number?

(3) What is the difference between a concrete number and an abstract number?

(4) Define notation and numeration.

(5) Write each of the following numbers in words:

(*a*) 980; (*b*) 605; (*c*) 28,284; (*d*) 9,006,042; (*e*) 850,317,002; (*f*) 700,004.

(6) Represent in figures the following expressions:

(*a*) Seven thousand, six hundred. (*b*) Eighty-one thousand, four hundred, two. (*c*) Five million, four thousand, seven. (*d*) One hundred eight million, ten thousand, one. (*e*) Eighteen million, six. (*f*) Thirty thousand, ten.

(7) What is the sum of $3,290 + 504 + 865,403 + 2,074 + 81 + 7$? Ans. 871,359.

(8) $709 + 8,304,725 + 391 + 100,302 + 300 + 909 =$ what? Ans. 8,407,336.

(9) Find the difference between the following:

(*a*) 50,962 and 3,338; (*b*) 10,001 and 15,339.

Ans. { (*a*) 47,624.
(*b*) 5,338.

(10) (*a*) $70,968 - 32,975 =$? (*b*) $100,000 - 98,735 =$?

Ans. { (*a*) 37,993.
(*b*) 1,265.

(11) The greater of two numbers is 1,004 and their difference is 49; what is their sum? Ans. 1,959.

(12) From 5,962 + 8,471 + 9,023 take 3,874 + 2,039.
Ans. 17,543.

(13) A man willed $125,000 to his wife and two children; to his son he gave $44,675, to his daughter $26,380, and to his wife the remainder. What was his wife's share?
Ans. $53,945.

(14) Find the products of the following:
(*a*) 526,387 × 7; (*b*) 700,298 × 17; (*c*) 217 × 103 × 67.

$$\text{Ans.} \begin{cases} (a) & 3{,}684{,}709. \\ (b) & 11{,}905{,}066. \\ (c) & 1{,}497{,}517. \end{cases}$$

(15) If your watch ticks once every second, how many times will it tick in one week? Ans. 604,800 times.

(16) If a monthly publication contains 24 pages in each issue, how many pages will there be in eight yearly volumes?
Ans. 2,304.

(17) An engine and boiler in a manufactory are worth $3,246. The building is worth three times as much, plus $1,200, and the tools are worth twice as much as the building, plus $1,875. (*a*) What is the value of the building and tools? (*b*) What is the value of the whole plant?

$$\text{Ans.} \begin{cases} (a) & \$34{,}689. \\ (b) & \$37{,}935. \end{cases}$$

(18) Solve the following by cancelation:

$$(a)\ \frac{72 \times 48 \times 28 \times 5}{96 \times 15 \times 7 \times 6} = ? \qquad (b)\ \frac{80 \times 60 \times 50 \times 16 \times 14}{70 \times 50 \times 24 \times 20} = ?$$

$$\text{Ans.} \begin{cases} (a) & 8. \\ (b) & 32. \end{cases}$$

(19) If a mechanic earns $1,500 a year for his labor, and his expenses are $968 per year, in what time can he save enough to buy 28 acres of land, at $133 an acre?
Ans. 7 years.

(20) A freight train ran 365 miles in one week, and 3 times as far, lacking 246 miles, the next week; how far di it run the second week? Ans. 849 miles

(21) If the driving wheel of a locomotive is 16 ft. in ci cumference, how many revolutions will it make in going fro Philadelphia to Pittsburg, the distance of which is 35 miles, there being 5,280 feet in one mile? Ans. 116,820 rev.

(22) What is the quotient of
(*a*) 589,824 ÷ 576? (*b*) 369,730,620 ÷ 43,911? (*c*) 2,527,525 ÷ 505? (*d*) 4,961,794,302 ÷ 1,234?

Ans. (*a*) 1,024.
(*b*) 8,420.
(*c*) 5,005.
(*d*) 4,020,903.

(23) A man paid $444 for a horse, wagon, and harness. If the horse cost $264 and the wagon $153, how much did the harness cost? Ans. $27.

(24) What is the product of
(*a*) 1,024 × 576? (*b*) 5,005 × 505? (*c*) 43,911 × 8,420?

Ans. (*a*) 589,824.
(*b*) 2,527,525.
(*c*) 369,730,620.

(25) If a man receives 30 cents an hour for his wages, how much will he earn in a year, working 10 hours a day and averaging 25 days per month? Ans. $900.

(26) What is a fraction?

(27) What are the terms of a fraction?

(28) What does the denominator show?

(29) What does the numerator show?

(30) How do you find the value of a fraction?

(31) Is $\frac{13}{8}$ a proper or an improper fraction, and why?

(32) Write three mixed numbers.

(33) Reduce the following fractions to their lowest terms: $\frac{4}{8}$, $\frac{4}{16}$, $\frac{8}{32}$, $\frac{32}{64}$. Ans. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$.

(34) Reduce 6 to an improper fraction whose denominator is 4. Ans. $\frac{24}{4}$.

(35) Reduce $7\frac{7}{8}$, $13\frac{5}{16}$, and $10\frac{3}{4}$ to improper fractions.
Ans. $\frac{63}{8}$, $\frac{213}{16}$, $\frac{43}{4}$.

(36) What is the value of each of the following: $\frac{13}{2}$, $\frac{17}{4}$, $\frac{69}{16}$, $\frac{16}{8}$, $\frac{67}{64}$? Ans. $6\frac{1}{2}$, $4\frac{1}{4}$, $4\frac{5}{16}$, 2, $1\frac{3}{64}$.

(37) Solve the following:

(*a*) $35 \div \frac{5}{16}$; (*b*) $\frac{9}{16} \div 3$; (*c*) $\frac{17}{2} \div 9$; (*d*) $\frac{113}{64} \div \frac{7}{16}$; (*e*) $15\frac{3}{4} \div 4\frac{3}{8}$.

$$\text{Ans.} \begin{cases} (a) & 112. \\ (b) & \frac{3}{16}. \\ (c) & \frac{17}{18}. \\ (d) & 4\frac{1}{28}. \\ (e) & 3\frac{3}{5}. \end{cases}$$

(38) $\frac{1}{8} + \frac{2}{8} + \frac{5}{8} = ?$ Ans. 1.

(39) $\frac{1}{4} + \frac{3}{8} + \frac{5}{16} = ?$ Ans. $\frac{15}{16}$.

(40) $42 + 31\frac{5}{8} + 9\frac{7}{16} = ?$ Ans. $83\frac{1}{16}$.

(41) An iron plate is divided into four sections; the first contains $29\frac{3}{4}$ square inches; the second, $50\frac{5}{8}$ square inches; the third, 41 square inches, and the fourth, $69\frac{3}{16}$ square inches. How many square inches are in the plate?
Ans. $190\frac{9}{16}$ sq. in

(42) Find the value of each of the following:

$$(a)\ \frac{7}{\frac{3}{16}};\quad (b)\ \frac{\frac{15}{32}}{\frac{5}{8}};\quad (c)\ \frac{\frac{4+3}{2+6}}{5}. \qquad \text{Ans.} \begin{cases} (a) & 37\frac{1}{3}. \\ (b) & \frac{3}{4}. \\ (c) & \frac{7}{40}. \end{cases}$$

(43) The numerator of a fraction is 28, and the value of the fraction $\frac{7}{8}$; what is the denominator? Ans. 32.

(44) What is the difference between (*a*) $\frac{7}{8}$ and $\frac{7}{16}$? (*b*) 13 and $7\frac{7}{16}$? (*c*) $312\frac{9}{16}$ and $229\frac{5}{32}$?

$$\text{Ans.} \begin{cases} (a) & \frac{7}{16}. \\ (b) & 5\frac{9}{16}. \\ (c) & 83\frac{13}{32}. \end{cases}$$

(45) If a man travels $85\frac{5}{12}$ miles in one day, $78\frac{9}{15}$ miles in another day, and $125\frac{17}{35}$ miles in another day, how far did he travel in the three days? Ans. $289\frac{211}{420}$ miles.

(46) From $573\frac{4}{5}$ tons take $216\frac{5}{8}$ tons. Ans. $357\frac{7}{40}$.

(47) At $\frac{3}{8}$ of a dollar a yard, what will be the cost of $9\frac{1}{4}$ yards of cloth? Ans. $3\frac{15}{32}$ dollars.

(48) Multiply $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{7}{11}$ of $\frac{19}{20}$ of 11 by $\frac{7}{8}$ of $\frac{5}{6}$ of 45.
Ans. $109\frac{13}{128}$.

(49) How many times are $\frac{2}{3}$ contained in $\frac{3}{4}$ of 16?
Ans. 18 times.

(50) Bought $211\frac{1}{4}$ pounds of old lead for $1\frac{7}{8}$ cents per pound. Sold a part of it for $2\frac{1}{2}$ cents per pound, receiving for it the same amount as I paid for the whole. How many pounds did I have left? Ans. $52\frac{13}{16}$ pounds.

(51) Write out in words the following numbers: .08, .131, .0001, .000027, .0108, and 93.0101.

(52) How do you place decimals for addition and subtraction?

(53) Give a rule for multiplication of decimals.

(54) Give a rule for division of decimals.

(55) State the difference between a fraction and a decimal.

(56) State how to reduce a fraction to a decimal.

(57) Reduce the following fractions to equivalent decimals: $\frac{1}{2}$, $\frac{7}{8}$, $\frac{5}{32}$, $\frac{65}{100}$, and $\frac{125}{1000}$.

$$\text{Ans.}\left\{\begin{array}{l} .5. \\ .875. \\ .15625. \\ .65. \\ .125. \end{array}\right.$$

(58) Solve the following:

(*a*) $\frac{32.5 + .29 + 1.5}{4.7 + 9}$; (*b*) $\frac{1.283 \times \overline{8 + 5}}{2.63}$;

(*c*) $\frac{\overline{589 + 27} \times \overline{163 - 8}}{25 + 39}$; (*d*) $\frac{\overline{40.6 + 7.1} \times (3.029 - 1.874)}{6.27 + 8.53 - 8.01}$.

$$\text{Ans.}\begin{cases}(a) & 2.5029.\\ (b) & 6.3418.\\ (c) & 1{,}491.875.\\ (d) & 8.1139.\end{cases}$$

(59) How many inches in .875 of a foot? Ans. 10½ in.

(60) What decimal part of a foot is $\frac{3}{16}$ of an inch?
Ans. .015625.

(61) A cubic inch of water weighs .03617 of a pound. What is the weight of a body of water whose volume is 1,500 cubic inches? Ans. 54.255 lb.

(62) If by selling a carload of coal for $82.50, at a profit of $1.65 per ton, I make enough to pay for 72.6 ft. of fencing at $.50 a foot, how many tons of coal were in the car?
Ans. 22 tons.

(63) Divide 17,892 by 231, and carry the result to four decimal places. Ans. 77.4545 +.

(64) Find the value of the following expression when the result is carried to three decimal places:

$$\frac{74.26 \times 24 \times 3.1416 \times 19 \times 19 \times 350}{33{,}000 \times 12 \times 4} = ? \quad \text{Ans. } 446.619-.$$

(65) Express (*a*) .7928 in 64ths; (*b*) .1416 in 32ds; (*c*) .47915 in 16ths.

$$\text{Ans.}\begin{cases}(a) & \frac{51}{64}.\\ (b) & \frac{5}{32}.\\ (c) & \frac{8}{16}.\end{cases}$$

(66) Work out the following examples:

(*a*) $709.63 - .8514$; (*b*) $81.963 - 1.7$; (*c*) $18 - .18$; (*d*) $1 - .001$; (*e*) $872.1 - (.8721 + .008)$; (*f*) $(5.028 + .0073) - (6.704 - 2.38)$.

Ans. (*a*) 708.7786.
(*b*) 80.263.
(*c*) 17.82.
(*d*) .999.
(*e*) 871.2199.
(*f*) .7113.

(67) Work out the following:

(*a*) $\frac{7}{8} - .807$; (*b*) $.875 - \frac{3}{8}$; (*c*) $(\frac{5}{32} + .435) - (\frac{21}{100} - .07)$; (*d*) What is the difference between the sum of 33-millionths and 17-thousandths, and the sum of 53-hundredths and 274-thousandths?

Ans. (*a*) .068.
(*b*) .5.
(*c*) .45125.
(*d*) .786967.

(68) What is the sum of .125, .7, .089, .4005, .9, and .000027? Ans. 2.214527.

(69) $927.416 + 8.274 + 372.6 + 62.07938 = ?$
Ans. 1,370.36938.

(70) Add 17-thousandths, 2-tenths, and 47-millionths.
Ans. .217047.

(71) Find the products of the following expressions:

(*a*) $.013 \times .107$; (*b*) $203 \times 2.03 \times .203$; (*c*) $2.7 \times 31.85 \times (3.16 - .316)$; (*d*) $(107.8 + 6.541 - 31.96) \times 1.742$.

Ans. (*a*) .001391.
(*b*) 83.65427.
(*c*) 244.56978.
(*d*) 143.507702.

(72) Solve the following:

(*a*) $(\frac{7}{16} - .13) \times \overline{.625 + \frac{5}{8}}$; (*b*) $(\frac{19}{32} \times .21) - (.02 \times \frac{3}{16})$; (*c*) $(\frac{13}{4} + .013 - 2.17) \times \overline{13\frac{1}{4} - 7\frac{5}{16}}$.

Ans. (*a*) .384375.
(*b*) .1209375.
(*c*) 6.4896875.

(73) Solve the following:

(*a*) $.875 \div \frac{1}{2}$; (*b*) $\frac{7}{8} \div .5$; (*c*) $\dfrac{.375 \times \frac{1}{4}}{\frac{5}{16} - .125}$.

$$\text{Ans.} \begin{cases} (a)\ 1.75. \\ (b)\ 1.75. \\ (c)\ .5. \end{cases}$$

(74) Find the value of the following expression:

$$\frac{1.25 \times 20 \times 3}{\dfrac{87 + (11 \times 8)}{459 + 32}}.$$

Ans. $210\frac{3}{7}$.

(75) From 1 plus .001 take .01 plus .000001.

Ans. .990999.

ARITHMETIC.

(ARTS. 182–349.)

(76) What is 25 per cent. of 8,428 lb.? Ans. 2,107 lb.

(77) What is 1 per cent. of $100? Ans. $1.

(78) What is $\frac{1}{2}$ per cent. of $35,000? Ans. $175.

(79) What per cent. of 50 is 2? Ans. 4%.

(80) What per cent. of 10 is 10? Ans. 100%.

(81) Solve the following:

(*a*) Base = $2,522 and percentage = $176.54. What is the rate? (*b*) Percentage = 16.96 and rate = 8 per cent. What is the base? (*c*) Amount = 216.7025 and base = 213.5. What is the rate? (*d*) Difference = 201.825 and base = 207. What is the rate?

Ans. (*a*) 7%.
(*b*) 212.
(*c*) $1\frac{1}{2}$%.
(*d*) $2\frac{1}{2}$%.

(82) A farmer gained 15% on his farm by selling it for $5,500. What did it cost him? Ans. $4,782.61.

(83) A man receives a salary of $950. He pays 24% of it for board, $12\frac{1}{2}$% of it for clothing, and 17% of it for other expenses. How much does he save in a year? Ans. $441.75.

(84) If $37\frac{1}{2}$ per cent. of a number is 961.38, what is the number? Ans. 2,563.68.

(85) A man owns $\frac{3}{4}$ of a property. 30% of his share is worth $1,125. What is the whole property worth?
Ans. $5,000.

(86) What sum diminished by 35% of itself equals $4,810?
Ans. $7,400.

(87) A merchant's sales amounted to \$197.55 on Monday, and this sum was $12\frac{1}{2}\%$ of his sales for the week. How much were his sales for the week? Ans. \$1,580.40.

(88) The distance between two stations on a certain railroad is 16.5 miles, which is $12\frac{1}{2}\%$ of the entire length of the road. What is the length of the road? Ans. 132 miles.

(89) After paying 60% of my debts, I find that I still owe \$35. What was my whole indebtedness? Ans. \$87.50.

(90) Reduce 28 rd. 4 yd. 2 ft. 10 in. to inches.
Ans. 5,722 in.

(91) Reduce 5,722 in. to higher denominations.
Ans. 28 rd. 4 yd. 2 ft. 10 in.

(92) How many seconds in 5 weeks and 3.5 days?
Ans. 3,326,400 sec.

(93) How many pounds, ounces, pennyweights, and grains are contained in 13,750 gr.?
Ans. 2 lb. 4 oz. 12 pwt. 22 gr.

(94) Reduce 4,763,254 links to miles.
Ans. 595 mi. 32 ch. 54 li.

(95) Reduce 764,325 cu. in. to cu. yd.
Ans. 16 cu. yd. 10 cu. ft. 549 cu. in.

(96) What is the sum of 2 rd. 2 yd. 2 ft. 3 in.; 4 yd. 1 ft. 9 in.; 2 ft. 7 in.? Ans. 3 rd. 2 yd. 2 ft. 1 in.

(97) What is the sum of 3 gal. 3 qt. 1 pt. 3 gi.; 6 gal. 1 pt. 2 gi.; 4 gal. 1 gi.; 8 qt. 5 pt.? Ans. 16 gal. 3 qt. 2 gi.

(98) What is the sum of 240 gr. 125 pwt. 50 oz. and 3 lb.?
Ans. 7 lb. 8 oz. 15 pwt.

(99) What is the sum of 11° 16′ 12″; 13° 19′ 30″; 20° 25″; 26′ 29″; 10° 17′ 11″? Ans. 55° 19′ 47″.

(100) What is the sum of 130 rd. 5 yd. 1 ft. 6 in.; 215 rd. 2 ft. 8 in.; 304 rd. 4 yd. 11 in.? Ans. 2 mi. 10 rd. 5 yd. 7 in.

(101) What is the sum of 21 A. 67 sq. ch. 3 sq. rd. 21 sq. li.; 28 A. 78 sq. ch. 2 sq. rd. 23 sq. li.; 47 A. 6 sq. ch. 2 sq. rd. 18 sq. li.; 56 A. 59 sq. ch. 2 sq. rd. 16 sq. li.; 25 A. 38 sq. ch. 3 sq. rd. 23 sq. li.; 46 A. 75 sq. ch. 2 sq. rd. 21 sq. li.?
Ans. 255 A. 3 sq. ch. 14 sq. rd. 122 sq. li.

(102) From 20 rd. 2 yd. 2 ft. 9 in. take 300 ft.
Ans. 2 rd. 1 yd. 2 ft. 9 in.

(103) From a farm containing 114 A. 80 sq. rd. 25 sq. yd., 75 A. 70 sq. rd. 30 sq. yd. are sold. How much remains?
Ans. 39 A. 9 sq. rd. 25¼ sq. yd.

(104) From a hogshead of molasses, 10 gal. 2 qt. 1 pt. are sold at one time, and 26 gal. 3 qt. at another time. How much remains? Ans. 25 gal. 2 qt. 1 pt.

(105) If a person were born June 19, 1850, how old would he be August 3, 1892? Ans. 42 yr. 1 mo. 14 da.

(106) A note was given August 5, 1890, and was paid June 3, 1892. What length of time did it run?
Ans. 1 yr. 9 mo. 28 da.

(107) What length of time elapsed from 16 min. past 10 o clock A. M., July 4, 1883, to 22 min. before 8 o'clock P. M., Dec. 12, 1888? Ans. 5 yr. 5 mo. 8 da. 9 hr. 22 min.

(108) If 1 iron rail is 17 ft. 3 in. long, how long would 51 rails be, if placed end to end? Ans. 53 rd. 1½ yd. 9 in.

(109) Multiply 3 qt. 1 pt. 3 gi. by 4.7.
Ans. 4 gal. 2 qt. 1.7 gi.

(110) Multiply 3 lb. 10 oz. 13 pwt. 12 gr. by 1.5.
Ans. 5 lb. 10 oz. 6 gr.

(111) How many bushels of apples are contained in 9 bbl., if each barrel contains 2 bu. 3 pk. 6 qt.
Ans. 26 bu. 1 pk. 6 qt.

(112) Multiply 7 T. 15 cwt. 10.5 lb. by 1.7.
Ans. 13 T. 3 cwt. 67.85 lb.

(113) Divide 358 A. 57 sq. rd. 6 sq. yd. 2 sq. ft. by 7.
Ans. 51 A. 31 sq. rd. 8 sq. ft.

(114) Divide 282 bu. 3 pk. 1 qt. 1 pt. by 12.
Ans. 23 bu. 2 pk. 2 qt. ¼ pt.

(115) How many iron rails, each 30 ft. long, are required to lay a railroad track 23 miles long? Ans. 8,096 rails.

(116) How many boxes, each holding 1 bu. 1 pk. and 7 qt., can be filled from 356 bu. 3 pk. and 5 qt. of cranberries?
Ans. 243 boxes.

(117) If 16 square miles are equally divided into 62 farms, how much land will each contain?

Ans. 165 A. 25 sq. rd. 24 sq. yd. 3 sq. ft. 80 + sq. in.

(118) What is the square of 108? Ans. 11,664.

(119) What is the cube of 181.25? Ans. 5,954,345.703125.

(120) What is the fourth power of 27.61?

Ans. 581,119.73780641.

(121) Solve the following: (*a*) 106^2; (*b*) $(182\frac{1}{8})^2$; (*c*) $.005^2$; (*d*) $.0063^2$; (*e*) 10.06^2.

Ans.
(*a*) 11,236.
(*b*) 33,169.515625.
(*c*) .000025.
(*d*) .00003969.
(*e*) 101.2036.

(122) Solve the following: (*a*) 753^3; (*b*) 987.4^3; (*c*) $.005^3$; (*d*) $.4044^3$.

Ans.
(*a*) 426,957,777.
(*b*) 962,674,279.624.
(*c*) .000000125.
(*d*) .066135317184.

(123) What is the fifth power of 2? Ans. 32.

(124) What is the fourth power of 3? Ans. 81.

(125) What are the values of: (*a*) 67.85^2? (*b*) $967,845^2$? (*c*) $(\frac{3}{8})^2$? (*d*) $(\frac{1}{4})^2$?

Ans.
(*a*) 4,603.6225.
(*b*) 936,723,944,025.
(*c*) $\frac{9}{64}$.
(*d*) $\frac{1}{16}$.

(126) What is (*a*) the tenth power of 5? (*b*) The fifth power of 9?

Ans.
(*a*) 9,765,625.
(*b*) 59,049.

(127) Solve the following: (*a*) 1.2^4; (*b*) 11^6; (*c*) 1^7; (*d*) $.01^4$; (*e*) $.1^5$.

Ans.
(*a*) 2.0736.
(*b*) 1,771,561.
(*c*) 1.
(*d*) .00000001.
(*e*) .00001.

(128) Find the values of the following: (*a*) $.0133^3$; (*b*) 301.011^3; (*c*) $(\frac{1}{8})^3$; (*d*) $(3\frac{3}{4})^3$.

Ans. (*a*) .000002352637.
(*b*) 27,273,890.942264331.
(*c*) $\frac{1}{512}$.
(*d*) $52\frac{47}{64}$, or 52.734375.

(129) In what respect does evolution differ from involution?

NOTE.—In the answers to the following examples, a minus sign after a number indicates that the last digit is not quite as large as the number printed. Thus, 12.497 — indicates that the number really is 12.496 +, and that the 6 has been made a 7 because the next succeeding figure was 5 or greater. For example, had it been desired to use but three decimal places in example 121 (*b*), the answer would have been written 33,169.516 —.

(130) Find the square root of the following: (*a*) 3,486,784.401; (*b*) 9,000,099.4009; (*c*) .001225.

Ans. (*a*) 1,867.29 +.
(*b*) 3,000.017 —.
(*c*) .035.

(131) Extract the square root of (*a*) 10,795.21; (*b*) 73,008.04; (*c*) 90; (*d*) .09.

Ans. (*a*) 103.9.
(*b*) 270.2.
(*c*) 9.487 —.
(*d*) .3.

(132) Extract the cube root of (*a*) .32768; (*b*) 74,088; (*c*) 92,416; (*d*) .373248.

Ans. (*a*) .6894 +
(*b*) 42.
(*c*) 45.212 —.
(*d*) .72.

(133) Extract the cube root of 2 to six decimal places.

Ans. 1.259921 +.

(134) Extract the cube root of (*a*) 1,758.416743; (*b*) 1,191,016; (*c*) $\frac{4}{32}$; (*d*) $\frac{27}{512}$.

Ans. (*a*) 12.07.
(*b*) 106.
(*c*) $\frac{1}{2}$.
(*d*) $\frac{3}{8}$.

(135) Extract the cube root of 3 to six decimal places.

Ans. 1.442250 –.

(136) Solve the following: (*a*) $\sqrt{123.21}$; (*b*) $\sqrt{114.921}$; (*c*) $\sqrt{502,681}$; (*d*) $\sqrt{.00041209}$.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ 11.1.\\(b)\ 10.72+.\\(c)\ 709.\\(d)\ .0203.\end{array}\right.$$

(137) Solve the following: (*a*) $\sqrt[3]{.0065}$; (*b*) $\sqrt[3]{.021}$; (*c*) $\sqrt[3]{8,036,054,027}$; (*d*) $\sqrt[3]{.000004096}$; (*e*) $\sqrt[3]{17}$.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ .18663-.\\(b)\ .2759-.\\(c)\ 2,003.\\(d)\ .016.\\(e)\ 2.5713-.\end{array}\right.$$

(138) Solve the following: (*a*) $\sqrt[4]{6,561}$; (*b*) $\sqrt[6]{117,649}$; (*c*) $\sqrt[6]{.000064}$; (*d*) $\sqrt[3]{\frac{3}{8}}$.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ 9.\\(b)\ 7.\\(c)\ .2.\\(d)\ .72112+.\end{array}\right.$$

(139) Extract the square root of (*a*) $\frac{1225}{5476}$; (*b*) .3364; (*c*) .1; (*d*) $25.0\frac{3}{4}$; (*e*) $.000\frac{4}{9}$.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ \frac{35}{74}.\\(b)\ .58.\\(c)\ .31623-.\\(d)\ 5.00749+.\\(e)\ .02108+.\end{array}\right.$$

(140) (*a*) Extract the fourth root of 2 to four decimal places; (*b*) extract the sixth root of 6.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ 1.1892+.\\(b)\ 1.34801-.\end{array}\right.$$

(141) Extract the square root of (*a*) 3.1416 and (*b*) .7854 to four decimal places.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ 1.7725-.\\(b)\ .8862+.\end{array}\right.$$

(142) Extract the cube root of (*a*) 3.1416 and (*b*) .5236 to four decimal places.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ 1.4646-.\\(b)\ .8060-.\end{array}\right.$$

Find the value of x in the following:

(143) $11.7 : 13 :: 20 : x$. Ans. $22.22+$.

(144) (*a*) $20 + 7 : 10 + 8 :: 3 : x$; (*b*) $12^2 : 100^2 :: 4 : x$.

Ans. $\left\{\begin{array}{l}(a)\ 2.\\ (b)\ 277.7+.\end{array}\right.$

(145) (*a*) $\frac{4}{x} = \frac{7}{21}$; (*b*) $\frac{x}{24} = \frac{8}{16}$; (*c*) $\frac{2}{10} = \frac{x}{100}$; (*d*) $\frac{15}{45} = \frac{60}{x}$; (*e*) $\frac{10}{150} = \frac{x}{600}$.

Ans. $\left\{\begin{array}{l}(a)\ x = 12.\\ (b)\ x = 12.\\ (c)\ x = 20.\\ (d)\ x = 180.\\ (e)\ x = 40.\end{array}\right.$

(146) $x : 5 :: 27 : 12.5$ Ans. $10\frac{4}{5}$.

(147) $45 : 60 :: x : 24$. Ans. 18.

(148) $x : 35 :: 4 : 7$. Ans. 20.

(149) $9 : x :: 6 : 24$. Ans. 36.

(150) $\sqrt[3]{1,000} : \sqrt[3]{1,331} = 27 : x$. Ans. 29.7.

(151) $64 : 81 = 21^2 : x^2$. Ans. 23.625.

(152) $7 + 8 : 7 = 30 : x$. Ans. 14.

(153) A man whose steps measure 2 ft. 5 in. takes 2,480 steps in walking a certain distance. How many steps of 2 ft. 7 in. will be required for the same distance? Ans. 2,320 steps.

(154) If a horse travels 12 mi. in 1 hr. 36 min., how far will he travel at the same rate in 15 hr.? Ans. 112.5 mi.

(155) If a column of mercury 27.63 in. high weighs .76 of a pound, what will be the weight of a column of mercury having the same diameter, 29.4 inches high? Ans. .808+lb.

(156) If 2 gal. 3 qt. 1 pt. of water will last a man 5 da., how long will 5 gal. 3 qt. last him, if he drinks at the same rate? Ans. 10 da.

(157) Heat from a burning body varies inversely as the square of the distance from it. If a thermometer held 6 ft. from a stove shows a rise in temperature of 24 degrees, how many degrees rise in temperature would it indicate if held 12 ft. from the stove? Ans. 6°.

(158) If a pile of wood 12 ft. long, 4 ft. wide and 3 ft. high is worth $12, what is the value of a pile of wood 15 ft. long, 5 ft. wide and 6 ft. high? Ans. $37.50.

(159) If 100 gal. of water run over a dam in 2 hr., how many gallons will run over the dam in 14 hr. 28 min.? Ans. 723⅓ gal.

(160) If a cistern 28 ft. long, 12 ft. wide, 10 ft. deep holds 798 bbl. of water, how many barrels of water will a cistern hold that is 20 ft. long, 17 ft. wide, and 6 ft. deep? Ans. 484½ bbl.

(161) If a railway train runs 444 mi. in 8 hr. 40 min., in what time can it run 1,060 mi. at the same rate of speed? Ans. 20 hr. 41.44 min.

(162) If sound travels at the rate of 6,160 ft. in 5½ sec., how far does it travel in 1 min.? Ans. 67,200 ft.

(163) If 5 men by working 8 hours a day can do a certain amount of work, how many men by working 10 hours a day can do the same work? Ans. 4 men.

(164) If a man travel 540 miles in 20 days of 10 hours each, how many hours a day must he travel to cover 630 miles in 25 days? Ans. 9⅓ hr.

(165) Referring to example 4, Art. **349,** what is the horsepower of an engine whose cylinder is 30 inches in diameter, piston speed 660 feet per minute, and mean effective pressure 42 pounds per square inch? Ans. 594 horsepower.

(166) The weight of a cubic inch of cast iron is .261 pound Referring to Art. **345,** what is the weight of a solid cast iron cylinder whose diameter is 12 inches and length is 60 inches? Ans. 1,771.11 lb.

(167) Referring to Art. **348,** what is the centrifugal force of a 40-pound body revolving in a circle having a radius of 10 inches, at a speed of 18 feet per second? Ans. 484.7 lb.

ALGEBRA.

(ARTS. 350–524.)

(168) Change the fraction $-\frac{c-(a-b)}{c+(a+b)}$ so that the sign before the dividing line will be $+$.

(169) Factor the following: (a) $9x^4+12x^2y^2+4y^4$; (b) $49a^4-154a^2b^2+121b^4$; (c) $64x^2y^2+64xy+16$.

(170) Divide: (a) $3x^2+x+9x^3-1$ by $3x-1$; (b) $a^3-2ab^2+b^3$ by $a-b$; (c) $7x^3+58x-24x^2-21$ by $7x-3$.

$$\text{Ans.}\left\{\begin{array}{ll}(a) & 3x^2+2x+1.\\(b) & a^2+ab-b^2.\\(c) & x^2-3x+7.\end{array}\right.$$

(171) Why are letters used in Algebra, and in what ways do they differ from figures?

(172) Factor the expressions: (a) $4x^3y-12x^3y^2+8xy^3$; (b) x^4-y^4; (c) $8x^3-27y^3$.

$$\text{Ans.}\left\{\begin{array}{ll}(b) & (x^2+y^2)(x+y)(x-y).\\(c) & (2x-3y)(4x^2+6xy+9y^2).\end{array}\right.$$

(173) Multiply $3m^3+3n^3+10mn^2+10m^2n$ by $5m^2n^3-mn^4-5m^3n^2+3m^4n$.

$$\text{Ans.}\left\{\begin{array}{l}9m^7n+15m^6n^2-5m^5n^3+6m^4n^4\\+25m^3n^5+5m^2n^6-3mn^7.\end{array}\right.$$

(174) Raise to their indicated powers the following: $(2a^2bc^3)^4$, $(-3a^2b^2c)^5$ and $(-7m^3nx^2y^4)^2$.

(175) Factor: (a) $4a^2-b^2$; (b) $16x^{10}-1$; (c) $16x^6-8x^4y^2+x^2y^4$.

(176) Extract the square root of $4a^6-12a^5x+5a^4x^2+6a^3x^3+a^2x^4$. Ans. $2a^3-3a^2x-ax^2$.

(177) (a) Arrange $a^3b^2+2abc+3-7a^2b^3+6a^4b^4$ according to the decreasing powers of a; (b) according to the

increasing powers of b. (c) With $a^2 + 1 + 2a^3 + ax$ arranged according to the *increasing* powers of a, should the 1 be placed first or last, and why?

(178) Find the values of $\sqrt[4]{16a^{12}b^4c^8}$, $\sqrt[5]{-32a^{15}}$ and $\sqrt[3]{-1{,}728a^6d^{12}x^3y^9}$.

(179) (a) Enclose the first three and last three terms of $a - 2x + 4y - 3z - 2b + c$ in parentheses connected by a minus sign. (b) Place the expression $-3b - 4c + d - (2f - 3c)$ in brackets, preceded by a minus sign. (c) Indicate the subtraction of $2b - (3c + 2d) - a$ from x.

(180) Multiply: (a) $2x^3 + 2x^2 + 2x - 2$ by $x - 1$; (b) $x^2 - 4ax + c$ by $2x + a$, and (c) $-a^3 + 3a^2b - 2b^3$ by $5a^2 + 9ab$.

$$\text{Ans.} \left\{ \begin{array}{l} (a)\ 2x^4 - 4x + 2. \\ (b)\ 2x^3 - 7ax^2 + 2cx - 4a^2x + ac. \\ (c)\ -5a^5 + 6a^4b + 27a^3b^2 - 10a^2b^3 - 18ab^4. \end{array} \right.$$

(181) Find the sum of the following: (a) $4xyz - 3xyz - 5xyz$, $6xyz - 9xyz + 3xyz$. (b) $3a^2 + 2ab + 4b^2$, $5a^2 - 8ab + b^2$, $-a^2 + 5ab - b^2$, $18a^2 - 20ab - 19b^2$ and $14a^2 - 3ab + 20b^2$. (c) $4mn + 3ab - 4c$, $3x - 4ab + 2mn$ and $3m^2 - 4p$.

$$\text{Ans.} \left\{ \begin{array}{l} (a)\ -4xyz. \\ (b)\ 39a^2 - 24ab + 5b^2. \\ (c)\ 6mn - ab - 4c + 3x + 3m^2 - 4p. \end{array} \right.$$

(182) Find the reciprocal of 3.1416, .7854, and $\frac{1}{64.32}$.

(183) Perform the indicated additions:

(a) $\dfrac{x}{x-y} + \dfrac{x-y}{y-x}$; ($b$) $\dfrac{x^2}{x^2-1} + \dfrac{x}{x+1} - \dfrac{x}{1-x}$;

(c) $\dfrac{3a-4b}{7} - \dfrac{2a-b+c}{3} + \dfrac{13a-4c}{12}$.

$$\text{Ans.} \left\{ \begin{array}{l} (a)\ \dfrac{y}{x-y}. \\ (b)\ \dfrac{3x^2}{x^2-1}. \\ (c)\ \dfrac{71a-20b-56c}{84}. \end{array} \right.$$

(184) Resolve into their factors: (*a*) $45x^7y^{10} - 90x^5y^7 - 360x^4y^8$; (*b*) $a^2b^2 + 2abcd + c^2d^2$; (*c*) $(a+b)^2 - (c-d)^2$.

Ans. (*c*) $(a+b+c-d)\,(a+b-c+d)$.

(185) (*a*) Give an illustration, not contained in the text, which will explain the difference between positive and negative quantities. (*b*) In what respects are addition and subtraction different in Algebra from addition and subtraction in arithmetic?

(186) Divide:

(*a*) $\dfrac{2ax+x^2}{a^3-x^3}$ by $\dfrac{x}{a-x}$; (*b*) $\dfrac{6m^2n^2-3n}{4m^4n^2-4m^2n+1}$ by $\dfrac{3n}{4m^4n^2-1}$;

(*c*) $9+\dfrac{5y^2}{x^2-y^2}$ by $3+\dfrac{5y}{x-y}$.

$$\text{Ans.}\begin{cases}(a)\ \dfrac{2a+x}{a^2+ax+x^2}.\\(b)\ 2m^2n+1.\\(c)\ \dfrac{3x-2y}{x+y}.\end{cases}$$

(187) State which of the following trinomials are perfect squares: $1-2x^2+x^4$; $9m^2n^2-2mn+16$; $64+120x+25x^2$; $4x+1+4x^2$; $4a^2+20ab-25b^2$; $4x^2y^2-4y^4+x^4$.

(188) (*a*) What is the reciprocal of $\frac{24}{49}$? (*b*) Of what number is 700 the reciprocal?

(189) Find the least common multiple of $12xy\,(x^2-y^2)$, $2x^2\,(x^2+2xy+y^2)$, $3y^2(x-y)^2$ and $6(x^2+xy)$.

Ans. $12x^2y^2(x+y)^2(x-y)^2$.

(190) Multiply: (*a*) $2+4a-5a^2-6a^3$ by $7a^3$; (*b*) $4x^2-4y^2+6z^2$ by $3x^2y$; (*c*) $3b+5c-2d$ by $6a$.

$$\text{Ans.}\begin{cases}(a)\ 14a^3+28a^4-35a^5-42a^6.\\(b)\ 12x^4y-12x^2y^3+18x^2yz^2.\\(c)\ 18ab+30ac-12ad.\end{cases}$$

(191) (*a*) Explain in your own words the difference between a coefficient and an exponent. (*b*) How are coefficients and exponents treated in multiplication, and how in division? (*c*) What is the law of signs in multiplication?

(192) Remove the symbols of aggregation from the following:

(*a*) $2a - \{3b + [4c - 4a - (2a+2b)] + [3a - \overline{b+c}]\}$;

(*b*) $7a - \{3a - [(2a - 5a) + 4a]\}$;
(*c*) $a - \{2b + [3c - 3a - (a + b)] + [2a - (b + c)]\}$.

Ans. $\begin{cases} (a) \ 5a - 3c. \\ (b) \ 5a. \\ (c) \ 3a - 2c. \end{cases}$

(193) What are the factors of: (*a*) $x^3 + 8$? (*b*) $x^3 - 27y^3$? and (*c*) $xm - nm + xy - ny$? Ans. (*c*) $(x - n)(m + y)$.

(194) Extract the square root of $4x^4 + 8ax^3 + 4a^2x^2 + 16b^2x^2 + 16ab^2x + 16b^4$. Ans. $2x^2 + 2ax + 4b^2$.

(195) Reduce $\dfrac{c(a + b) + cd}{(a + b)c}$ to its simplest form.

(196) Combine the like terms of the expressions: (*a*) $x + y + z - (x - y) - (y + z) - (-y)$; (*b*) $(2x - y + 4z) + (-x - y - 4z) - (3x - 2y - z)$; (*c*) $a - [2a + (3a - 4a)] - 5a - \{6a - [(7a + 8a) - 9a]\}$.

Ans. $\begin{cases} (a) \ 2y. \\ (b) \ z - 2x. \\ (c) \ -5a. \end{cases}$

(197) State how you would read the following expressions: (*a*) $a^2x^2 + 2a^3b^5 - (a + b)$; (*b*) $\sqrt[3]{x} + y(a - n^2)^{\frac{2}{3}}$; (*c*) $(m + n)(m - n)^2\left(m - \dfrac{n}{2}\right)$.

(198) Divide $3a^3 + 2 - 4a^5 + 7a + 2a^6 - 5a^4 + 10a^2$ by $a^3 - 1 - a^2 - 2a$. Ans. $2a^3 - 2a^2 - 3a - 2$.

(199) Factor: (*a*) $x^8y^2 - 64x^2y^2$; (*b*) $a^2 - b^2 - c^2 + 1 - 2a + 2bc$; (*c*) $1 - 16a^2 + 8ac - c^2$.

Ans. $\begin{cases} (a) \ x^2y^2(x + 2)(x - 2)(x^2 + 2x + 4)(x^2 - 2x + 4). \\ (b) \ (a - 1 + b - c)(a - 1 - b + c). \\ (c) \ (1 + 4a - c)(1 - 4a + c). \end{cases}$

(200) How may the signs of all the terms of the denominator of a fraction be changed from + to − or from − to + without altering the value of the fraction?

(201) From $a^4 - b^4$ take $5a^3b - 7a^2b^2 + 5ab^3$, and from the result take $3a^4 - 4a^3b + 6a^2b^2 + 5ab^3 - 3b^4$.

Ans. $-2a^4 - a^3b + a^2b^2 - 10ab^3 + 2b^4$.

(202) (*a*) From $3a - 2b + 3c$ take $2a - 7b - c - b$. (*b*)

Subtract $x^3+y^3-xy^2$ from $2x^3-3x^2y+2xy^3$. (*c*) From $14a+4b-6c-3d$ take $11a-2b+4c-4d$.

$$\text{Ans.}\begin{cases}(a)\ a+6b+4c.\\(b)\ x^3-3x^2y+xy^2+2xy^3-y^3.\\(c)\ 3a+6b-10c+d.\end{cases}$$

(203) Find the numerical values of the following when $a=16$, $b=10$ and $x=5$: (*a*) $(ab^2x+2abx)4a$; (*b*) $2\sqrt{4a}-\frac{2bx}{a-b}+\frac{b-x}{x}$; (*c*) $(b-\sqrt{a})\,(x^3-b^2)\,(a^2-b^2)$.

$$\text{Ans.}\begin{cases}(a)\ 614{,}400.\\(b)\ \tfrac{1}{3}.\\(c)\ 23{,}400.\end{cases}$$

(204) Reduce to their simplest forms:

$$(a)\ \frac{15mxy^2}{75mx^2y^3};\quad (b)\ \frac{x^2-1}{4x(x+1)};\quad (c)\ \frac{(a^3+b^3)\,(a^2+ab+b^2)}{(a^3-b^3)\,(a^2-ab+b^2)}.$$

$$\text{Ans. } (c)\ \frac{a+b}{a-b}.$$

(205) Simplify:

$$(a)\ \frac{\dfrac{1}{1-x}-\dfrac{1}{1+x}}{\dfrac{1}{1-x}+\dfrac{1}{1+x}};\quad (b)\ \frac{\dfrac{a^2}{b^3}+\dfrac{1}{a}}{\dfrac{a}{b^2}-\dfrac{a-b}{ab}};$$

$$(c)\ \frac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$$

$$\text{Ans.}\begin{cases}(a)\ x.\\(b)\ \dfrac{a+b}{b}.\\(c)\ \dfrac{4}{3x+3}.\end{cases}$$

(206) Simplify $\frac{3+2x}{2-x}-\frac{2-3x}{2+x}+\frac{16x-x^2}{x^2-4}$. Ans. $\frac{1}{x+2}$.

(207) (*a*) Reduce $1+2x-\frac{4x-4}{5x}$ to a fractional form. (*b*) Change $\frac{3x^2+2x+1}{x+4}$ to a mixed quantity. (*c*) Multiply $1+\frac{4}{x}-\frac{5}{x^2}$ by $\frac{x-7}{x^2-8x+7}$.

$$\text{Ans.}\begin{cases}(a)\ \dfrac{10x^2+x+4}{5x}.\\(b)\ 3x-10+\dfrac{41}{x+4}.\\(c)\ \dfrac{x+5}{x^2}.\end{cases}$$

(208) Find the products of:

(*a*) $\frac{9m^2n^2}{8p^3q^3}$, $\frac{5p^2q}{2xy}$, and $\frac{24x^2y^2}{90mn}$; (*b*) $\frac{a^3-x^3}{a^3+x^3}$ and $\frac{(a+x)^2}{(a-x)^2}$;

(*c*) $3ax+4$ and $\frac{a^2}{9a^3x^2+24a^2x+16a}$.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ \dfrac{3mnxy}{4pq^2}.\\[2ex] (b)\ \dfrac{(a+x)\,(a^2+ax+x^2)}{(a-x)\,(a^2-ax+x^2)}.\\[2ex] (c)\ \dfrac{a}{3ax+4}.\end{array}\right.$$

(209) Divide: (*a*) $35m^3y+28m^2y^2-14my^3$ by $-7my$; (*b*) $4a^4-3a^5b-a^6b^2$ by a^4; (*c*) $4x^3-8x^5+12x^7-16x^9$ by $4x^2$.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ -5m^2-4my+2y^2.\\ (b)\ 4-3ab-a^2b^2.\\ (c)\ x-2x^3+3x^5-4x^7.\end{array}\right.$$

(210) (*a*) Write a monomial; a binomial; a polynomial. (*b*) In the expression, $a+2ab-b^2$, why cannot the indicated addition and subtraction be performed? (*c*) What operation is indicated between the quantities in $4ac^2d$?

(211) Multiply $\frac{a^2+c^2+ac}{a^2+b^2-c^2-2ab}$ by $\frac{a^2+c^2-b^2-2ac}{a^4c-ac^4}$.

Suggestion.—Factor the numerators and denominators before multiplying.

$$\text{Ans. } \frac{a+b-c}{ac(a-b+c)\,(a-c)}.$$

(212) Translate the following algebraic expressions into ordinary language: $\sqrt{\frac{a+b+c}{n}}+\sqrt{a}+\frac{b+c}{n}+\sqrt{a+b}+\frac{c}{n}+(a+b)c+a+bc$.

(213) Simplify: (*a*) $\frac{4x+5}{3}-\frac{3x-7}{5x}+\frac{9}{12x^2}$;

(*b*) $\frac{1}{2a(a+x)}+\frac{1}{2a(a-x)}$;

(*c*) $\frac{x}{y}+\frac{y}{x+y}+\frac{x^2}{x^2+xy}$.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ \dfrac{80x^3+64x^2+84x+45}{60x^2}.\\[2ex] (b)\ \dfrac{1}{a^2-x^2}.\\[2ex] (c)\ \dfrac{x+y}{y}.\end{array}\right.$$

(214) Find the least common multiple of: (a) $18ax^2$, $72ay^2$ and $12xy$; (b) $4(1+x)$, $4(1-x)$ and, $2(1-x^2)$; (c) $(a-b)(b-c)$, $(b-c)(c-a)$, $(c-a)(a-b)$

Ans. $\begin{cases} (a)\ 72ax^2y^2. \\ (b)\ 4(1-x^2). \\ (c)\ (a-b)(b-c)(c-a). \end{cases}$

(215) Factor $3x^6 - 3 + a - ax^6$.

Ans. $(x^2+x+1)(x^2-x+1)(x+1)(x-1)(3-a)$.

(216) Extract the square root of $x^4 + 4\frac{1}{4}x^2y^2 + x^3y + 2xy^3 + 4y^4$.

Ans. $x^2 + \frac{1}{2}xy + 2y^2$.

(217) What is (a) the arithmetic ratio of $x^4 - 1$ to $x + 1$? (b) The geometric ratio?

Ans. $\begin{cases} (a)\ x^4 - x - 2. \\ (b)\ (x-1)(x^2+1). \end{cases}$

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(218) (*a*) Express with radical signs: $x^{\frac{3}{4}}$; $3x^{\frac{1}{3}}y^{-\frac{3}{2}}$; $3x^{\frac{1}{6}}y^{-\frac{5}{6}}z^{\frac{1}{3}}$. (*b*) Clear $a^{-1}b^{\frac{1}{2}} + \frac{c^{-2}}{a+b} + (m-n)^{-1} - \frac{a^2b^{-2}c}{c^{-3}}$ of negative exponents. (*c*) Express with fractional exponents: $\sqrt[7]{x^6}$; $\sqrt[3]{x^{-4}}$; $(\sqrt[4]{b^5x^2})^3$.

(219) Introduce the coefficients under the radical signs in $3\sqrt{21}$, $a^2b\sqrt{b^3c}$ and $2x\sqrt[5]{x}$.

(220) A post has $\frac{1}{5}$ of its length in the earth, $\frac{3}{7}$ in the water, and 13 feet in the air. What is its length?

Ans. 35 feet

(221) The following formula appears in works on Heat: $t = \frac{W_1s_1t_1 + W_2s_2t_2}{W_1s_1 + W_2s_2}$. It is required to transform it so that t_2 will stand alone in the first member. In other words, solve for t_2.

Ans. $t_2 = \frac{(W_1s_1 + W_2s_2)\,t - W_1s_1t_1}{W_2s_2}$.

(222) A man performed a journey of 48 miles in a certain number of hours, but if he had traveled 4 miles more each hour, he would have performed the journey in 6 hours less time. How many miles did he travel per hour?

Ans. 4 miles.

(223) The formula $S = \sqrt[3]{\frac{CPD^2}{f\left(2 + \frac{D^2}{d^2}\right)}}$ has been used to calculate the diameter of the shafts for compound marine engines, where S is the diameter of shaft; C, the length of crank, D and d, the diameters of cylinders, and P, the steam pressure. (*a*) Transform this so that P will stand alone in

the first member. (*b*) Find the value of P when $S = 6$, $C = 10$, $D = 30$, $d = 18$, and $f = 864$.

$$\text{Ans.}\begin{cases}(a)\ P = \dfrac{(2d^2 + D^2) f S^3}{C D^2 d^2}.\\(b)\ P = 99.1, \text{ nearly.}\end{cases}$$

(224) Solve the equations:
(*a*) $3x + 6 - 2x = 7x$; (*b*) $5x - (3x - 7) = 4x - (6x - 35)$;
(*c*) $(x + 5)^2 - (4 - x)^2 = 21x$.

$$\text{Ans.}\begin{cases}(a)\ x = 1.\\(b)\ x = 7.\\(c)\ x = 3.\end{cases}$$

(225) Find the values of the following:
(*a*) $\sqrt{27} + 2\sqrt{48} + 3\sqrt{108}$; (*b*) $\sqrt[3]{128} + \sqrt[3]{686} + \sqrt[3]{16}$;
(*c*) $\sqrt{\frac{3}{8}} + \sqrt{\frac{1}{6}} + \sqrt{\frac{2}{27}}$.

$$\text{Ans.}\begin{cases}(a)\ 29\sqrt{3}.\\(b)\ 13\sqrt[3]{2}.\\(c)\ \frac{19}{36}\sqrt{6}.\end{cases}$$

(226) A vessel containing some water was filled by pouring in 42 more gallons; there was then seven times as much water in the vessel as at first. How much did the vessel hold? Ans. 49 gallons.

(227) Solve:
(*a*) $2\sqrt{3x + 4} - x = 4$; (*b*) $\sqrt{3x - 2} = 2(x - 4)$;
(*c*) $\sqrt{x + 16} = 2 + \sqrt{x}$.

$$\text{Ans.}\begin{cases}(a)\ x = 4.\\(b)\ x = 6 \text{ or } 2\frac{3}{4}.\\(c)\ x = 9.\end{cases}$$

(228) Solve:
(*a*) $\sqrt{3x - 5} = \dfrac{\sqrt{7x^2 + 36x}}{x}$; (*b*) $x^2 - (b - a)c = ax - bx + cx$; (*c*) $(x - 2)(x - 4) - 2(x - 1)(x - 3) = 0$.

$$\text{Ans.}\begin{cases}(a)\ x = 6 \text{ or } -2.\\(b)\ x = a - b \text{ or } c.\\(c)\ x = 1 \pm \sqrt{3}.\end{cases}$$

(229) Solve: (*a*) $\sqrt{x - 4ab} = \dfrac{(a + b)(a - b)}{\sqrt{x}}$;

$$(b)\ -\frac{1}{\sqrt{x + 1}} + \frac{1}{\sqrt{x - 1}} = \frac{1}{\sqrt{x^2 - 1}}.$$

$$\text{Ans.}\begin{cases}(a)\ x = (a + b)^2 \text{ or } -(a - b)^2.\\(b)\ x = 1\frac{1}{4}.\end{cases}$$

(230) Solve by substitution:

$$\left.\begin{aligned} 5x - 2y &= 51. \\ 19x - 3y &= 180. \end{aligned}\right\} \qquad \text{Ans.} \begin{cases} x = 9. \\ y = -3. \end{cases}$$

(231) Solve the following equations: (*a*) $2x^2 - 27x = 14$; (*b*) $x^2 - \frac{2x}{3} + \frac{1}{12} = 0$; (*c*) $x^2 + ax = bx + ab$.

$$\text{Ans.} \begin{cases} (a) \ x = 14 \text{ or } -\frac{1}{2}. \\ (b) \ x = \frac{1}{2} \text{ or } \frac{1}{6}. \\ (c) \ x = b \text{ or } -a. \end{cases}$$

(232) A crew that can pull at the rate of 12 miles an hour down the stream finds that it takes twice as long to row a given distance up stream as it does down stream. What is the rate of the current? Ans. 3 miles per hour.

(233) Solve the following:

(*a*) $\frac{10x + 3}{3} - \frac{6x - 7}{2} = 10(x - 1)$;

(*b*) $(a^2 + x)^2 = x^2 + 4a^2 + a^4$; (*c*) $\frac{x - 1}{x - 2} - \frac{x + 1}{x + 2} = \frac{3}{x^2 - 4}$.

$$\text{Ans.} \begin{cases} (a) \ x = 1\frac{1}{2}. \\ (b) \ x = 2. \\ (c) \ x = 1\frac{1}{2}. \end{cases}$$

(234) Solve the following equations, eliminating by addition or subtraction:

$$\left.\begin{aligned} 11x + 3y &= 100. \\ 4x - 7y &= 4. \end{aligned}\right\} \qquad \text{Ans.} \begin{cases} x = 8. \\ y = 4. \end{cases}$$

(235) Solve: (*a*) $y^{\frac{5}{3}} = 243$; (*b*) $x^{10} + 31x^5 - 10 = 22$; (*c*) $x^3 - 4x^{\frac{3}{2}} = 96$.

$$\text{Ans.} \begin{cases} (a) \ y = 27. \\ (b) \ x = 1 \text{ or } -2. \\ (c) \ x = 2\sqrt[3]{18} \text{ or } (-8)^{\frac{2}{3}} \end{cases}$$

(236) (*a*) What is the value of a^0? (*b*) What does $a^0 \div a^{-1}$ equal? (*c*) What does $\sqrt[7]{(3x^2 + 5xy^3 + 6x^2y)^7}$ equal when $x = 2$ and $y = 4$?

(237) What is the value of x in:

(*a*) $\frac{6x+1}{15}-\frac{2x-4}{7x-16}=\frac{2x-1}{5}$; (*b*) $\frac{ax^2}{c-bx}+a+\frac{ax}{b}=0$;

(*c*) $\frac{\sqrt{x}-3}{\sqrt{x}+7}=\frac{\sqrt{x}-4}{\sqrt{x}+1}$?

Ans. $\begin{cases} (a)\ x=-2. \\ (b)\ x=\frac{bc}{b^2-c}. \\ (c)\ x=25. \end{cases}$

(238) Simplify:

$$(a)\ \sqrt{\frac{3}{2}};\quad (b)\ \frac{3}{11}\sqrt{\frac{4}{7}};\quad (c)\ z\sqrt[3]{\frac{2x}{z}}.$$

Ans. $\begin{cases} (a)\ \frac{1}{2}\sqrt{6}. \\ (b)\ \frac{6}{77}\sqrt{7}. \\ (c)\ \sqrt[3]{2xz^2}. \end{cases}$

(239) Solve the equations:

(*a*) $\frac{9x+20}{36}=\frac{4(x-3)}{5x-4}+\frac{x}{4}$; (*b*) $ax-\frac{3a-bx}{2}=\frac{1}{2}$;

(*c*) $am-b-\frac{ax}{b}+\frac{x}{m}=0$.

Ans. $\begin{cases} (a)\ x=8. \\ (b)\ x=\frac{3a+1}{2a+b}. \\ (c)\ x=bm. \end{cases}$

(240) Solve the following equations:

$$\left.\begin{aligned} x+y&=13. \\ xy&=36. \end{aligned}\right\}$$

Ans. $\begin{cases} x=9,\ y=4. \\ x=4,\ y=9. \end{cases}$

(241) Solve the equations:

$$\left.\begin{aligned} x^3-y^3&=98. \\ x-y&=2. \end{aligned}\right\}$$

Ans. $\begin{cases} x=5,\ y=3. \\ x=-3,\ y=-5. \end{cases}$

(242) In the composition of a quantity of gunpowder, the niter was 10 lb. more than $\frac{2}{3}$ of the whole, the sulphur was $4\frac{1}{2}$ lb. less than $\frac{1}{6}$ of the whole, and the charcoal was 2 lb. less than $\frac{1}{7}$ of the niter. What was the amount of gunpowder? Ans. 69 lb.

(243) The hind and fore wheels of a wagon have circumferences of 16 and 14 feet, respectively. How far has the carriage advanced when the fore wheels have made 51 revolutions more than the hind wheels? Ans. 5,712 feet.

(244) Find the values of x in the following:

(*a*) $5x^2 - 9 = 2x^2 + 24$; (*b*) $\frac{3}{4x^2} - \frac{1}{6x^2} = \frac{7}{3}$;

(*c*) $\frac{x^2}{5} - \frac{x^2 - 10}{15} = 7 - \frac{50 + x^2}{25}$.

$$\text{Ans.} \begin{cases} (a) & x = \pm\sqrt{11}. \\ (b) & x = \pm\frac{1}{2}. \\ (c) & x = \pm 5. \end{cases}$$

(245) Solve by comparison:

$$\left.\begin{array}{l} 4x + 3y = 48. \\ 5y - 3x = 22. \end{array}\right\} \qquad \text{Ans.} \begin{cases} x = 6. \\ y = 8. \end{cases}$$

(246) Two trains start at the same time to run 1,200 miles. One runs 10 miles an hour faster than the other and arrives 10 hours sooner; what was the speed of each, supposing it to be uniform? Ans. 30 and 40 miles an hour.

(247) Solve:

$$\left.\begin{array}{l} 2x - \frac{y-3}{5} - 4 = 0. \\ 3y + \frac{x-2}{3} - 9 = 0. \end{array}\right\} \qquad \text{Ans.} \begin{cases} x = 2. \\ y = 3. \end{cases}$$

(248) Find the product of the following:
(*a*) $\sqrt[3]{2} \times \sqrt[5]{3}$; (*b*) $\sqrt[4]{2ax} \times \sqrt[3]{ax^2}$; (*c*) $2\sqrt{xy} \times 3\sqrt[5]{x^3y}$.

$$\text{Ans.} \begin{cases} (a) & \sqrt[15]{864}. \\ (b) & \sqrt[12]{8a^7x^{11}}. \\ (c) & 6\sqrt[10]{x^{11}y^7}. \end{cases}$$

(249) A can do a piece of work in 5 days, B in 6 days, and C in $7\frac{1}{2}$ days; in what time will they do it, working together?
Ans. 2 days.

(250) A person has two horses, and a saddle worth \$10. If the saddle be put on the first horse, his value becomes double that of the second; but if the saddle be put on the second horse, his value will not amount to that of the first horse by \$13. What is the value of each horse?
Ans. \$56 and \$33.

(251) If A should give B \$5 he would then have \$6 less than B; but if he received \$5 from B, three times his money would be \$20 more than four times B's. How much money did each have?

$$\text{Ans.}\begin{cases}\text{A, \$31.}\\ \text{B, \$27.}\end{cases}$$

(252) Solve the following equations;

(*a*) $x^2 - 6x = 16$; (*b*) $x^2 - 7x = 8$; (*c*) $9x^2 - 12x = 21$.

$$\text{Ans.}\begin{cases}(a)\ x = 8 \text{ or } -2.\\ (b)\ x = 8 \text{ or } -1.\\ (c)\ x = 2\tfrac{1}{3} \text{ or } -1.\end{cases}$$

(253) Find the values of the following:

$$\left(c^{-\frac{2}{5}}\right)^{-\frac{1}{3}};\ \left(m\sqrt{n^3}\right)^{-\frac{1}{3}};\ \left(c\,d^{-2}\right)^{\frac{1}{a}}.$$

(254) A wine merchant has two kinds of wine, one worth 90 cents a quart, and the other 50 cents a quart. How much of each must be put in a mixture of 60 quarts, that the mixture may be worth 75 cents a quart?

$$\text{Ans.}\begin{cases}37\tfrac{1}{2}\text{ qt. of 90-cent wine.}\\ 22\tfrac{1}{2}\text{ qt. of 50-cent wine.}\end{cases}$$

(255) What fraction is that whose numerator being doubled, and denominator being increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$. Ans. $\frac{4}{5}$.

(256) There is a number consisting of two digits, which is equal to four times the sum of those digits; and if 18 be added to the number, the digits will be inverted. What is the number?

NOTE.—Remember that any number, as 28, equals $20 + 8 = 10 \times 2 + 8$. Ans. 24.

(257) When 4 is added to the greater of two numbers, the greater number is $3\frac{1}{4}$ times the less; but when 8 is added to the less, the less is one-half the greater. What are the two numbers? Ans. 48 and 16.

LOGARITHMS.

(ARTS. 618–667.)

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(258) Solve, using logarithms,

$$x = 351.36 \times 100 \times 24\left[1 - \left(\tfrac{200}{100}\right)^{.29078}\right].$$

NOTE.—In logarithmic work negative quantities are used as though they were positive, the sign of the result being determined independently.

Ans. $x = -188{,}300$.

(259) What are the logarithms of the following numbers: (*a*) 2,376? (*b*) .6413? (*c*) .0002507?

(260) Divide the following by using logarithms: (*a*) 755.4 ÷ .00324; (*b*) .05555 ÷ .0008601; (*c*) 4.62 ÷ .6448.

$$\text{Ans.}\begin{cases}(a)\ 233{,}150.\\(b)\ 64.584.\\(c)\ 7.1648.\end{cases}$$

(261) Find the value of x, by using logarithms, in

$$x^{.74} = \frac{238 \times 1000}{.0042^{.6602}}.$$

Ans. $x = 2{,}432{,}700{,}000$.

(262) Divide $\sqrt[5]{.00743}$ by $\sqrt[6]{.006}$.

Ans. 1,893.6.

(263) Multiply together the following by using logarithms: 1,728, .00024, .7462, 302.1 and 7.6094. Ans. 711.40.

(264) Calculate the value of $\dfrac{\sqrt{5.954} \times \sqrt[3]{61.19}}{\sqrt[5]{298.54}}$. Ans. 3.0759.

(265) Calculate the value of $\sqrt[7]{.0532864}$. Ans. .65780.

(266) Obtain the values of: (*a*) $32^{4.8}$; (*b*) $.76^{3.62}$, and (*c*) $.84^{.38}$.

$$\text{Ans.}\begin{cases}(a)\ 16{,}777{,}000.\\(b)\ .37028.\\(c)\ .93590.\end{cases}$$

(267) Calculate the value of $\sqrt[6]{\frac{1}{249}} \div \sqrt[5]{\frac{23}{71}}$.

Ans. .49950.

(268) Find the numbers corresponding to the following logarithms: .81293, 2.52460, $\overline{1}.27631$.

(269) Find the value of v_1 in $p\,v^{1.41} = p_1 v_1^{1.41}$, when $p = 134.7$, $v = 1.495$, and $p_1 = 16.421$. Ans. 6.6504.

(270) What is the value of

$$\sqrt[5]{\frac{7.1895 \times 4,764.2^2 \times 0.00326^5}{.000489 \times 457^3 \times .576^2}}?$$ Ans. .020786.

(271) In the formula $p = 960,000 \frac{t^{2.18}}{l\,d}$, find the value of p, when $t = \frac{3}{16}$, $l = 120$, and $d = 2\frac{1}{4}$. Ans. 92.480.

(272) Referring to example 271, what is the value of t, when $p = 160$, $l = 132$ and $d = 2$? Ans. .23863.

GEOMETRY AND TRIGONOMETRY.

(ARTS. 668–827.)

(273) If one of the angles formed by one straight line meeting another straight line equals $\frac{4}{5}$ of a right angle, what is the other angle equal to? Ans. $1\frac{1}{5}$ right angles.

(274) If a number of straight lines meet a given straight line at a given point, all being on the same side of the given line, so as to form six equal angles, what is the size of one angle? Ans. $\frac{1}{3}$ of a right angle.

(275) The diametrical pitch of a gear wheel is the number of teeth in the wheel divided by the diameter of the wheel. If the pitch is 4, and the diameter of the gear is 12 inches, what is the size of an angle formed by drawing lines from the center to the middle points of two adjacent teeth? Ans. $\frac{1}{12}$ of a right angle.

(276) If a triangle has two equal angles, what kind of a triangle is it?

(277) In an equilateral heptagon one of the sides equals 3 inches; what is the length of the perimeter? Ans. 21 in.

(278) The perimeter of a regular decagon is 40 inches; what is the length of a side? Ans. 4 in.

(279) What is one angle of a regular dodecagon equal to? Ans. $1\frac{2}{3}$ right angles.

(280) A triangle has three equal angles; what is it called?

(281) Can a triangle be formed with three lines whose lengths are 12 inches, 7 inches, and 4 inches? Give reasons for your opinion.

(282) Can a quadrilateral be formed with lines whose lengths are 20 inches, 9 inches, 4 inches, and 7 inches? Give reasons,

(283) A certain triangle has two equal angles. If, from the vertex of the other angle, a perpendicular is drawn to the side opposite, which is 7 inches long, what are the lengths of the two parts of the side thus divided by the perpendicular?

(284) The shortest distance from a given point to a given line is 9 inches; the distances from this point to the two extremities of the line are 12 inches and 15 inches; what is the length of the line? Ans. 19.94 in.

(285) The sum of two angles of a right-angled triangle is $\frac{5}{4}$ of a right angle; what is the other angle equal to? Ans. $\frac{3}{4}$ of a right angle.

(286) What is one of the angles of an equiangular octagon equal to? Ans. $1\frac{1}{2}$ right angles.

(287) In a right-angled triangle one acute angle equals $\frac{5}{8}$ of a right angle; what is the other angle equal to? Ans. $\frac{3}{8}$ of a right angle.

(288) Given, three points, A, B, and C, and the distance from A to B equal to $1\frac{1}{2}$ inches, from B to C $1\frac{1}{2}$ inches, and from C to A 2 inches; pass a circle through these three points.

(289) The chord of an arc in a circle whose radius is 6 inches is 4 inches long; what is the length of the chord of half the arc? Ans. 2.03 in.

(290) If the diameter of the circle in the last problem had been 6 inches, what would have been the length of the chord of half the arc? Ans. 2.14 in.

(291) The diameter of a plane section of a sphere is 6 inches, and its height is 2 inches; what is the diameter of the sphere? Ans. $6\frac{1}{2}$ in

(292) The length of a perpendicular from the center of a circle to a chord is $5\frac{3}{4}$ inches; if the diameter of the circle is 17 inches, what is the length of the chord? Ans. 12.52 in.

(293) The sides of an inscribed angle intercept three-fourths of the circumference; how many quadrants are there in the angle? Ans. $1\frac{1}{2}$ quadrants.

(294) How many equal sectors are there in a circle, if each sector measures $\frac{2}{7}$ of a right angle? Ans. 14 sectors.

(295) If the perimeter of a regular inscribed octagon is 24 inches, and the length of the perpendicular from the center to one of the sides is 3.62 inches, what is the diameter of the circle in which the octagon is inscribed?

Ans. 7.84 in.

(296) Two equal circles intersect so that the common chord of the two arcs of intersection measures $10\frac{1}{2}$ inches. If the circles are struck with a 13-inch radius, what is the greatest distance between the two intersecting arcs?

Ans. 2.2 in.

(297) In the last example, if the radius of one circle is 13 inches, and of the other 8 inches, what is the greatest distance between the arcs? Ans. 3.07 in.

(298) If the height of a plane section of a sphere is $3\frac{1}{4}$ inches, and the diameter of the sphere is 14 inches, what is the diameter of the flat surface of the section?

Ans. 11.82 in.

(299) What part of a circle is an arc of 19° 19′ 19″? Express it decimally. Ans. .053672 of a circle.

(300) What part of a quadrant would an angle of 19° 19′ 19″ be? Express it decimally. Ans. .214688 of a quadrant.

(301) A regular decagon is inscribed in a circle whose diameter is 23 inches; what is the perimeter of the decagon?

Ans. 71 in., nearly.

(302) What is the difference between 90° and 35° 24′ 25.8″?

(303) In a right-angled triangle $A\,B\,C$, the hypotenuse $A\,B = 17.69$ feet, and the side $A\,C = 9$ ft. 9 in.; find the other three parts.

$$\text{Ans.} \begin{cases} 56^\circ\ 33'\ 15''. \\ 33^\circ\ 26'\ 45''. \\ 14 \text{ ft. } 9 \text{ in.} \end{cases}$$

(304) Add 159° 27′ 34.6″, 25° 16′ 8.7″, and 3° 48′ 53″.

(305) Find the sine, cosine, and tangent of 17° 27′ 37″.

(306) In a triangle $A\,B\,C$, $A\,B = 26$ feet 7 inches, $A\,C = 40$ feet, and the included angle $A = 36° 20' 43''$; find the remaining parts.

Ans. $\begin{cases} C = 40° 16' 53''. \\ B = 103° 22' 24''. \\ B\,C = 24 \text{ ft. } 4.4 \text{ in.} \end{cases}$

(307) Find the sine, cosine and tangent of 63° 4′ 51.8″.

(308) Sine = .27038, cosine = .27038, and tangent = 2.27038; find the corresponding angles.

(309) A polygon of eleven sides is called an undecagon. If a regular undecagon whose perimeter is 4 feet 3 inches be inscribed in a circle, what is the size of an angle formed by drawing radii to the extremities of one of the sides? Also, what is the radius of the circle?

Ans. $\begin{cases} 32° 43' 38.2''. \\ \text{Radius} = 8.23 \text{ in.} \end{cases}$

(310) One angle of a triangle is 47° 13′ 29″; what are the other two angles if one of them is twice the given angle?

Ans. $\begin{cases} 38° 19' 33''. \\ 94° 26' 58''. \end{cases}$

(311) One angle of a triangle is 75° 48′ 17″; what are the other two angles if one of them is half as large as the given angle?

Ans. $\begin{cases} 37° 54' 8.5''. \\ 66° 17' 34.5''. \end{cases}$

(312) In a triangle $A\,B\,C$, the side $A\,B = 16$ feet 5 inches; the side $B\,C = 13$ feet $6\frac{1}{2}$ inches, and the angle $A = 54° 54' 54''$; find the remaining parts.

Ans. $\begin{cases} B = 42° 19' 36''. \\ C = 82° 45' 30''. \\ A\,C = 11 \text{ ft. } 1\frac{3}{4} \text{ in.} \end{cases}$

(313) If one-third of an angle of a certain triangle = 14° 47′ 10″, what are the angles, one of the other two being two and one-half times the given angle?

Ans. $\begin{cases} 24° 44' 45''. \\ 44° 21' 30.'' \\ 110° 53' 45''. \end{cases}$

(314) In a right-angled triangle *A B C*, the two sides are 437 feet and 792 feet in length; find the hypotenuse and the two acute angles.

$$\text{Ans.}\begin{cases}28^\circ\ 53'\ 18''.\\ 61^\circ\ 6'\ 42''.\\ 904\text{ ft. }6\tfrac{3}{4}\text{ in.}\end{cases}$$

(315) Find by trigonometry and prove by geometry that the angle between two adjacent sides of a regular octagon inscribed in a circle is 135°. If the perimeter of the octagon is 56 feet, what is the diameter of the circle?

Diameter = 18 feet $3\frac{1}{2}$ inches.

(316) Draw a diagram showing the sine, cosine, and tangent of 67° 8′ 49″.

(317) Given, the tangent of a certain angle = 3. (*a*) Draw a diagram showing an angle having this tangent, and mark its sine and cosine. (*b*) Give their values from the tables.

(318) If the cosine of an angle is .39278, what are the actual lengths of the cosine, tangent, and sine of the same angle in a circle whose diameter is $4\frac{3}{4}$ times as large?

(319) In a triangle *A B C*, the angle $A = 29^\circ\ 21'$; angle $C = 76^\circ\ 44'\ 18''$, and the side $A\,C = 31$ feet 10 inches; find the other three parts.

$$\text{Ans.}\begin{cases}B\,C = 16\text{ ft. }3\text{ in.}\\ A\,B = 32\text{ ft. }3\text{ in.}\\ B = 73^\circ\ 54'\ 42''.\end{cases}$$

(320) A regular decagon is inscribed in a circle whose radius is $9\frac{3}{4}$ inches; what is the perimeter of the decagon?

Ans. 60.26 inches.

(321) In the above question, what is the difference between the perimeter of the decagon and the circle; also, what is the difference of their areas?

$$\text{Ans.}\begin{cases}1\text{ in.}\\ 19.26\text{ sq. in.}\end{cases}$$

(322) The area of a circle is 89.42 square inches; what is its diameter and circumference? What is the length of a side of the largest regular hexagon that could be inscribed in it?

Length of side = 5.335 in.

(323) The distance between two parallel sides of a wrought iron octagon bar is 2 inches; what is the weight of

a bar 10 feet long, a cubic inch of wrought iron weighing 0.282 pound? Ans. 112 lb. 2 oz.

(324) The outside and inside diameters of a cast iron spherical shell are 16 inches and 12 inches; what is its weight, a cubic inch of cast iron weighing 0.261 pound? Ans. 323.61 lb.

(325) The length of an arc of a circle is $5\frac{13}{32}$ inches by measurement. If the number of degrees in the arc is 27, what is the diameter of the circle? Ans. 22.95 in.

(326) What is the difference between a plane figure containing 7 square inches and one 7 inches square? If both figures are perfect squares, what are the lengths of the sides?

(327) (*a*) What is the area of a circle whose diameter is $17\frac{1}{64}$ inches? (*b*) What is the length of an arc of 16° 7′ 21″ in the above circle? Ans. Length of the arc = 2.394 in.

(328) What is the area of an ellipse whose axes are 12 inches and 8 inches? What is its perimeter?

(329) What is the entire surface of a cone whose base is 7 inches in diameter, and whose altitude is 11 inches? Ans. 165.41 sq. in.

(330) What is the height of a cone having the same volume and diameter as a 10-inch sphere? Ans. 20 in.

(331) What is the height of a cylinder having the same volume and diameter as a 12-inch sphere? Ans. 8 in.

(332) (*a*) What is the area of a triangle whose base is $9\frac{1}{2}$ inches, and whose altitude is 12 inches? (*b*) If the angle which one side forms with the base is 79° 22′, what is the perimeter of the triangle? Ans. Perimeter = 35.73 in.

(333) The diagonal of a trapezium is 11 inches; the lengths of the perpendiculars from the opposite vertexes upon this diagonal are $4\frac{1}{4}$ inches and 7 inches; what is the area of the trapezium?

(334) The length of a chord of a segment in a circle

whose diameter is 10 inches is $6\frac{3}{4}$ inches; what is the area of the segment and the number of degrees in its arc?

$$\text{Ans.}\begin{cases}84^\circ\ 54'\ 28.6''.\\ 6.074\text{ sq. in.}\end{cases}$$

(335) What is the convex area of a pyramid whose slant height is 17 inches, the perimeter of its base being 63 inches?

(336) What is the volume and entire area of a frustum of a cone whose upper base is 12 inches and lower base is 18 inches in diameter, and whose altitude is 14 inches?

$$\text{Ans.}\begin{cases}2{,}506.84\text{ cu. in.}\\ 1{,}042.3\text{ sq. in.}\end{cases}$$

(337) What is the area of the surface of a sphere 27 inches in diameter? Ans. 2,290.2 sq. in.

(338) Wishing to make some dumb-bells to weigh 20 pounds each exclusive of the handle, the balls to be equal spheres, what must be the diameter of the balls, a cubic inch of cast iron weighing 0.261 pound? Ans. 4.18 in.

(339) What is the volume of an engine cylinder, in cubic feet, whose diameter is 19 inches, and whose stroke is 24 inches? Ans. 3.938 cu. ft.

(340) The chord of the arc of a segment is 14 inches long, and the height of the segment is 2 inches; what is the radius? Ans. $13\frac{1}{4}$ in.

(341) The cylinders of a compound engine are 19 and 31 inches in diameter, and the stroke is 24 inches; if the clearance at each end in the small cylinder is 14% of the **stroke,** and in the large cylinder 8% of the **stroke,** (*a*) what is the total volume in cubic feet of the steam in the small cylinder during one stroke? (*b*) In the large cylinder? (*c*) What is the ratio between the two?

$$\text{Ans.}\begin{cases}4.489\text{ cu. ft.}\\ 11.321\text{ cu. ft.}\\ \text{Ratio} = 2.522 : 1.\end{cases}$$

(342) In the above example the pipe which connects the small or high-pressure cylinder to the large or low-pressure cylinder is 8 inches in diameter and 7 feet long. (*a*) What

is its volume in cubic feet? (*b*) What is the ratio of its volume to that of the high-pressure cylinder?

Ans. $\begin{cases} 2.443 \text{ cu. ft.} \\ \text{Ratio} = 0.544 : 1. \end{cases}$

(343) (*a*) What is the volume and area of a cylindrical ring whose outside diameter is 16 inches and inside diameter 13 inches? (*b*) If made of cast iron, what is its weight? Ans. Weight = 21 lb.

(344) If all the dimensions in Fig. 91, Art. **790,** be doubled, what will be its area? Ans. 453.92 sq. in.

(345) The altitude of a parallelopipedon is 18 inches; its base is a square, one edge measuring $5\frac{1}{4}$ inches; what is its convex area, entire area, and volume?

Ans. $\begin{cases} 378 \text{ sq. in.} \\ 433.125 \text{ sq. in.} \\ 496.125 \text{ cu. in.} \end{cases}$

(346) What is the convex area and entire area of a hexagonal pyramid, the slant height being 37 feet, and one edge of the base measuring 12 feet?

Ans. $\begin{cases} 1{,}332 \text{ sq. ft.} \\ 1{,}706.112 \text{ sq. ft.} \end{cases}$

(347) If the altitude of the pyramid in the last problem had been 37 feet, what would have been its volume?

Ans. 4,614 cu. ft.

(348) How many yards of Brussels carpeting, 27 inches wide, will it take to cover a room 15 feet by 18 feet?

Ans. 40 yd.

(349) How many square yards of plaster will it take to cover the sides and ceiling of a room 16 × 20 feet, and 11 feet high, with four windows, each 7 × 4 feet, and three doors, each 9 × 4 feet over all, the baseboard coming 6 inches above the floor? Ans. $95\frac{7}{9}$ sq. yd.

(350) What is the area of a sector if the chord of the arc is $6\frac{1}{3}$ inches long, and the diameter of the circle is 10 inches?

Ans. 18.95 sq. in.

(351) What is the area in square feet of a parallelogram whose base is 129 inches long, if the shortest distance between the base and side opposite is 7 feet?

(352) The parallel sides of a trapezoid are 15 feet 7 inches, and 21 feet 11 inches long; the altitude is 7 feet 8 inches. (*a*) What is the area of the trapezoid? (*b*) What is the length of a side of an equilateral triangle having the same area?

$$\text{Ans.} \begin{cases} 143.75 \text{ sq. ft.} \\ 18 \text{ ft. } 2.64 \text{ in.} \end{cases}$$

(353) (*a*) What would be the length of a side of a square having the same area as the trapezoid in the last problem? (*b*) The diameter of a circle? (*c*) How much shorter is the circumference of the circle than the perimeter of the square?

$$\text{Ans.} \begin{cases} 11.99 \text{ ft.} \\ 13\tfrac{1}{2} \text{ ft.} \\ 5 \text{ ft. } 6.6 \text{ in.} \end{cases}$$

(354) In a triangle $A\,B\,C$, $A\,B = 24$ feet, $B\,C = 11$ feet 3 inches, and $A\,C = 18$ feet; required, the three angles.

$$\text{Ans.} \begin{cases} A = 26^\circ\ 28'\ 5''. \\ B = 45^\circ\ 29'\ 23''. \\ C = 108^\circ\ 2'\ 32''. \end{cases}$$

ELEMENTARY MECHANICS.

(ARTS. 828–966.)

(355) A ball, thrown horizontally by the hand, has a velocity of 500 ft. per second. If the ground is level, and the distance from the ground to the hand at the instant the ball leaves the hand is 5 ft. 6 in., how far will the ball go before striking the ground? Ans. 292.42 ft.

(356) An engine fly-wheel, 80 in. in diameter, makes 160 revolutions per minute; what is the velocity in feet per second of a point on the rim? Ans. 55.85 ft. per sec.

(357) In the last example, through how many degrees, minutes and seconds will a point on the rim turn in one-seventh of a second? Ans. $137^\circ\ 8'\ 34\frac{2}{7}''$.

(358) The fly-wheel of an engine drives a pulley, rigidly connected to a drum on which an elevator rope winds. The fly-wheel is 4 ft. in diameter and makes 54 revolutions per minute; the diameter of the pulley is 36 in., and of the drum, 18 in.; (*a*) how long will it take the elevator to reach the top of a building 100 ft. high? (*b*) If required to reach it in 30 seconds, how many revolutions should the fly-wheel make per minute? Ans. $\begin{cases} (a)\ 17.68 \text{ sec.} \\ (b)\ 31.83 \text{ R. P. M.} \end{cases}$

(359) Define and give an example of uniform motion; of variable motion.

(360) Define force. Name five different kinds of forces.

(361) Explain what you understand by inertia.

(362) Is inertia a force? If so, why?

(363) What is weight? How is it measured?

(364) In order that the effect of a force upon a body may be compared with that of another force acting on the same body, what three conditions must be fulfilled?

(365) What is motion? What is rest?

(366) Can a body be in motion with respect to one body and at rest with respect to another? Give examples.

(367) Where will a body weigh the more, on the top of a high mountain or at the bottom of a deep valley, the bottom of the valley being as far below sea-level as the top of the mountain is above?

(368) If the top of a mountain is 31,680 feet above sea-level, what would a body weighing 20,000 lb. at sea-level weigh at the top of the mountain? Take the radius of the earth at sea-level as 3,960 miles. Ans. 19,939 lb. 8½ oz.

(369) If the body in the last example had been dropped in a hole just big enough to let it fall to the bottom, and the bottom of the hole was 2 miles below sea-level, how much would it have weighed at the bottom of the hole?

Ans. 19,989 lb. 14.4 oz.

(370) State the three laws of motion.

(371) What is acceleration?

(372) What do you understand by initial velocity?

(373) Why is it dangerous to jump from a moving train?

(374) Why is it that a man cannot lift himself by pulling his boot straps?

(375) Explain how forces are represented by lines.

(376) What is meant by the expression, "the resultant of several forces"?

(377) If a line 5 in. long represents a force of 20 lb., (*a*) how long must the line be to represent a force of 1 lb.? (*b*) Of 6¼ lb.?

(378) What do you understand by the components of a force?

(379) If a body be acted upon by two equal forces, one due east and the other due south, in what direction will the body move? What is the direction of the resultant of the two forces?

(380) Find the point of suspension of a rectangular cast iron lever 4 ft. 6 in. long, 2 in. deep and ¾ in. thick, having

weights of 47 lb. and 71 lb. hung from each end, in order that there may be equilibrium. Take the weight of a cubic inch of cast iron as .261 lb.

SUGGESTION.—First find the weight of the lever; then consider this weight to be concentrated at the center of gravity of the lever, and combine it with the other two weights in the same manner as though there were three weights.

Ans. $\begin{cases} \text{Short arm} = 22.343 \text{ in.} \\ \text{Long arm} = 31.657 \text{ in.} \end{cases}$

Solve the two following examples by the method of triangle of forces, and parallelogram of forces, and mark the direction of the resultant:

(381) Two forces act upon a body at a common point; one with a force of 75 lb., and the other with a force of 40 lb.; if the angle between them is 60°, and both forces act towards the body, what is the value of the resultant?

Ans. 101.12 lb.

(382) In the last example, if one force (the one of 75 lb.) acts away from the body, and the other towards it, what is the resultant? Ans. 65 lb.

(383) If two forces of 27 lb. and 46 lb., respectively, act in exactly opposite directions upon a body, what is the resultant?

(384) The entire solar system is moving through space at the rate of 18 miles per second; (*a*) what is its velocity in miles per hour? (*b*) How far will it go in one day?

(385) Two bodies, starting from the same point, move in opposite directions, one at the rate of 11 ft. per second, and the other at the rate of 15 miles per hour; (*a*) what will be the distance between them at the end of 8 minutes? (*b*) How long before they will be 825 ft. apart?

Ans. $\begin{cases} (a) \text{ 3 miles.} \\ (b) \text{ 25 seconds.} \end{cases}$

(386) If six forces act towards the center of gravity of a body at angles of 30°, 45°, 135°, 210°, 225°, and 300°, whose magnitudes are 75, 47, 61, 32, 53, and 98 pounds, respectively, what is the value of their resultant? Solve by method of polygon of forces. Ans. $45\frac{3}{4}$ lb.

(387) Suppose that the velocity of a steamboat in still water is 10 miles per hour, and that it is placed in a river flowing 4 miles per hour; also, that a man is walking the deck from stern to bow at the rate of 3 miles per hour. (*a*) What is the velocity of the boat when headed up stream? (*b*) When headed down stream? (*c*) Of the man in each case?

(388) A peg in the wall is pulled by two strings, one with a force of 21 lb., at an angle to the vertical of 45°, and the other with a force of 28 lb., at an angle of 60°; what is the value and direction of the resultant when the forces are on opposite sides of the vertical line? Use the method of parallelogram of forces. Ans. 30.34 lb.

(389) A force of 87 lb. acts at an angle of 23° to the horizontal; what are its horizontal and vertical components? Find, first, by the method of triangle of forces, and, second, by trigonometry. Ans. $\begin{cases} 80.084 \text{ lb.} \\ 33.994 \text{ lb.} \end{cases}$

(390) A weight of 325 pounds rests upon a smooth inclined plane, as shown in Fig. 119, Art. **884.** If the angle of the plane is 15°, (*a*) what is the perpendicular pressure against it? (*b*) What force would it be necessary to exert parallel to the plane, to keep it from sliding downwards, there being no friction? Solve by trigonometry, and also by the method of the triangle of forces.

Ans. $\begin{cases} (a)\ 313.93 \text{ lb.} \\ (b)\ 84.12 \text{ lb.} \end{cases}$

(391) If the weight of a body is 125 lb., what is its mass?

(392) If the mass of a body is 53.7, what is its weight?

(393) (*a*) Is the mass of a body always the same? (*b*) If the mass of a body on the earth's surface is 25, what would be its mass at the center of the earth? (*c*) On the surface of the moon?

(394) A body on the earth's surface weighs 141 lb.; (*a*) at what point above the surface will it weigh 100 lb.? (*b*) At what point below the surface? Take the earth's radius as 4,000 miles. Ans. $\begin{cases} (a)\ 749.736 \text{ miles.} \\ (b)\ 1{,}163.12 \text{ miles.} \end{cases}$

(395) If a body were dropped from a balloon 1 mile above the earth's surface, (*a*) how long a time would it require to fall to the earth? (*b*) What would be its velocity when it struck?

Ans. { (*a*) 18.12 seconds.
(*b*) 582.76 ft. per sec.

(396) In the last example, if the body weighed 160 lb., what would be the kinetic energy on striking the earth?

Ans. 844,799 ft.-lb.

(397) If a cannon ball were fired vertically upward with a velocity of 2,360 ft. per second, (*a*) how high would it go? (*b*) How long would it take to fall to the earth?

Ans. { (*a*) 16.4 miles.
(*b*) 2 min. 26.77 sec.

(398) The earth turns round once in 24 hours. If it were a perfect sphere 8,000 miles in diameter, how far would a point on its surface travel in one minute?

(399) If a projectile weighing 400 pounds be fired from a cannon with a velocity of 1,875 ft. per second, at a target 6 ft. distant, (*a*) what will be its kinetic energy, on striking the target, in foot-pounds? (*b*) In foot-tons? (*c*) If it penetrates but 6 in., what will be its striking force?

Ans. { (*a*) 21,863,339.55 ft.-lb.
(*b*) 10,931.67 ft.-tons.
(*c*) 43,726,679 lb.

(400) If the acceleration due to gravity were 20 ft. per second, instead of 32.16 ft. per second, how much longer would it take a body to fall to the earth from a height of 200 ft. than it does now?

Ans. 0.9454 second.

(401) What do you understand by center of gravity?

(402) What do you understand by specific gravity?

(403) (*a*) What is the density of a cubic foot of a body occupying a space of 800 cu. in., and weighing 500 lb.? (*b*) What is its specific gravity?

Ans. { (*a*) 33.582.
(*b*) 17.28.

(404) A body has been falling freely for 5 seconds what is its velocity?

(405) Suppose that a body, under certain conditions, were to fall freely for 3 seconds, and then fall uniformly with the velocity it had at the end of the third second for 6 seconds longer, how far would it fall? Ans. 723.6 ft.

(406) The weight of the head and piston of a steam hammer, together with the piston rod, is 8 tons. If it falls 8 ft. and compresses a mass of iron $\frac{1}{2}$ in., what is the force of the blow? Ans. 1,536 tons.

(407) Explain what you understand by centrifugal force.

(408) If a cast iron sphere 4 in. in diameter be revolved in a circle, in which the distance from the center of the sphere to the center of the circle is 15 in., what will be the tension of the string, the sphere making 60 revolutions per minute? Ans. 13.38 lb.

(409) The outside diameter of an engine fly-wheel is 80 in.; width of face, 26 in.; average thickness of rim, 5 in.; revolutions per minute, 175; what is the centrifugal force tending to burst the rim? Ans. 121,394+ lb.

(410) If a body weighs one pound at a distance of 100 miles from the center of the earth, (*a*) what will it weigh at the surface? (*b*) At 100 miles above the surface? Take the earth's radius as 4,000 miles.

Ans. $\begin{cases} (a) & 40 \text{ lb.} \\ (b) & 38.072 \text{ lb.} \end{cases}$

(411) What would be the horsepower of a machine that could raise 10,746 lb. 354 ft. in 10 minutes?

Ans. 11.5275 H. P.

(412) How far above the surface of the earth will a 2-lb. ball weigh 3 oz.? Ans. 9,064 miles.

(413) What is the range of a projectile thrown horizontally, 50 ft. above a level plain, the initial velocity being 140 ft. per second? Ans. 246.87 ft.

(414) A projectile has an initial velocity of 30 ft. per second. How far below the horizontal line of direction will it strike a body 10 ft. away? Ans. 1 ft. 9.44 in.

(415) What do you understand by moment of a force? Illustrate it.

(416) Illustrate your idea of a couple.

(417) Can a couple have a single resultant force?

(418) Where is the center of gravity of a triangle located?

(419) If $A\,B\,C\,D$ is a quadrilateral, and $A\,B = 10$ in., $B\,C = 8$ in., $C\,D = 7$ in., $D\,A = 9$ in., and the angle between $A\,B$ and $B\,C$ is 90°, where is the center of gravity of the figure? Determine it graphically.

(420) Where is the center of gravity of a regular pentagon, the length of a side being seven inches?

(421) If the weight of the balls shown in Fig. 128 were $W_1 = 21$ lb., $W_2 = 15$ lb., $W_3 = 17$ lb., and $W_4 = 9$ lb., where would the center of gravity be, the distance between the centers of W_1 and W_2 being 34 in. between W_2 and W_3, 25 in.; between W_3 and W_4, 40 in., and between W_4 and W_1, 18 in.?

(422) Find the center of gravity of a square board of uniform thickness whose sides are 14 in. in length, and having one of its corners cut off at a distance of 4 in., measured from that corner each way; or, what is the same thing, cutting off a right-angled triangle whose sides, including the right angle, are 4 in. long.

(423) A bookbinder has a press, the screw of which has 4 threads to the inch. It is worked by a lever 15 in. long, to which is applied a force of 25 lb.; (*a*) what will be the pressure if the loss by friction is 5,000 lb.? (*b*) What would be the theoretical pressure? (*c*) What is the efficiency in this case?

$$\text{Ans.}\begin{cases}(a)\ 4{,}424.8\text{ lb.}\\(b)\ 9{,}424.8\text{ lb.}\\(c)\ 46.95\%.\end{cases}$$

(424) How would you determine whether a body was stable or not if placed in a certain position?

(425) If a prism 10 inches square and 22 inches long on its axis has been so cut that, when placed upon its base, its axis makes an angle of 60° with the horizontal, will it stand or fall? Consider the plane of the upper base as being at right angles to the axis.

(426) If the force moves through a distance of 5 ft. 6 in., while the weight is moving 6 in., (*a*) what is the velocity ratio of the machine? (*b*) What weight would a force of 5 lb. applied to the power arm raise?

(427) In the last example, if the efficiency were 65%, what weight could be raised?

(428) The length of a lever is 5 ft.; where must the fulcrum be placed, so that a weight of 35 lb. at one end may balance one of 180 lb. at the other end?

(429) In a block and tackle the theoretical force necessary to raise a weight of 1,000 lb. is 50 lb.; (*a*) what is the velocity ratio? (*b*) How many pulleys are there? (*c*) If the actual force necessary to raise the load is 95 lb., what is the efficiency? Ans. (*c*) 52.63%.

(430) The nuts on a cylinder head are tightened by a wrench whose handle is 20 in. long. If the force exerted upon the wrench is 60 lb., and the efficiency of the combination is 40%, what pressure will the nut exert against the head (or, in other words, what is the tension of the stud), the pitch of the screw being ⅛ of an inch? Ans. 24,127.5 lb.

(431) The base of an inclined plane is 20 ft. in length and its height is 5 ft.; (*a*) what force acting parallel to the plane will balance a weight of 1,580 lb.? (*b*) What force acting parallel to the base would balance this weight?

Ans. $\begin{cases} (a)\ 383.2\text{ lb.} \\ (b)\ 395\text{ lb.} \end{cases}$

(432) Find what the weights *W* and *W′* must be to produce equilibrium when the levers shown in Fig. 1 are suspended from the ring *A*.

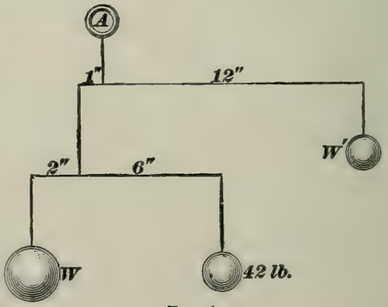

FIG. 1.

(433) If in Fig. 141, the power arms *P F* equal 14, 21 and 19 inches, respectively, and the weight arms

$W F$ equal $2\frac{1}{2}$, $3\frac{1}{4}$ and $2\frac{3}{8}$ inches, respectively, what force applied at P will raise a load of 725 lb.? Ans. 3.032 lb.

(434) In Fig. 144, suppose the radius of the wheel A is 15 in.; of C, 12 in., of E, 20 in.; of the drum F, 5 in.; of the pinion D, $3\frac{1}{2}$ in., and of B, 3 in.; (*a*) what load would a force of 35 lb. applied at P raise? (*b*) What is the velocity ratio? (*c*) If the weight actually raised was 1,932 lb., what is the efficiency? Ans. Efficiency = 80.5%.

(435) A frame having the shape of an equilateral triangle measuring 15 in. on each edge is suspended in a horizontal position, weights of 12, 15, and 18 lb., respectively, being hung from each corner. Where is the point of suspension that the frame may remain horizontal? Solve graphically, and measure the perpendicular distances from the point of suspension to each edge of the triangle.

(436) A stone weighing 500 lb. is balanced on the edge of the roof of a building 75 ft. high; (*a*) what is its potential energy? (*b*) If all of its potential energy could be changed into kinetic energy, without any loss through friction or heat, how many horsepower would be developed, on the supposition that the work was done in the same time that it would take the stone to fall freely to the ground? Ans. (*b*) 31.57 H.P.

(437) A cubic foot of a certain kind of stone weighs 127 lb.; what is its specific gravity? Ans. 2.032.

(438) The specific gravity of bismuth is 9.823; what is the weight of a cubic inch? Ans. .3553 lb.

(439) In the differential pulley shown in Fig. 149, the radius of the larger pulley is $6\frac{1}{2}$ in., and of the smaller pulley, $5\frac{3}{4}$ in.; what weight will a force P of 60 lb. raise if the efficiency of the mechanism is 48%? Ans. 499.2 lb.

(440) If a hammer whose head weighs $1\frac{1}{2}$ lb., strikes a nail with a velocity of 25 ft. per second, driving it $\frac{3}{8}$ of an inch into the wood, what is the force of the blow?

Ans. 466.42 lb.

(441) If 4 cu. ft. of copper alloy weigh a ton (2,000 lb.), (*a*) what is its specific gravity? (*b*) What is the weight of a cu. in.? Ans. (*a*) Sp. Gr. 8.

(442) If the distance between the center line of the handle and the axis of the drum shown in Fig. 143 is 14½ in., and the diameter of the drum is 5 in., what load will a force of 30 lb. exerted on the handle at *P* raise?

(443) If the coefficient of friction is .21, what force would be required to move a body weighing 75 lb.?

(444) If a man raises a weight of 900 lb. 150 feet in 15 minutes, by means of a fixed and movable pulley, (*a*) how much work has he done? (*b*) What part of a horsepower is this equivalent to? Ans. (*b*) $\frac{3}{11}$ H.P.

(445) In the last example, what horsepower would the man have actually expended if the resistance due to friction had been 36% of the load? Ans. .3709 H.P.

(446) If a force of 18 lb. is just sufficient to move a weight of 88 lb. along a horizontal plane, what is the coefficient of friction? Ans. .2045.

(447) If 3 cu. ft. of a certain material weigh 1,200 lb., what is its density? Ans. 12.438.

(448) An iron plate rests upon four supports; upon it is placed a weight of 125 lb.; a compressed spring, placed under the plate directly under the center of gravity of the plate and weight, exerts an upward pressure of 47½ lb. What is the pressure upon each support?

(449) Find the resultant of the forces acting in Fig. 2—all acting towards the same point.

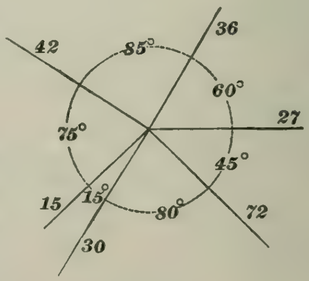

FIG. 2.

(450) The distance between the center line of the handle and the axis of the drum in Fig. 143 is 12 in., and the diameter of the drum is 4½ in. The free end of the rope, forming part of a block and tackle having 6 pulleys, is wound up on this drum. How great a weight can be lifted by the pulleys if a force of 30 lb. is exerted on the handle? Ans. 960 lb.

(451) (*a*) What is the velocity ratio in the last example? (*b*) If the weight actually lifted by a force of 30 lb. was 790 lb., what is the efficiency?

Ans. $\begin{cases} (a)\ 32. \\ (b)\ 82.29\%. \end{cases}$

(452) It is desired to raise a weight by means of a pulley fixed overhead, the free end of the rope passing over another pulley, fixed to the floor. If the resistance due to friction is 24% of the load lifted, (*a*) what force would be necessary to raise a weight of 475 lb.? (*b*) What is the efficiency?

Ans. $\begin{cases} (a)\ 589 \text{ lb.} \\ (b)\ 80.64\%. \end{cases}$

(453) Explain your idea of work, power, horsepower, and kinetic energy. If a constant force of 6 pounds can cause a body weighing 60 pounds to move a distance of 25 feet in $2\frac{1}{2}$ seconds, what is (*a*) the work done? (*b*) The power expended?

HYDROMECHANICS.

(ARTS. 967–1038.)

(454) An iron sphere is sunk in the ocean to a depth of two miles. The diameter of the sphere is 20 inches; what is the total pressure upon it? Ans. 5,908,971 lb.

(455) A hollow sphere weighs 125 lb. in air and $83\frac{1}{2}$ lb. in water. If its specific gravity is 7.2, what is its volume?
Ans. 1,147.4 cu. in.

(456) What should be the diameter of a pipe, 2,800 feet long, that will discharge 225,000 gallons of water per hour, under a head of 26 feet? Calculate to the nearest inch.
Ans. 16 in.

(457) A squirt-gun has a hole in it $\frac{3}{32}$ of an inch in diameter. It is held vertically upwards, and a pressure of 50 lb. is applied to the piston, which is $\frac{7}{8}$ of an inch in diameter. Neglecting all resistances, (*a*) how high will the water rise? (*b*) If held horizontally 10 ft. from the ground, what will be its range?

$$\text{Ans.} \begin{cases} (a)\ 191.6 \text{ ft.} \\ (b)\ 87.54 \text{ ft.} \end{cases}$$

(458) A weir, whose top is 3 ft. 6 in. below the surface of the water, is 2 ft. deep and 30 in. broad; (*a*) what is the actual mean velocity? (*b*) What is the discharge in cubic feet per second? (*c*) In gallons per hour?

$$\text{Ans.} \begin{cases} (a)\ 10.44 \text{ ft. per sec.} \\ (b)\ 52.21 \text{ cu. ft. per sec.} \\ (c)\ 1{,}405{,}910.9 \text{ gal. per hour.} \end{cases}$$

(459) A pipe 12,000 ft. long and $7\frac{1}{2}$ in. in diameter discharges water under a head of 76 ft.; what is the discharge in gallons per minute? Ans. 447.6 gal.

(460) In the last example, (*a*) what is the velocity of discharge in feet per minute? (*b*) What is the discharge in cubic feet per second?

Ans. { (*a*) 195 ft. per min.
(*b*) 1 cu. ft., nearly.

(461) An 8-in. pipe has a hole in it $\frac{1}{2}$ of an inch in diameter; what would be the theoretical velocity of efflux, if the surface of the water were 10 ft. above the center of the hole?

Ans. 25.36 ft. per sec.

(462) What must be the necessary head in order that a $6\frac{1}{2}$ in. pipe 1,500 ft. long shall discharge 42,000 gallons of water per hour? Ans. 42.48 ft.

(463) A vertical cylinder having a diameter of 20 in., and a length inside of 36 in., is filled with water. A pipe having a diameter of $\frac{3}{8}$ of an inch is screwed into the upper head and fitted with a piston weighing 10 oz., on which is laid a weight of 25 lb. If the end of the pipe is 10 ft. above the level of the water in the cylinder, (*a*) what is the pressure per square inch on the bottom of the cylinder? (*b*) On the top? (*c*) What equivalent weight laid on the lower cylinder head would replace the pressure it sustains?

Ans. { (*a*) 237.75 lb. per sq. in.
(*b*) 236.45 lb. per sq. in.
(*c*) 74,691.54 lb.

(464) If, in the last example, a hole 1 inch in diameter be drilled through the cylinder wall midway of its length, and covered by a flat plate in such a manner that the water cannot leak out, what will be the pressure against the plate?

Ans. 186.22 lb.

(465) A piece of wood weighs $11\frac{1}{4}$ oz. in air. It is attached to a piece of marble weighing 5 lb. in air and 3 lb. 2 oz. in water. Both, together, weigh 2 lb. 9 oz. in water. (*a*) What is the specific gravity of the wood? (*b*) Of the marble?

Ans. { (*a*) .555.
(*b*) 2.667.

(466) What is the mean velocity of efflux from a straight pipe 4 in. in diameter and 4,000 ft. long, under a head of 120 ft.? Ans. 5.4 ft. per sec.

(467) If the length of the pipe, in the last example, had been 2,000 ft., what would the mean velocity have been in feet per second? Ans. 7.8 ft. per sec.

(468) A cylinder fitted with a piston is used as a lifting cylinder by passing a rope over a pulley and fastening one end to the piston rod. The piston is moved by means of water obtained from the city reservoir, and a gauge attached to a pipe near the cylinder shows the pressure to be 90 lb. per sq. in. The diameter of the cylinder is 19 in., and of the pipe $\frac{1}{2}$ of an inch. If friction be neglected, (*a*) how great a weight can be raised? (*b*) How great a weight could be raised if the pipe were $\frac{1}{4}$ of an inch in diameter?

Ans. (*a*) 25,517.6 lb.

(469) A 10-inch pipe 5,280 ft. long is required to deliver water with a velocity of 8 feet per second; (*a*) what is the necessary head? (*b*) What is the discharge in gallons per hour?

Ans. $\begin{cases} (a) \text{ 130.73 ft.} \\ (b) \text{ 117,504 gal. per hour.} \end{cases}$

(470) What is the actual velocity of discharge from a small, square-edged orifice in the side of a vessel, if the water at the center of the orifice has a pressure of 30 lb. per sq. in.? Ans. 65.34 ft. per sec.

(471) The upper base of a cylinder submerged in water is 40 feet below the surface. The diameter of the cylinder is 20 inches, the altitude is 36 inches, and the bases are parallel. If the bases are horizontal, (*a*) what is the upward pressure of the water on the cylinder? (*b*) The downward pressure?

Ans. $\begin{cases} (a) \text{ 5,862.85 lb.} \\ (b) \text{ 5,453.82 lb.} \end{cases}$

(472) A bottle weighs 2 lb. in air and 10 oz. in water. A pound of sugar is put into the bottle, and the bottle then weighs 16 oz. in water. What is the specific gravity of the sugar? Ans. 1.6.

(473) A jet of water issues with a velocity of 33 feet per second; what theoretical head is necessary to give it this velocity? Ans. 16.931 ft.

(474) A weir having a depth of 15 in. and a breadth of 21 in. has its top on a level with the upper surface of the water. (*a*) How many gallons will it discharge per hour? (*b*) What is the actual mean velocity in feet per second?

Ans. { (*a*) 216,551 gal., nearly.
(*b*) 3.676 ft. per sec.

(475) A 3-in. pipe 6,000 ft. long is required to deliver water at a velocity of 12 ft. per sec. What head will be necessary? Ans. 1,040.37 ft.

(476) If the weight of 40 cu. in. of lead in air is 16.4 lb., (*a*) how much will it weigh in water? (*b*) If a piece weighing 2 lb. be cut off, what will be the volume of the remaining portion?

Ans. { (*a*) 14.953 lb.
(*b*) 35.122 cu. in.

(477) A vessel having an elliptical base is filled with water. The area of the upper surface of the water is 47 sq. in., and the long and short diameters of the base are 13½ in. and 9 in., respectively. If a pressure of 12 lb. per sq. in. is applied to the upper surface, and the depth of the water is 20 in., (*a*) what is the total downward pressure? (*b*) The upward pressure?

Ans. { (*a*) 1,214.144 lb.
(*b*) 564 lb.

(478) In the last example, suppose that a flat rectangular plate 5 in. by 8 in. were so placed on the bottom of the vessel as to make an angle of 53 degrees with the base, one of the narrow edges resting upon the base. (*a*) What would be the perpendicular pressure on the plate? (*b*) The horizontal pressure? (*c*) The vertical pressure?

Ans. { (*a*) 504.314 lb.
(*b*) 402.76 lb.
(*c*) 303.5 lb.

(479) A 5-in. pipe discharges water with a velocity of 7.2 ft. per second. How many gallons will it discharge in one day? Ans. 634,478 gal., nearly.

(480) A 5½-in. pipe discharges 38,000 gallons of water per hour; what is the mean velocity in feet per second?

Ans. 8.5526 ft. per sec.

(481) A piece of brass tubing is 1 ft. long; its inside diameter is 2 in., and its outside diameter, $2\frac{1}{2}$ in. If its weight in air is 6 lb. 5 oz., what is its specific gravity?

Ans. 8.23.

(482) If the pipe which supplies a city hydrant with water is 6 in. in diameter, and the vertical height of the reservoir is 180 ft. above the pipe, (*a*) what is the pressure on a section of the pipe 1 ft. in length tending to separate one half from the other, due to the head only? (*b*) What is the pressure per sq. in. at the hydrant?

$$\text{Ans.}\begin{cases}(a) & 5{,}624.64 \text{ lb.}\\(b) & 78.12 \text{ lb. per sq. in.}\end{cases}$$

(483) A weir, whose top is on a level with the upper surface of the water, is 27 in. broad and 36 in. deep; (*a*) what is the actual discharge in cubic feet per second? (*b*) What is the theoretical discharge?

$$\text{Ans.}\begin{cases}(a) & 38.44 \text{ cu. ft.}\\(b) & 62.5 \text{ cu. ft.}\end{cases}$$

(484) If the surface of the water in a 6-in. pipe is 45 ft. above the discharge orifice, which is $1\frac{1}{2}$ in. in diameter, (*a*) what will be the theoretical velocity of efflux? (*b*) If the upper surface of the water sustains an additional pressure of 10 lb. per sq. in., what will be the velocity of efflux?

$$\text{Ans.}\begin{cases}(a) & 53.9 \text{ ft. per sec.}\\(b) & 66.28 \text{ ft. per sec.}\end{cases}$$

(485) What is the discharge in gallons per second from a 6-in. pipe, if the mean velocity is 7.5 ft. per second?

Ans. 11.016 gal. per sec.

(486) A hollow iron cylinder is 27 in. long over all; its outside diameter is 14 in.; inside diameter, 13 in., and the ends are $\frac{1}{4}$ of an inch thick. If placed in water, will it sink or float?

(487) An empty bottle weighing 1 lb. 5 oz. is filled with water, and then weighs 2 lb. When filled with linseed oil, it weighs 1 lb. 15.34 oz. What is the specific gravity of the oil?

Ans. .94.

(488) What is the velocity of discharge through a short tube whose length is twice the diameter of the orifice, if the

pressure of the water at the point of discharge is 41 lb. per sq. in.? Ans. 76.39 ft. per sec.

(489) If the head were constant, and the tube in the last example were $1\frac{1}{2}$ in. in diameter, (*a*) what would be the discharge in cubic feet per minute? (*b*) What would be the theoretical discharge? (*c*) What is the ratio between (*a*) and (*b*), or, in other words, the efficiency?

Ans. (*a*) 46.77 cu. ft.
(*b*) 57.39 cu. ft.
(*c*) 1 : .815.

(490) In the last example, suppose that the discharge had been through a square-edged orifice having the same area as the short tube; (*a*) what would the discharge have been per second? (*b*) The theoretical discharge? (*c*) The efficiency?

Ans. (*a*) .5884 cu. ft.
(*b*) .9568 cu. ft.
(*c*) 1 : .615.

(491) The base of a vessel is an ellipse whose long and short diameters are 9 in. and 5 in., respectively, and the depth of the water is 6 ft.; (*a*) what is the theoretical velocity of efflux through a 2-in. circular hole in the bottom of the vessel? (*b*) What is the pressure on the base per square inch?

Ans. (*a*) 19.722 ft. per sec.
(*b*) 2.6 lb. per sq. in.

(492) A cross-section of the upper end of a vessel filled with water is an ellipse whose axes are 6 in. and 4 in.; the lower end is circular and has a diameter of 15 in.; the depth of the water in the vessel is 24 in.; what is the total pressure upon the base when a weight of 132 lb. is laid upon the upper surface? Ans. 1,390.9 lb.

(493) A 4-in. pipe discharges 12,000 gal. per hour; what is the velocity of discharge in feet per second?

(494) The cylinder of a hydraulic press is 10 in. in diameter. The plunger is forced outwards by means of a small pump which supplies the press cylinder with water, its piston being $\frac{1}{2}$ in. in diameter, and its stroke $1\frac{1}{2}$ in. If a force of 100 lb. be applied to the pump piston, (*a*) how great

a force can it exert on the plunger? (*b*) What is the velocity ratio between the piston and the plunger? (*c*) How far does the plunger advance for one stroke of the piston?

Ans. (*a*) 40,000 lb.
(*b*) 400 : 1.
(*c*) .00375 in.

(495) How many gallons per minute will a weir 14 in. by 20 in. discharge if the top of the weir is 9 ft. below the upper surface of the liquid, (*a*) when the long side is vertical? (*b*) When the short side is vertical?

Ans. (*a*) 13,502 gal.
(*b*) 13,323 gal.

(496) What is the mean velocity for both cases of the last example?

Ans. (*a*) 15.47 ft. per sec.
(*b*) 15.27 ft. per sec.

(497) The weight necessary to sink a Nicholson's hydrometer to a fixed point on the rod is 2 lb. 8½ oz. The weight necessary to sink the hydrometer to this point, when a piece of slate is in the basket, is 1 lb. 11 oz., and when the slate is in the upper pan 12 oz.; (*a*) what is the specific gravity of the slate? (*b*) What is its volume?

Ans. (*a*) 1.9.
(*b*) 25.92 cu. in.

(498) What is the theoretical mean velocity of discharge through a weir whose depth is 3 ft., and whose top is level with the upper surface of the water? Ans. 9.26 ft. per sec.

(499) The surface of the water contained in a vessel is 19 ft. above the ground; (*a*) what is the range of the water issuing from an orifice 4 ft. 9 in. from the top? (*b*) How far below the surface is the other point of equal range? (*c*) What is the greatest range?

Ans. (*a*) 16.454 ft.
(*c*) 19 ft.

(500) A 5-in. pipe 1,300 ft. long discharges water under a head of 25 ft.; what is the number of gallons discharged per hour? Ans. 17,350 gal.

(501) What values of f would you use for $v_m = 2.37$, 3.19, 5.8, 7.4, 9.83, and 11.5, respectively?

(502) What would be the total pressure on a cube, one edge of which measures 10½ in., if sunk 3½ miles below sea-level? Ans. 5,443,383 lb.

(503) A spherical shell whose inside diameter is 19 in. is filled with water, which is subjected to a pressure of 80 lb. per sq. in. What is the pressure tending to separate one half of the sphere from its opposite half? Neglect the weight of the water. Ans. 22,682 lb.

PNEUMATICS.

(ARTS. 1039–1088.)

(504) What do you understand by *tension of gases?*

(505) A cylinder filled with compressed air supports a column of mercury 4 feet high. (*a*) What is the tension of the air in pounds per square inch? (*b*) In atmospheres? Take the weight of a cubic inch of mercury in all cases as .49 pound.

Ans. $\begin{cases} (a) & 23.52 \text{ lb.} \\ (b) & 1.6 \text{ atmos.} \end{cases}$

(506) By reason of a partial vacuum, a column of water 19 feet in height is supported by the atmosphere. (*a*) How many inches of vacuum does the gauge show, and (*b*) what is the pressure above the mercury in pounds per square inch?

Ans. $\begin{cases} (a) & 16.828 \text{ in.} \\ (b) & 6.454 \text{ lb. per sq. in.} \end{cases}$

(507) A closed vessel, fitted with a piston, contains coal gas under a pressure of three atmospheres. If the piston is so moved that the volume is 2½ times its former volume, what is the tension of the gas in pounds per square inch? The temperature is the same in both cases.

Ans. 17.64 lb. per sq. in.

(508) A certain quantity of air, under a pressure of 1½ atmospheres and a temperature of 75°, weighs 7.14 pounds. It is put in an empty vessel in which one cubic foot of air weighs .08 pound. (*a*) What is the new volume? (*b*) The temperature, the pressure remaining the same? (*c*) The original volume?

Ans. $\begin{cases} (a) & 89.25 \text{ cu. ft.} \\ (b) & 283.887^\circ. \\ (c) & 64.188 \text{ cu. ft.} \end{cases}$

(509) The temperature of the discharged air of an air compressor, the tension of which is 40 pounds per square

inch, is 120°; when it has cooled down to the temperature of the surrounding air, which is 55°, what is its tension ?
Ans. 35.517 lb. per sq. in.

(510) What is the weight of a cubic foot of air at 60°, under a pressure of one atmosphere ? Ans. .076296 lb.

(511) The total pressure upon a body is 175,000 pounds per square foot; what is the equivalent pressure in atmospheres ? Ans. 82.672 atmos.

(512) Three gases, oxygen, hydrogen, and nitrogen, are mixed together in a vessel containing 40 cubic feet. The volume and tension of the oxygen are 12 cubic feet and one atmosphere, respectively; of the hydrogen, 10 cubic feet and two atmospheres; of the nitrogen, 8 cubic feet and three atmospheres. The temperature of the separate gases and of the mixture remaining the same throughout, what is the tension of the mixture ? Ans. 20.58 lb. per sq. in.

(513) In the last example, suppose the volume of the mixture is not known, and that the tension is required to be 23 pounds per square inch; what is the volume of the mixture ? Ans. 35.79 cu. ft.

(514) A balloon is filled with 10,000 cubic feet of hot air at a temperature of 280°. If the temperature of the surrounding air is 77°, and the weight of the balloon and fixtures is 100 pounds, how great a weight will it lift ? The tension of the hot air is one atmosphere. Ans. 102.68 lb.

(515) A vessel containing 13 cubic feet of air having a temperature of 73° and a tension of one atmosphere is placed in communication with another vessel containing 18 cubic feet of air, at a temperature of 53° and a tension of 30 pounds per square inch. What is the new temperature if the tension of the mixture is 20 pounds per square inch ?
Ans. −20.65°.

(516) What is a vacuum ? Illustrate it.

(517) An air pump produces a vacuum of $\frac{1}{40}$ of an inch; what is the pressure upon a square foot ? Ans. 1.764 lb.

(518) A horizontal cylinder, closed at one end and open

at the other, is exactly fitted with a piston having a hole in it to allow the confined air to escape. The length of the cylinder is 6 feet, and the diameter is 40 inches. The piston weighs 325 pounds, and is moved inwards until the length of the space between the cylinder head and the piston is 40 inches. (*a*) How great a force will be necessary to pull the piston out of the cylinder after the hole has been plugged, if the coefficient of friction is 14% ? (*b*) To shove it in until the length of the enclosed space is 6 inches ? Assume that the temperature remains constant.

Ans. { (*a*) 8,255.55 lb.
(*b*) 104,723.612 lb.

(519) There are 8.47 cubic feet of air, under a pressure of 38 pounds per square inch. If $4\frac{1}{2}$ cubic feet be removed, what will be the tension of the remainder, the temperature remaining the same? Ans. 17.812 lb. per sq. in.

(520) A vessel containing 3 cubic feet of gas weighing .5 pound under a pressure of one atmosphere has compressed into it enough more of the gas to make it weigh 1 pound and 6 ounces; the temperature remaining the same, what is the new tension of the gas in pounds per square inch? Ans. 40.425 lb. per sq. in.

(521) If 4,516 cubic inches of gas having a temperature of 260° are cooled down to a temperature of 80°, the pressure remaining the same, what is the volume?

Ans. 1.96 cu. ft.

(522) If 55 cubic feet of air, under a pressure of $1\frac{1}{4}$ atmospheres, have a temperature of 88°, what is the weight?

Ans. 4.977 lb.

(523) Two vessels, the volumes of which are each $7\frac{1}{2}$ cubic feet, are filled with air; the temperature is the same in both, but the tension in one is two atmospheres, and in the other 40 pounds per square inch. If all of the air in one vessel is compressed into the other, what is the tension of the mixture after it has cooled down to the original temperature? Ans. 69.4 lb. per sq. in.

(524) A solid block of wood, 48″ × 36″ × 24″, weighs in air 1,200 pounds; how much will it weigh in vacuum? The temperature of the air is 60°. Ans. 1,201.83 lb.

(525) A double-acting steam pump is required to force water to a height of 127 feet; the height of the suction is 16 feet. Allowing 25% for friction, etc., (*a*) what must be the horsepower of a steam engine to drive this pump, if the diameter of the plunger is 9 inches, stroke 12 inches, and number of strokes per minute 125? (*b*) How many gallons could be discharged per hour?

Ans. (*a*) 19.942 H. P.
(*b*) 24,784.3 gal.

(526) In the last example, if the pump is single-acting, and the number of strokes per minute 100, what will be the discharge in gallons per hour? Ans. 9,914.4 gal.

(527) If you are told that the vacuum gauge of a condenser shows 23 inches vacuum, what do you understand by it? What is the pressure in the condenser?

(528) What is a pressure of one atmosphere equivalent to in pounds per square foot? Ans. 2,116.8 lb. per sq. ft.

(529) If the weight of 3 cubic feet of air at a certain temperature and under a pressure of 30 pounds per square inch is .27 pound, what is the weight of 1 cubic foot under a pressure of 65 pounds per square inch and at the same temperature? Ans. 0.195 lb.

(530) In the last example, what is the temperature of the air? Ans. 439.6°.

(531) What are the absolute temperatures corresponding to 32°, 212°, 62°, 0° and − 40°?

(532) Three and one-half pounds of air, under a pressure of 10 atmospheres, occupy a volume of 4 cubic feet. What is the temperature? Ans. − 6.583°.

(533) Fifteen cubic feet of oxygen, having a tension of 63 pounds per square inch, and 19 cubic feet of nitrogen, having a tension of three atmospheres, are mixed together in a vessel the volume of which is 25 cubic feet. The temperature of both gases and of the mixture being the same, what is the tension of the mixture? Ans. 71.316 lb,

(534) One pound of air has a temperature of 80° and a volume of 10 cubic feet; what is its tension?

Ans. 20 lb. per sq. in.

(535) If an indicator card taken from a condensing engine shows a pressure of $12\frac{1}{2}$ pounds below the atmosphere, how many inches of vacuum will the vacuum gauge show?

(536) A vacuum of 27 inches will support a column of water of what height? Ans. 30.6 ft.

(537) A certain vessel has a volume of 6.7 cubic feet. A vacuum gauge attached to it shows $17\frac{1}{2}$ inches. (*a*) How much air at atmospheric pressure will it be necessary to admit to have the vacuum gauge show 5 inches? (*b*) to show 0 inches? Ans. (*a*) $2.79\frac{1}{6}$ cu. ft.

(538) A certain vessel contains 11 cubic feet of gas weighing 2.4 pounds. If put in communication with a second vessel from which all the air has been removed, and which has a volume of 25 cubic feet, what will be the weight of a cubic foot of the gas, the temperature remaining constant?

Ans. $\frac{1}{15}$ lb.

(539) The air contained in a closed vessel, under a pressure of 12 pounds per square inch, is heated from 60° to 300°; what is its tension? Ans. 17.54 lb.

(540) What is the weight of a cubic foot of air at 212°, under a pressure of one atmosphere? Ans. .059039 lb.

(541) The diameter and stroke of the piston of an air compressor is 20 inches and 32 inches, respectively. If the discharge valve opens when the piston has completed 26 inches of its stroke, (*a*) what is the volume? (*b*) the weight? (*c*) the tension of the air discharged? Take the temperature of the outside air as 75°, and the temperature at discharge as 125°.

Ans. (*a*) 1,884.96 cu. in. = 1.0908 cu. ft.
(*b*) .43143 lb.
(*c*) 85.727 lb. per sq. in.

(542) Nineteen cubic feet of air, having a tension of 12 pounds per square inch, are mixed in a vessel which holds 30 cubic feet, with 21 cubic feet of air from another vessel. If

the tension of the mixture is 35 pounds per square inch, what was the tension of the second volume of gas?

Ans. 39.14 lb. per sq. in.

(543) A vessel containing 45 cubic feet of air at a temperature of 60° and a tension of 13 pounds per square inch is emptied into another vessel containing 60 cubic feet of air at a temperature of 80° and a tension of 17 pounds per square inch. What is the tension of the mixture if its temperature is 72°? Ans. 26.723 lb. per sq. in.

(544) What is a *partial vacuum?* If enough air is admitted to the vacuum chamber to cause the column of mercury to be $4\frac{1}{2}$ inches shorter than the barometer column, how many inches of vacuum will the gauge show?

(545) By reason of a partial vacuum, a column of alcohol 16 feet high is supported. How many inches will the vacuum gauge show? Ans. 11.337 in.

(546) The stroke and diameter of the piston of a blowing engine (one form of an air compressor) are each 80 inches. The valves are so set that they will open for discharge when the tension of the compressed air becomes 9 pounds above the atmosphere. (*a*) At what point of the stroke will the valves open? (*b*) How many cubic feet of air having this tension will be discharged during one stroke of the piston, the temperature being constant throughout?

$$\text{Ans.} \begin{cases} (a)\ 30.38 \text{ in.} \\ (b)\ 144.34 \text{ cu. ft.} \end{cases}$$

(547) A certain quantity of air under a pressure of $3\frac{1}{2}$ atmospheres weighs 13 pounds. After expanding under a constant temperature, the weight of the same volume is only 2 pounds. What is the tension of the air?

Ans. 7.915 lb. per sq. in.

(548) The stroke of the piston in an air compressor is 60 inches. When the piston has traveled 50 inches, what is the tension (the temperature at discharge being 130°) of the enclosed air, assuming that the delivery valves do not open until this point is reached? The original temperature is 60°. Ans. 100.07 lb. per sq. in.

(549) A pound of air has a temperature of 127° and a tension of 27 pounds per square inch. What is its volume?
Ans. 8.055 cu. ft.

(550) The weight of a certain body of air having a tension of 4,000 pounds per square foot and a temperature of 100° is .5 pounds. What is its volume? Ans. 3.735 cu. ft.

(551) Forty cubic feet of air having a temperature of 100° and a tension of 90 pounds per square inch are mixed with 57 cubic feet having a temperature of 130° and a tension of 80 pounds per square inch. The tension of the mixture is 120 pounds per square inch and the temperature is 110°. What is the volume? Ans. 67.248 cu. ft.

(552) If a bottle fitted with a rubber bulb, as shown in Fig. 199, be filled with water until the air in the bulb occupies a space of 20 cubic inches under a tension of exactly one atmosphere, what will be the total pressure on the bottom of the bottle caused by squeezing the bulb until the space inside is only $\frac{1}{4}$ of its original volume? The opening in the neck of the bulb is $\frac{1}{4}$ of a square inch, the bottom of the bottle is 3 inches in diameter, and the depth of the water is 12 inches. Ans. 314.793 lb.

(553) Four cubic feet of air is heated under a constant pressure from 40° to 115°. What is the resulting volume?
Ans. 4.6 cu. ft.

STRENGTH OF MATERIALS.

(ARTS. 1089–1179.)

(554) (*a*) What are the advantages and disadvantages of cast iron as a material for construction purposes? (*b*) Of steel? (*c*) Of wrought iron? (*d*) For what purposes are each of the above largely used?

(555) Define the following: (*a*) Stress; (*b*) strain; (*c*) elasticity; (*d*) coefficient of elasticity; (*e*) ultimate strength.

(556) In how many ways may stress be applied to a body? Give an example of each.

(557) How much will a wrought-iron rod 2″ in diameter and 10 feet long be elongated by a pull of 40 tons?

Ans. .122 in.

(558) A steel rod 7½″ long and ½″ in diameter is elongated .009″ by a pull of 7,000 pounds; what is the coefficient of elasticity?

Ans. 29,708,853 lb. per sq. in.

(559) What force is required to produce an elongation of .006″ in a cast-iron bar 1½″ × 2″ and 9 feet long?

Ans. 2,500 lb.

(560) A wooden rod 3″ in diameter is elongated .05″ by a force of 2,000 pounds; what was its original length?

Ans. 265.07 in.

(561) What is the minimum diameter of a wrought-iron bolt which is to withstand a steady pull of 6 tons with safety?

Ans. 1.054 in.

(562) How long must a cast-iron bar be, if supported vertically at its upper end, to break under its own weight?

Ans. 6,400 feet.

(563) The diameter of a wrought-iron bolt is $\frac{3}{4}''$; what should be the thickness of the bolt head in order that the bolt may be equally strong in tension and in shear; in other words, what should be the thickness of the head in order that the force required to pull the bolt apart shall be just equal to the force required to strip the head from the bolt ?
Ans. .206 in.

(564) What safe load may be borne by a brick pier, the cross-section of which is $2\frac{1}{2} \times 3\frac{1}{2}$ feet ? Ans. 105 tons.

(565) In Fig. 3, the force P is $1\frac{3}{4}$ tons and acts at an angle of 30° with the horizontal. The width b of the timber tie rod is 4 inches. Find the safe length a necessary to prevent shearing on the surface a b.
Ans. 10.1 in.

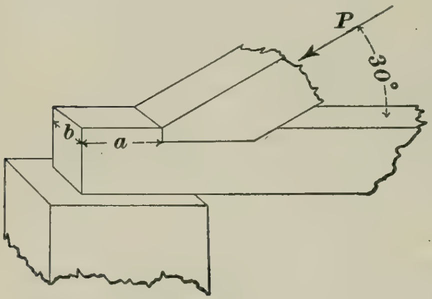

FIG. 3.

(566) Show that a cylindrical boiler is twice as liable to rupture along a longitudinal seam as along a transverse seam.

(567) Assuming that the strength of the boiler plate is reduced 40% owing to the area lost by the rivet holes, what should be the thickness of the plate of a wrought-iron boiler 4 feet in diameter which sustains a steam pressure of 120 lb. per sq. in ? Assume factor of safety for steady stress.
Ans. .35 in.

(568) What should be the thickness of a 6″ cast-iron water pipe to carry a steady pressure of 200 lb. per sq. in. ?
Ans. .18 in.

(569) What should be the thickness of a wrought-iron boiler tube 3″ in diameter, 12 ft. long, and exposed to an external steam pressure of 130 lb. per sq. in.? Use a factor of safety of 10. Ans. .272 in.

(570) A cast-iron cannon of 4″ bore is subjected to an internal pressure of 2,000 lb. per sq. in. on being fired; what should be the thickness of the metal in order that it shall not be subjected to a stress of more than 2,800 lb. per sq. in.?

Ans. 5 in.

(571) A structure of the nature of a simple beam is 100 feet long between the supports, and carries three equal loads of 1,200 lb. each, at distances from the left support of 40, 60, and 80 ft. (*a*) Find the maximum bending moment; (*b*) the shear at a distance of 30 ft. from the left support; (*c*) the maximum shear.

Ans. { (*a*) 748,800 in.-lb.
(*b*) 1,440 lb.
(*c*) – 2,160 lb.

(572) Find graphically the magnitude, direction, and position of the resultant of the forces shown in Fig. 4.

NOTE.—In Figs. 4 and 6, the line mn is drawn simply to locate the forces. Thus, in Fig. 4, F_1 makes an angle of 47½° with mn; F_2 makes an angle of 72½° with mn, and so on. The distance between the intersection of F_1 and F_2 with mn is 1 inch; the distance between the intersections of F_3 and F_2 is 1¼ inches, etc.

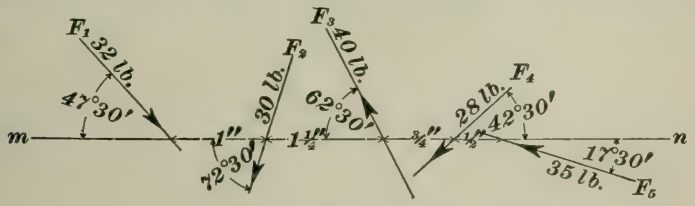

FIG. 4.

(573) A beam is loaded as in Fig. 5. Find the reaction of the supports, and draw the shear diagram.

(574) (*a*) Where is the maximum bending moment of the above beam? (*b*) What is it in inch-tons? (*c*) In inch-pounds?

Ans. { (*b*) 288 in.-tons.
(*c*) 576,000 in.-lb.

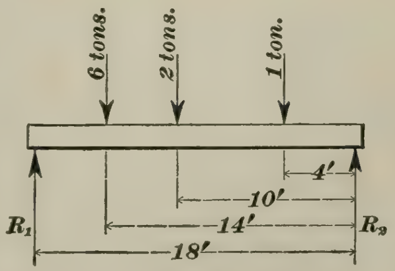

FIG. 5.

(575) Define the following: Simple beam; cantilever; restrained beam; continuous beam.

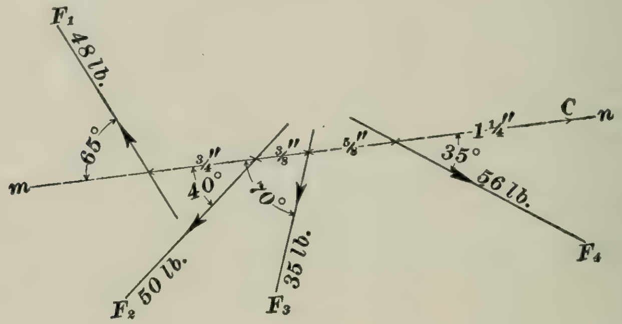

FIG. 6.

(576) Find the resultant moment of the forces of Fig. 6 about the point C.

(577) A rectangular white pine beam 22 feet long, carries a load of 4 tons, 8 feet from one end. Assuming the depth of the beam to be double the width, what should be its dimensions to safely support the load?

Ans. Depth = $17\frac{1}{3}$ in.; breadth = $8\frac{2}{3}$ in.

(578) The beam of Fig. 7 is loaded as shown and has, in addition, a uniform load of 40 pounds per foot. (*a*) Find the maximum shear; (*b*) the maximum bending moment; (*c*) the reactions of the supports.

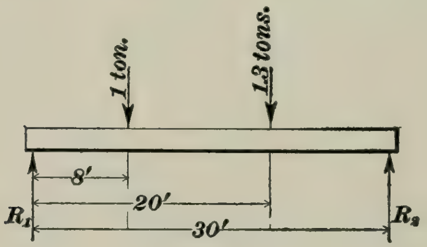

FIG. 7.

Ans.
(*a*) 2,930 lb.
(*b*) 320,000 in.-lb.
(*c*) 2,930 and 2,870 lb.

(579) The moment of inertia of a wrought-iron I beam 12″ deep is 280; what uniform load will it sustain? The stress is assumed to be steady. The beam is 20 ft. long and rests upon two supports.

Ans. 875 lb. per ft.

(580) A hollow circular cast-iron beam 8 feet long rests upon two supports. The inside diameter is 5 in. and the outside diameter 6½ in.; what is the maximum safe load that may be concentrated at its center? Ans. 4,624 lb.

(581) A hemlock floor-beam is 2″ × 10″ and 16 feet long; (*a*) what distributed load will it carry? (*b*) What distributed load would it carry if laid with the 10″ side horizontal?

Ans. (*a*) 78.12 lb. per ft.
(*b*) 15.6 lb. per ft.

(582) Find the deflections of the beams of (*a*) Example 579, (*b*) Example 580, and (*c*) Example 581 (*a*).

Ans. (*a*) .45 in.
(*b*) .1 in., nearly.
(*c*) .461 in.

(583) The piston of a steam engine is 14 in. in diameter; the steam pressure is 80 lb. per sq. in. Assuming that the total pressure on the piston comes on the crank-pin at the dead points, and considering the crank-pin as a cantilever uniformly loaded, what should be its diameter if 4 inches long and made of wrought iron? Take a factor of safety of 10. Ans. 3.82 in.

(584) What load can be safely sustained by a cast-iron column 14 feet long and 6″ in diameter, with flat ends?

Ans. 120,872 lb.

(585) What should be the dimensions of a piece of timber 30 ft. long, with flat ends, to support a load of 7 tons, the cross-section being in the form of a square?

Ans. 9⅝ in. square.

(586) A cylindrical steel connecting-rod 7½ ft. long is subjected to a maximum stress of 21,000 pounds. Assuming it to be a column with both ends hinged, and subjected to shocks, determine its diameter to the nearest ⅛″. Ans. 2⅝ in.

(587) A wrought-iron piston rod has a diameter of 2″ and a length of 4 feet. Assuming it to be a column with one end flat and the other rounded, what is the allowable diameter of the piston if the steam pressure is 60 pounds per sq. in.? The rod is subjected to shocks. Ans. 15⅞ in.

(588) A beam is 6″ × 8″ and 10 feet long, the 8″ side being vertical. Another beam of the same material is 4″ × 12″ and 16 feet long, the 12″ side being vertical. (*a*) What is the ratio between the maximum loads the two beams are capable of supporting? (*b*) What is the ratio between the deflections in the two cases, the manner of loading being the same?

Ans. $\begin{cases} (a) & 1\frac{1}{15} : 1. \\ (b) & .549 : 1. \end{cases}$

(589) (*a*) What should be the diameter of a round wrought-iron shaft to transmit 40 H. P. at 120 R. P. M.? (*b*) To transmit 80 H. P. at 100 R. P. M.?

Ans. $\begin{cases} (a) & 3.739 \text{ in.} \\ (b) & 4.65 \text{ in.} \end{cases}$

(590) What must be the diameter of a steel engine shaft to transmit 4,000 H. P. at 50 R. P. M.? Ans. 14.22 in.

(591) What H. P. can be transmitted by a steel shaft 4″ square, making 80 R. P. M.? Ans. 71.775 H. P.

(592) What horsepower can be transmitted by a hollow wrought-iron shaft making 100 R. P. M., the outside diameter being $7\frac{1}{2}$ in., and the inside diameter 5 in.?

Ans. 717.7 H. P.

(593) (*a*) What weight may be safely sustained by a hemp rope 8″ in circumference? (*b*) What should be the diameter of an iron wire rope of 7 strands to safely sustain a stress of 6,000 pounds? (*c*) What should be the circumference of a steel hoisting rope which is required to lift a load of $6\frac{2}{3}$ tons?

Ans. $\begin{cases} (a) & 6{,}400 \text{ lb.} \\ (b) & 1.054 \text{ in.} \\ (c) & 3.65 \text{ in.} \end{cases}$

(594) (*a*) An open-link chain is made of $\frac{7}{8}$″ iron; what safe load will it sustain? (*b*) What should be the diameter of the iron composing the links of a stud-link chain which is to sustain a load of 4 tons?

Ans. $\begin{cases} (a) & 9{,}187.5 \text{ lb.} \\ (b) & .667 \text{ in.} \end{cases}$

(595) A steel shaft 2″ in diameter is supported by hangers 10 feet apart; if a pulley be placed midway between the hangers, what should be the maximum allowable belt pull on

the pulley in order that the shaft shall not deflect more than $\frac{1}{8}''$? Ans. 327.25 lb.

SUGGESTION.—Assume the shaft to be a restrained beam, loaded in the middle.

(596) An engine shaft rests on bearings 5 feet apart; midway between the bearings the shaft carries a fly-wheel weighing 3 tons. (*a*) Find the size of the shaft to withstand this load. (*b*) Find the size of the shaft on the assumption that the engine makes 80 R. P. M. and develops 75 H. P. The shaft is made of steel. Factor of safety 10.

Ans. { (*a*) $4\frac{1}{4}$ in., nearly.
(*b*) $4\frac{5}{8}$ in., nearly.

(597) A white oak cantilever is loaded as shown in Fig. 8, and has in addition a uniform load of 80 pounds per foot. (*a*) Determine graphically the shear diagram and the maximum bending moment. (*b*) Find the necessary dimensions of the cross-section of the beam, assuming the depth to be $2\frac{1}{2}$ times the breadth, and the stress to be steady.

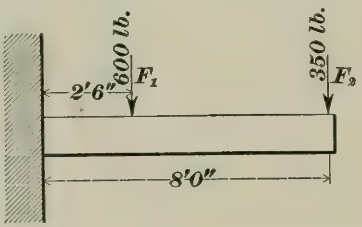

FIG. 8.

Ans. { (*a*) 82,320 in.-lb.
(*b*) $b = 3.7$ in.; $d = 9\frac{1}{4}$ in.

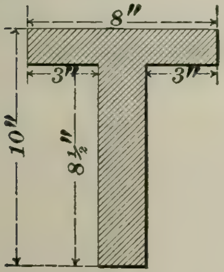

FIG. 9.

(598) A steel beam of the cross-section shown in Fig. 9 rests upon two supports 35 feet apart; (*a*) what concentrated load will it sustain at the center, the stress being considered as varying? (*b*) What total uniform load will it sustain, the stress being steady?

Ans. { (*a*) 7,246 lb.
(*b*) 20,290 lb.

(599) (*a*) The moment of inertia of a rectangle is 72; the area of the rectangle is 24 sq. in.; what is the value of r in formula **72,** Art. **1154?** (*b*) What are the dimensions of the rectangle? (*c*) The moment of inertia of a circle about

an axis through its center is $\frac{\pi d^4}{64}$; the area of the circle is $\frac{\pi d^2}{4}$; what is the radius of gyration of the circle?

$$\text{Ans.} \begin{cases} (a) \ 1.732. \\ (b) \ b = 4 \text{ in.}; \ d = 6 \text{ in.} \\ (c) \ \frac{1}{4} d. \end{cases}$$

(600) A cast-iron sphere 8″ in diameter is subjected to a steady internal pressure of 100 lb. per sq. in.; what should be its minimum thickness to safely withstand this pressure?
Ans. .06 in.

(601) A beam 28 feet long, uniformly loaded with 60 pounds per foot, rests upon two supports which are placed 5 feet from each end. Determine graphically the maximum shear and the maximum bending moment.
Ans. Bending moment = 20,160 in.-lb.

(602) Assuming the above beam to be made of white pine, find the dimensions of its cross-section, the stress being steady. The beam is to be rectangular.

(603) A wrought-iron boiler tube $2\frac{1}{2}$″ in diameter and 9 feet long has a thickness of .2″; what external pressure can the tube withstand, using a factor of safety of 10?
Ans. 106.43 lb. per sq. in.

(604) A crane chain is required to lift 5 tons; what should be the diameter of the iron composing the links of the chain, which is of the open-link variety? Ans. .913 in.

(605) A gear-wheel 4 feet in diameter is keyed to a shaft. The force acting tangent to the pitch circle of the gear-wheel is 350 pounds; what should be the diameter of the shaft if made of steel? Ans. 2.84 in., say $2\frac{7}{8}$ in.

(606) The head of an engine cylinder, 12″ in diameter, is fastened on by 10 wrought-iron bolts. In order to make the joint steam tight, a safe stress of only 2,000 pounds per sq. in. is allowed. The steam pressure being 90 pounds per sq. in., what should be the minimum diameter of the bolts, that is, the diameter at the root of the thread? Ans. .8 in.

(607) A beam 18 feet long has a load of 1 ton placed at each end. The supports are placed 5 feet from each end. (See Fig. 10.) (*a*) Draw the shear diagram and find the maximum bending moment. (*b*) If made of cast iron and circular in cross-section, what should be its diameter, the stress being steady?

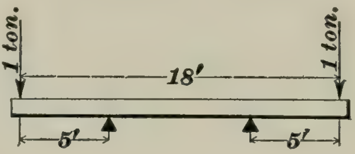

FIG. 10.

Ans. { (*a*) 120,000 in.-lb.
(*b*) 5.784 in.

(608) A beam 2″ × 6″ in cross-section and 18 feet long deflects .3″; how much will a beam 3″ × 8″ and 12 feet long deflect under the same load? The long side is placed vertically in both cases. Ans. .025 in.

(609) A wrought-iron key, or cotter, is used to fasten a rod, as shown in Fig. 11. There is a pull of 20,000 pounds on the rod. Assuming the breadth of the key to be four times the thickness, find its dimensions to safely resist the stress. Use a factor of safety of 10.

Ans. { Breadth, 2.828″.
Thickness, .707″.

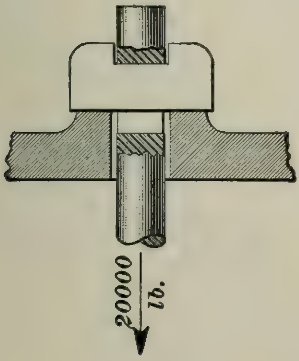

FIG. 11.

(610) A steel engine shaft 12″ in diameter, resting in bearings 54″ apart, carries a fly-wheel weighing 30 tons, midway between the bearings; what is the deflection of the shaft? Consider the shaft as a simple beam. Ans. .0064 in.

(611) (*a*) A 7-strand steel wire rope has a diameter of $1\frac{1}{4}''$; what load will it carry safely? (*b*) Find the circumference of a hemp rope to carry $1\frac{3}{4}$ tons.

Ans. { (*a*) 14,062.5 lb.
(*b*) 5.92 in.

(612) What stress would be required to rupture a cast-iron cylinder which is 8″ in diameter and 6″ thick?

Ans. 12,000 lb.

(613) A beam which is 100 feet long carries four equal loads of 1,200 pounds each, at distances of 20, 40, 60, and 80 feet from the left support. (*a*) Find the maximum bending moment and (*b*) the shear at both supports.

Ans. $\begin{cases} (a)\ 864{,}000 \text{ in.-lb.} \\ (b)\ 2{,}400 \text{ lb.} \end{cases}$

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